Nonlinear Systems

Lecture 26 05/02/13

Relative degree

SISO
$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

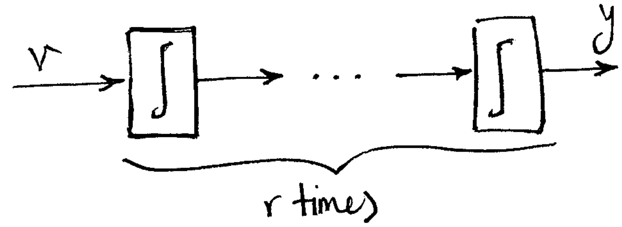
Differentiate the output up until input appears in the equation for y.

If a system has a well defined relative degree r, then it is input-output linearizable.

$$y^{(r)} = L_f^r h(x) + \underbrace{L_g L_f^{r-1} h(x)}_{\neq 0 \text{ (in a neighbourhood of the equilibrium)}} . u$$

$$u = \frac{1}{L_g L_f^{r-1} h(x)} \left\{ -L_f^r h(x) + v \right\} \qquad (1)$$

$$y^{(r)} = v$$



$$v = -K_{r-1} y^{(r-1)} - \cdots - K_0 y \qquad (2)$$

Where $K_0, \ldots, K_{r-1}$ are selected s.t.

$$f(s) = s^r + K_{r-1} s^{r-1} + \cdots + K_1 s + K_0$$

is a Hurwitz polynomial, then the output dynamic will be "well behaved"

$$y^{(r)} + K_{r-1} y^{(r-1)} + \cdots + K_1 \dot{y} + K_0 y = 0$$

Equivalent question:

Does controller (1)-(2) provide asymptotic stability of ⊛?

Not necessarily !!!

(1),(2) renders (n-r) dimensional manifold:

$$h(x) = L_f h(x) = \cdots = L_f^{r-1} h(x) = 0$$

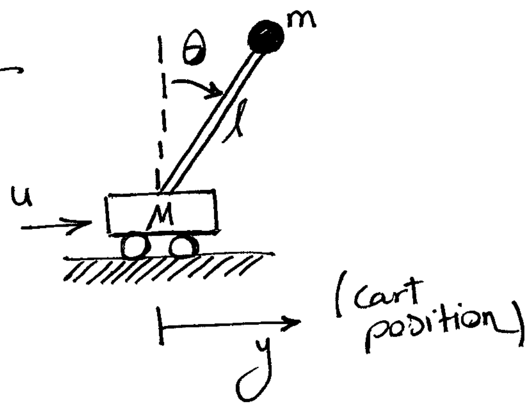
invariant and attractive.

$$y = \dot{y} = \cdots = y^{(r-1)} = 0$$

The dynamics restricted to this manifold are called ZERO DYNAMICS. This dynamics determines whether $\bar{x}=0$ is stable.

If $\bar{x}=0$ of the zero dynamics is asymptotically stable, then the entire system is called minimum phase.

Ex



4 states $\begin{bmatrix} y \\ \dot{y} \\ \theta \\ \dot{\theta} \end{bmatrix}$

model:

$$\ddot{y} = \frac{1}{\frac{M}{m} + \sin^2\theta}\left(\frac{1}{m}u + \dot{\theta} l \sin\theta - \frac{1}{2} g \sin 2\theta\right)$$

$$\ddot{\theta} = \frac{1}{l\left(\frac{M}{m} + \sin^2\theta\right)}\left(-\frac{1}{m} u\cos\theta - \dot{\theta}^2 \frac{l}{2}\sin 2\theta + \frac{M+m}{m} g \sin\theta\right)$$

Note! input u

output y

relative degree 2

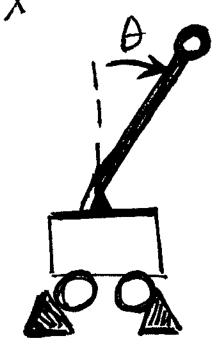
zero dynamics : $y(t) \equiv 0 \Rightarrow y^{(j)}(t) \equiv 0$

$j = 0, 1, 2, \ldots$

$$u = m\left\{\frac{1}{2} g \sin 2\theta - \dot{\theta} l \sin\theta\right\}$$

now by pluging this into $\ddot{\theta}$ equation + "exciting" algebra

$$\ddot{\theta} = \frac{g}{l} \sin\theta$$



falling like a rock, and we don't have any idea that this is going on from the measured output.

(θ dynamics is not observable from cart position.)

Non minimume phase (because zero dynamics is not stable)

Back to linear systems (SISO)

$$\dot{x} = Ax + Bu$$
$$y = cx$$

when $D \neq 0$ $\longrightarrow$ input can influence output at any frequency. We would never have roll off in the frequency response.

$$\dot{y} = c\dot{x} = c(Ax + Bu)$$
$$= cAx + cBu$$

if $cB \neq 0$ $\longrightarrow$ relative degree 1

if not, keep differentiating

$$\ddot{y} = c\ddot{x} = cA\dot{x} = cA^2x + \underbrace{cAB}_{\neq 0}u \Rightarrow \text{r.d. } 2$$

$$\dddot{y} = cA^2\dot{x} = cA^3x + \underbrace{cA^2B}_{\neq 0}u \Rightarrow \text{r.d. } 3$$

$\vdots$

$$cA^{r-1}B$$

$$[\; C \;]\left[A^{r-1} \right]\left[B \right]$$

The physical interpretation of this is the coefficients of the discrete time impulse response.

Aside:

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = Cx_k$$

$$y_k = CA\underbrace{x_{k-1}}_{0} + CBu_{k-1}$$

$$y_1 = Cx_1 = CA\cancel{x_0}^{\,0} + CB\cancel{u_0}^{\,1}$$

$$y_2 = CAB\cancel{u_0}^{\,1}$$

Impulse response of DT systems.

<u>Markov Parameters</u>

$\dot{x} = Ax + Bu$

$y = Cx$

positive real then $\longrightarrow$ r.d. $= 1$

$\downarrow$

$\exists P = P^T > 0$ s.t. $A^T P + PA < 0$ (1)

$\boxed{PB = C^T}$ (2)

P is positive definite $\Rightarrow$ $B^T P B = B^T C^T = (CB)^T > 0$

If transfer function PR $\iff$ r.d $= 1$

difference btw # poles & zeros $= 1$