Nonlinear Systems

Lecture 25 04/30/13

Input to state stability

Linear Systems

$\dot{x} = Ax + Bu$

stability of $\dot{x} = Ax$ guarantees boundedness of the state for bounded inputs

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

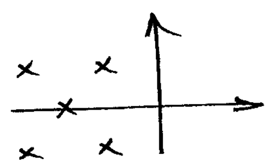
Vector norm

$$\|x(t)\| \leq \underbrace{\|e^{At}\|}_{\text{induced norm (largest sing. value of matrix exp.)}}\|x_0\| + \int_0^t \|e^{A(t-\tau)}\|\|B\|\|u(\tau)\|\,d\tau$$

$$\leq K \cdot e^{-\alpha t}\|x_0\| + \|B\| \sup_{0\leq\tau\leq t}\|u(\tau)\| \int_0^t Ke^{-\alpha\tau}\,d\tau$$

$$\Rightarrow \|x(t)\| \leq \underbrace{K e^{-\alpha t} \|x_0\|}_{\text{effect of initial cond.}} + \underbrace{\frac{K}{\alpha} \|B\| \sup_{0 \leq \tau \leq t} \|u(\tau)\|}_{\text{effect of inputs}}$$

Derived under the assumption that $\dot{x} = Ax$ is stable (e-values of A in the LHP)



Unfortunately, for nonlinear systems this property doesn't hold.

Ex (Counter example)

$$\dot{x} = -x + xu$$

note! $\dot{x} = -x$ is stable but any u with $|u(t)| > 1$ $\forall t$ is going to generate unbounded response.

e.g. $u(t) = 2$

$$\dot{x} = -x + x \cdot 2 \Rightarrow \dot{x} = x \Rightarrow x(t) = e^{t} x_0$$

Def. A system $\dot{x} = f(x,u)$ is input-to-state stable (ISS) if:

$$\|x(t)\| \leq \underbrace{\beta(\|x_0\|, t)}_{\text{Class KL function}} + \underbrace{\gamma\left(\sup_{0\leq\tau\leq t}\|u(\tau)\|\right)}_{\text{Class K function}}$$

For linear systems
$$\beta(r,t) = K e^{-\alpha t} \cdot r$$
$$\gamma(s) = \frac{K}{\alpha}\|B\| \cdot s$$

Implications of ISS:

1) $\dot{x} = f(x,u)$ is ISS $\Rightarrow$ $\dot{x} = f(x,0)$ is globally asymptotically stable

2) If $u(t) \xrightarrow{t\to\infty} 0 \Rightarrow x(t) \xrightarrow{t\to\infty} 0$

A dissipation like inequality for ISS:

If there are class K_∞ functions $\alpha_i(.)$; $i=1,2,3,4$ and a cts differentiable function $V(x)$ st.

$$\alpha_1(\|x\|) \leqslant V(x) \leqslant \alpha_2(\|x\|)$$

$$\dot{V} = \frac{\partial V}{\partial x} f(x, u) \leqslant -\alpha_3(\|x\|) + \alpha_4(\|u\|)$$

proof $\rightarrow$ (Khalil)

Ex $\quad \dot{x} = -x^p + x^q u$

p is an odd integer

ISS if $p > q$

$$V(x) = \tfrac{1}{2} x^2 \Rightarrow \dot{V} = x \dot{x} = -x^{p+1} + x^{q+1} u$$

* Young's inequality $\quad a.b \leqslant \frac{\alpha^r}{r} |a|^r + \frac{1}{s \alpha^s} |b|^s$

$r > 1$
$s > 1$ $\quad$ & $\quad (r-1)(s-1) = 1 \; ; \; \alpha > 0$

$$\Rightarrow \dot{V} = -x^{p+1} + x^{q+1} u$$

$$x^{q+1} u \leqslant \frac{\alpha^r}{r} |x|^{(q+1)r} + \frac{1}{s\alpha^r} |u|^s$$

Choose: $r = \dfrac{p+1}{q+1} > 1$; $s = 1 + \dfrac{1}{r-1}$

and α st. $\dfrac{\alpha^r}{r} = \dfrac{1}{2}$

$$\dot{V} \leqslant \underbrace{\frac{-1}{2}|x|^{p+1}}_{\alpha_3(|x|)} + \underbrace{\frac{1}{s\alpha^s}|u|^s}_{\alpha_4(\|u\|)}$$

Note! p has to be strictly larger than q
(otherwise no ISS, $\rightarrow \dot{x} = -x + xu$)

Feedback Linearization

Important notions: input-output linearization
relative degree
zero dynamics

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

$\left.\begin{array}{l} u(t) \in \mathbb{R} \\ y(t) \in \mathbb{R} \end{array}\right\}$ scalars (SISO nonlinear system) relativ degree:

Number of times that we need to differentiate the output to "see" the input (for input to appear in the output eqn)

$$y = h(x) \Rightarrow \dot{y} = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}(f(x) + g(x)u)$$

$$= \underbrace{\frac{\partial h}{\partial x} f(x)}_{L_f h(x)} + \underbrace{\frac{\partial h}{\partial x} g(x)}_{L_g h(x)} u$$

Lee derivative of func h in direction of f

If $L_g h(x) \neq 0$ in an open set containing the equilibrium then relative degree (r.d.) = 1

If not keep differentiating

$$\ddot{y} = \frac{\partial L_f h(x)}{\partial x}\dot{x} = \underbrace{L_f L_f h(x)}_{L_f^2 h(x)} + \underbrace{L_g L_f h(x)}\cdot u$$

same as before

<u>Def.</u> System $\dot{x} = f(x) + g(x)u$ has relative degree r if in
$y = h(x)$

a neighborhood of the equilibrium:

$$L_g L_f^{i-1} h(x) = 0 \quad ; \quad i = 1, 2, \dots, r-1$$

$$L_g L_f h(x) \neq 0$$

<u>Ex</u>

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^3 + u$$
$$y = x_1$$

$\dot{y} = \dot{x}_1 = x_2$ no input $\rightarrow$ keep differentiating

$\downarrow$

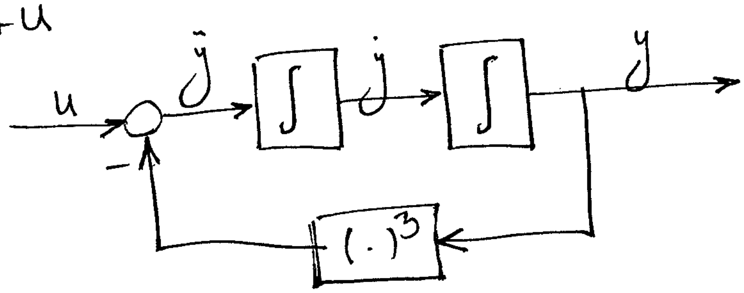
means r.d. at least bigger than one

$\ddot{y} = \dot{x}_2 = -x_1^3 + \underline{\underline{u}} \rightarrow$ r.d. $= 2$

$\ddot{y} = -y^3 + u$

$\downarrow$

now easy to deal with



choose $\quad u = -K_1\dot{y} - K_0 y + y^3$ → bad from design point of view

but give linear dynamic response

Linearization by means of fbk.

Ex2 $\quad y = x_2$

$\dot{y} = \dot{x}_2 = -x_1^3 + u$

r.d. 1