Nonlinear Systems

Lecture 14 03/12/13

Last time:

- Comparison functions
- Stability of time varying systems

K & KL functions

$$\begin{array}{ccc} K & & KL \\ \downarrow & & \downarrow \\ \alpha(r) & & \beta(r,s) \\ \downarrow & & \downarrow \quad \searrow \\ [0,\infty) & & [0,\infty) \quad [0,\infty) \end{array}$$

r: norm of an initial condition
s: $t-t_0$
} system theoretic interpretation

Can use K & KL functions to characterize <u>uniform</u> stability and <u>uniform</u> asymptotic stability.

Uniform stability:

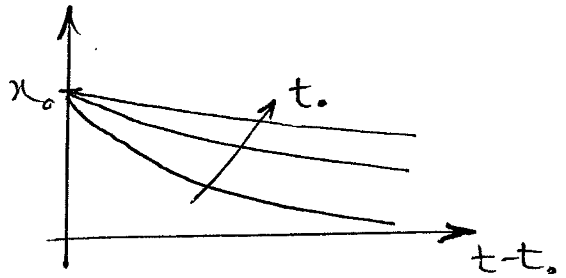
$$\|x(t)\| \leq \alpha(\|x_0\|)$$

Uniform AS.:

$$\|x(t)\| \leq \beta(\|x_0\|, t-t_0)$$

Ex $\quad \dot{x} = \dfrac{-x}{1+t}$

$$x(t) = \frac{1+t_0}{1+t} x_0 = \frac{x_0}{1 + \frac{t-t_0}{t_0+1}}$$

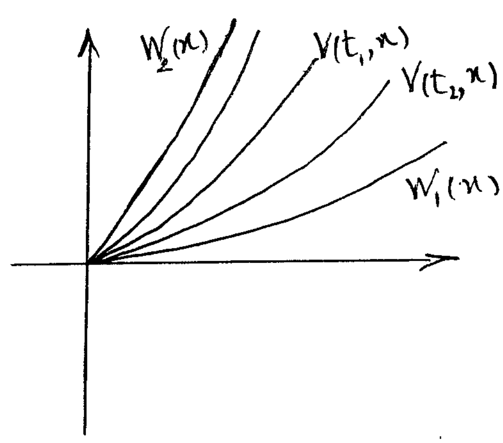


Lyapunov theory for time-varying systems :

$V(t, x)$

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f(t,x)$$

$$\alpha_1(\|x\|) \leq W_1(x) \leq V(t,x) \leq W_2(x) \leq \alpha_2(\|x\|)$$



Ex.

$$V(t,x) = x^T \cdot P(t) \cdot x$$

Thm. 1) If $W_1(x) \leq V(t,x) \leq W_2(x)$ and $\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq 0$ for some positive definite functions W_1 and W_2 on domain D that includes the origin $\bar{x}=0$, then $\bar{x}=0$ is uniformly stable

2) If further $\dot{V}(t,x) \leq -W_3(x)$ $\forall x \in D$ for some positive definite W_3, then $\bar{x}=0$ is uniformly asymptotically stable.

3) If $D = \mathbb{R}^n$ and W_1 is radially unbounded, then $\bar{x}=0$ is globally uniformly AS.

4) If $W_i(x) = K_i \|x\|^a$ for some constants $K_1, K_2, K_3 > 0$ then $\bar{x}=0$ is exponentially stable. $\left(\beta(\|x_0\|, t-t_0) = C\|x_0\| e^{-\lambda(t-t_0)} \quad c, \lambda > 0 \right)$

Proof: 1. $\alpha_1(\|x\|) \leq W_1(x) \leq V(t,x) \leq W_2(x) \leq \alpha_2(\|x\|)$

Aside

Any positive definite function can be bounded as below

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

we thus have:

$$\dot{V}(x,t) \leq 0 \Rightarrow V(x(t),t) \leq V(x(t_0),t_0)$$

$$\Rightarrow \alpha_1(\|x\|) \leq V(x(t),t) \leq V(x(t_0),t_0) \leq \alpha_2(\|x_0\|)$$

$$\Rightarrow \alpha_1(\|x(t)\|) \leq \alpha_2(\|x_0\|) \qquad (*)$$

We have: $\alpha \in K_{[0,a)}$

$\alpha^{-1} \in K_{[0,a)}$

e.g. $\alpha(x) = x^2$

$y = x^2$

$x = \alpha^{-1}(y)$

$\alpha^{-1}(y) = \sqrt{y} = y^{1/2}$

Now if we act α^{-1} on $(*)$ we have

$\alpha_1^{-1} \Big/ \alpha_1(\|x(t)\|) \leq \alpha_2(\|x_0\|)$

$$\|x(t)\| \leq (\alpha_1^{-1} \circ \alpha_2)(\|x_0\|) = \alpha_1^{-1}(\alpha_2(\|x_0\|))$$

Lemma 4.2 (Khalil):

- Inverse of class-K function is well defined locally (globally if $\alpha \in K_\infty$) and is class K.

- The composition of class-K functions is class-K.

e.g. think of $x^2 \; x^{1/3} = x^{2/3}$

$\alpha : \alpha_1^{-1} \circ \alpha_2$ is class K

$\|x(t)\| \leq \alpha(\|x(t_0)\|) \Rightarrow$ US (Uniform Stability)

2. $\dot{V}(x,t) \leq -W_3(x) \leq -\alpha_3(\|x\|)$

Note! • $V(x,t) \leq \alpha_2(\|x\|)$

$\Rightarrow \alpha_2^{-1}(V(x,t)) \leq \|x\|$

• $\alpha_1(\|x\|) \leq V(x,t) \Rightarrow \|x\| \leq \alpha_1^{-1}(V(x,t))$

$\alpha_3(\alpha_2^{-1}(V)) \leq \alpha_3(\|x\|)$

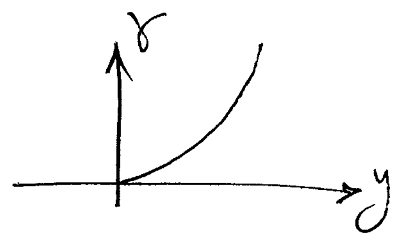
$\Rightarrow \dot{V}(x,t) \leq -W_3(x) \leq -\alpha_3(\|x\|) \leq -\alpha_3(\alpha_2^{-1}(V)) := -\gamma(V)$

$\dot V(x,t) \leq -\gamma(V(x,t))$

γ : class K

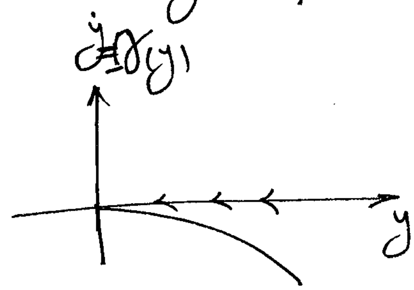
let us examine

$\dot y = -\gamma(y)$



γ: class-K

$y(t)$: scalar (non-negative)



Fact: $y(t) = \beta(y(t_0), t - t_0)$

class KL (Khalil)

Ex $\dot y = -Ky \Rightarrow y(t) = y_0 e^{-K(t-t_0)}$

$K > 0$

$\beta(r,s) = re^{-Ks}$

$\dot y = -Ky^2 \Rightarrow y(t) = \dfrac{y_0}{Ky_0(t-t_0)+1}$

$$\beta(r,s) = \frac{r}{Krs+1}$$

If $\quad \dot{V} \leq -\gamma(V)$

$$\dot{y} = -\gamma(V) \Rightarrow y(t) = \beta(y_0, t-t_0)$$

$$\Downarrow$$

$$V(x,t) \leq \beta(V(x_0,t_0), t-t_0)$$

We have accomplished a bound on $V(x,t)$ where

$$\alpha_1(\|x\|) \leq V(x,t) \leq \beta(V(x_0,t_0), t-t_0) \leq \alpha_2(\|x_0\|)$$

$$\alpha_1(\|x\|) \leq \beta(\alpha_2(\|x_0\|), t-t_0)$$

$$\|x(t)\| \leq \alpha_1^{-1}\underbrace{(\beta(\alpha_2(\|x_0\|), t-t_0))}_{\text{again from Khalil Lemma 4.2}}$$

$$\|x(t)\| \leq \tilde{\beta}(\|x_0\|, t-t_0)$$

(Khalil Lemma 4.2)

$$\alpha_1(\beta(\alpha_2(r), s)) = \tilde{\beta}(r,s) \longrightarrow KL$$

3. If $W_1(x)$ is radially unbounded it follows that $\alpha_1 \in K_\infty \Rightarrow \alpha_1^{-1}$ exists globally

4. $\alpha_i(\|x\|) = K_i \|x\|^a$

$$\boxed{\gamma(V)} = \alpha_3(\alpha_2^{-1}(V)) = \alpha_3\left(\left(\frac{V}{K_2}\right)^{1/a}\right) = K_3\left(\left(\frac{V}{K_2}\right)^{1/a}\right)^a = \boxed{\frac{K_3}{K_2} V}$$

if $V = K_i \|x\|^a = \alpha_2(\|x\|)$

$$\|x\| = \left(\frac{V}{K_i}\right)^{1/a} = \alpha_i^{-1}(V)$$

$$\Rightarrow \quad \dot{y} = -\frac{K_3}{K_2} y$$

$$y(t) = y_0 e^{-K_3/K_2 (t-t_0)}$$