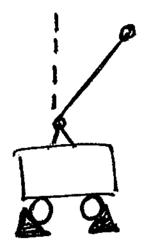
Nonlinear Systems

Lecture 27 05/07/13

Last time

- Input-output linearization
- zero-dynamics



y = position of cart

Zero dynamics : dynamics of inverted pendulum

→ Example of a system that doesn't have a globally defined relative degree :

$$\dot{x}_1 = x_2 + x_3^3$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = u$$
$$y = x_1 \quad \Rightarrow \quad \dot{y} = \dot{x}_1 = x_2 + x_3^3$$
$$\ddot{y} = \dot{x}_2 + 3x_3^2\dot{x}_3 = x_3 + 3x_3^2 u$$

$$\hat{y} = x_3 + 3x_3^2 u$$

vanishes when $x_3 = 0$

this system doesn't have well defined relative degree

this was an example of a system that doesn't have a well-defined relative degree ($x_3 = 0$)

when $x_3 = 0$ we would not be able to determine the influence of u, input, on $\hat{y}$ or the derivative of x_1.

recall : $v \longrightarrow \boxed{\int} \longrightarrow \cdots \longrightarrow \boxed{\int} \longrightarrow y$

$$y^{(r)} = L_f^r h(x) + \underbrace{L_g L_f^{r-1} h(x)}_{\neq 0} . u$$

$$u = \frac{1}{L_g L_f^{r-1} h(x)} \left\{ -L_f^r h(x) + \underline{\underline{v}} \right\}$$

$\downarrow$ useful framework for output tracking that shows limitations in ideal conditions. However, issue of robustness can be problematic when nonlinearities are not well characterized.

"Normal Form"

Translation: set of coordinates that displays zero dynamics.

Thm

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

If the above system has a well defined relative degree $r \leq n$ then there exist new variables

$$z \in \mathbb{R}^{n-r}$$
$$\xi \in \mathbb{R}^{r}$$

such that $T(x) = \begin{bmatrix} z \\ \xi \end{bmatrix}$; $T(0) = 0$ is a diffeomorphism (T^{-1} exists ; T, T^{-1} are cts, diff'ble) and the dynamics in new coordinates have the form:

$$\dot{z} = f_0(z, \xi)$$
$$\dot{\xi}_1 = \xi_2$$
$$\dot{\xi}_2 = \xi_3 \qquad\qquad (*)$$
$$\vdots$$
$$\dot{\xi}_r = b(z, \xi) + a(z, \xi)u$$

In particular:

$$\xi := \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \end{bmatrix}, \quad \begin{aligned} b(z,\xi) &= L_f^r h(x) \\ a(z,\xi) &= L_g L_f^{r-1} h(x) \end{aligned}$$

and $\quad \dot{z} = f_0(z,0)$ is the zero dynamics. $\blacksquare$

z should be linearly indep. of ξ and achieve $\dot{z} = f_0(z,\xi)$

ξ variables represent the output

I/O linearizing controller:

$$u = \frac{1}{a(z,\xi)}\left(-b(z,\xi) + v\right) \quad \dots (1)$$

$$v = -K_1\xi_1 - \dots - K_r\xi_r \quad \dots (2)$$

gives

$\dot{\xi} = A_\xi \xi$ where
$$A_\xi = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 \\ -K_1 & -K_2 & -K_3 & \cdots & -K_r \end{bmatrix}$$
is in the companion (controller-canonical) form & is Hurwitz!

If our objective is local asymptotic stability, then we can use linearization of $(*)$ with (1), (2) :

$$\begin{bmatrix} \dot{z} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_0}{\partial z}\Big|_0 & \frac{\partial f_0}{\partial \xi}\Big|_0 \\ 0 & A_\xi \end{bmatrix} \begin{bmatrix} z \\ \xi \end{bmatrix}$$

therefore the stability of (local asymptotic stability) this matrix is equivalent to $\frac{\partial f_0}{\partial z}\Big|_0$ being Hurwitz. (stability of zero dynamics).

If you have input to state stability you will have global asymptotic stability as well.

→ ξ is not important at this point when studying local stability properties.

Ex

$$\dot{x}_1 = x_2 \qquad (1)$$

$$\dot{x}_2 = \alpha x_3 + u \qquad (2)$$

$$\dot{x}_3 = \beta x_3 - u \qquad (3)$$

$$y = x_1$$

$$\dot{y} = \dot{x}_1 = x_2$$

$$\ddot{y} = \dot{x}_2 = \alpha x_3 + u$$

relative degree is 2. (globally well defined relative degree since nothing comes infront of u)

$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{\xi}_1 = \xi_2$$

$$\dot{\xi}_2 = \alpha x_3 + u$$

Now to determine zero dynamics $\longrightarrow$ x_3 by itself is not enough b/c input should not enter into zero dynamics.

Note:

a) We have identified ξ- part of the system, but still need to find $\dot{z} = f_0(z, \xi)$.

b) $x_3 \neq z$ (because u enters into the equations for x_3)

c) $y \equiv 0 \Rightarrow \dot{y} \equiv 0 \Rightarrow \xi_1 \equiv 0$
$\ddot{y} \equiv 0 \qquad \xi_2 \equiv 0$

$$\Downarrow$$

$u = -\alpha x_3$ (for $y \equiv 0$)

$\hookrightarrow \dot{x}_3 = \beta x_3 - u$

$\Rightarrow \dot{x}_3 = (\alpha+\beta) x_3$

stability determined by sign of $(\alpha+\beta)$
- $\alpha+\beta > 0 \rightarrow$ unstable
- $\alpha+\beta < 0 \rightarrow$ locally asymptotically stable

Construction of z:

$$(2)+(3) \Rightarrow \frac{d(x_2+x_3)}{dt} = (\alpha+\beta)\, x_3$$

$$z := x_2 + x_3 = \xi_2 + x_3$$

$$x_3 = z - \xi_2 \Rightarrow \dot{z} = \underbrace{(\alpha+\beta)}_{\frac{\partial f_0}{\partial z}\big|_0} z - (\alpha+\beta)\xi_2$$