Nonlinear Systems

Lecture 15 03/14/13

P1 from midterm

$$\dot{x}_1 = -x_1 + x_2^2 \qquad\qquad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2$$

$$\dot{x}_2 = -2x_1x_2 - x_2$$

a) e.p. $\quad 0 = -\bar{x}_1 + \bar{x}_2^2 \longrightarrow \bar{x}_1 = \bar{x}_2^2$

$\quad 0 = -2\bar{x}_1\bar{x}_2 - \bar{x}_2 = -(2\bar{x}_1+1)\bar{x}_2$

$\Downarrow$

$0 = -(2\bar{x}_2^2+1)\bar{x}_2$

$\bar{x}_2 = 0$

$\bar{x}_1 = 0$

$\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a unique e.p.

b) linearization $\quad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

but nothing can be said about global properties!

use Lyapunov based argument

$V(x) = \frac{1}{2}ax_1^2 + \frac{1}{2}bx_2^2 \quad a,b>0 \quad$ (g.p.d, radially unbounded)

$\dot{V}(x) = ax_1\dot{x}_1 + bx_2\dot{x}_2 = -ax_1^2 + ax_1x_2^2 - 2bx_1x_2^2 - bx_2^2$

Note! if $a = 2b > 0 \Rightarrow \dot{V}(x) = -2bx_1^2 - bx_2^2 < 0$

Stability of time varying systems:

Ex $\dot{x} = -g(t)x^3$; $g(t) \geq 1 \quad \forall t$

propose: $V(x) = \frac{1}{2}x^2 \Rightarrow W_1(x) = W_2(x) = \frac{1}{2}x^2$

$\dot{V} = x\dot{x} = -g(t)x^4 \leq -x^4 \Rightarrow W_3(x) = x^4$

Note! no exponential stability because we don't have $W_i(x) = K_i \|x\|^a$ $\longrightarrow$ there is a difference in the powers between W's.

Uniform global asymptotic stability but <u>not</u> exponential!

$a_1 = a_2 = 2$; $a_3 = 4$

Time invariant:

$$x(t) = \operatorname{sign}(x_0)\sqrt{\frac{x_0^2}{1+2(t-t_0)x_0^2}}$$

Algebraic Decay to zero (slower than exponential)

If Linearization is unstable @ the origin we can conclude that we do not have exponential decay. In other words, nonlinearities cannot affect stability (locally) and cause stability.

Note! linearization of $\dot{x} = -x^3$ is given by $\dot{x} = 0 \cdot x$

$\Rightarrow$ signal for the lack of exponential stability of $\bar{x} = 0$ of the nonlinear system.

* Lyapunov functions for time varying linear systems:

$\dot{x} = A(t)\, x$ $\qquad\qquad$ a) $0 < K_1 \|x\|^2 \leq V(x,t) \leq K_2 \|x\|^2$

$V(x) = x^T P(t)\, x$

$$V(x,t) = x(t)^T P(t)\, x(t)$$

$$\dot{V}(x,t) = \dot{x}^T(t) P(t) x(t) + \underbrace{x^T(t) \dot{P}(t) x(t)}_{\partial V / \partial t} + x^T(t) P(t) \dot{x}(t)$$

$$= x^T(t) \underbrace{\left[\dot{P}(t) + A^T(t) P(t) + P(t) A(t) \right]}_{-Q(t)} x(t)$$

$$= -x^T(t)Q(t)x(t) \leq -K_3\|x\|^2 \quad ; \quad K_3 > 0$$

$\Leftrightarrow$

<u>Aside</u> if there is $K_3 > 0$ s.t. $0 < K_3\|x\|^2 \leq x^T Q(t) x$

So we had

a) $0 < K_1\|x\|^2 \leq V(x,t) \leq K_2\|x\|^2$

$$0 < K_1 I \leq P(t) \leq K_2 I \quad \text{for all } t$$

$$0 < K_1 x^T x \leq x^T P(t) x \leq K_2 x^T x$$

b) $\dot{V}(x) = -x^T Q(t) x \quad ; \quad 0 < K_3 I \leq Q(t) \quad \forall t$

c) $\dot{P}(t) + A^T(t)P(t) + P(t)A(t) = -Q(t)$

then we can conclude

uniform global exponential stability.

The converse is true :

Suppose $\bar{x} = 0$ of $\dot{x} = A(t)x$ is uniformly exp. stable, $A(t)$ is cts and bounded, $Q(t) = Q^T(t)$ is cts, and $0 < K_3 I \leq Q(t) \leq K_4 I$

Then there is $P(t) = P^T(t)$ st.

Differential equation (DLE) Lyapunov $\leftarrow$ $\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + Q(t) = 0$, and

$$0 < K_1 I \leq P(t) \leq K_2 I$$ [redacted]

Recall :
$$A^T P + PA + Q = 0 \quad \Rightarrow P = \int_0^\infty e^{A^T t} Q e^{At} dt$$
A : Hurwitz

For the time varying case we build upon this and replace state transition matrix instead of e^{At}.

Therefor the sol'n to (DLE) is:

$$P(t) = \int_t^\infty \Phi^T(\tau, t) Q(\tau) \Phi(\tau, t) d\tau$$

Φ : state transition matrix

Question $\dot{x} = A(t) x$

$$Re(\lambda_i(A(t))) < 0 \xrightarrow{?} \text{exp. stability.}$$

Ex $A(t) = \begin{bmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\sin t \cos t \\ 1 - \frac{3}{2}\sin t \cos t & -1 + \frac{3}{2}\sin^2 t \end{bmatrix}$

$$\lambda_{1,2}(A(t)) = -\frac{1}{4} \pm j\frac{\sqrt{7}}{4}$$

But if you look at the state transition matrix:

$$\Phi(t,0) = \begin{bmatrix} e^{\frac{1}{2}t}\cos t & e^{-t}\sin t \\ e^{-\frac{1}{2}t}\sin t & e^{-t}\cos t \end{bmatrix}$$

$\Rightarrow$ There is no $K, \lambda > 0$ s.t. $\|\Phi(t,t_0)\| \leq K e^{-\lambda(t-t_0)}$