Lecture 05 Feb. 02

Ages ago:

- 1st order bifurcation
- Intro to phase portraits of 2nd order systems.

Today:

- Continuation of phase portraits of 2nd order systems.
- Hartman - Grobman Thm
- Bendixon Thm
  (Conditions for absence of periodic orbits of 2nd order systems) → (just stands for 2nd order systems)

$$\dot{x} = Ax \xrightarrow{T} \dot{z} = \bar{A}z$$

(a) $\bar{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$; $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$

(b) $\bar{A} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$; $\lambda \in \mathbb{R}$

(c) $\lambda_{12} = \alpha \pm j\beta \rightarrow \bar{A} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$

(a) can show

$$\boxed{z_2 = C\, z_1^{\frac{\lambda_2}{\lambda_1}}}$$

$$\dot{z}_i = \lambda_i z_i \Rightarrow z_i(t) = e^{\lambda_i t} z_i(0) \Rightarrow \boxed{\text{if } z_i(0)=0 \rightarrow z_i(t)=0}$$

$i = 1, 2$

want $z_2 = z_2(z_1) \Rightarrow$ have to eliminate t!

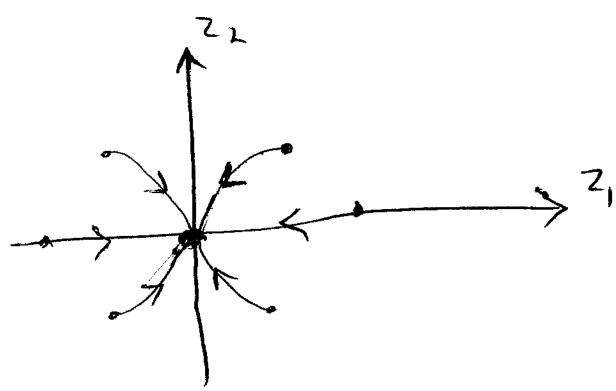
$$\frac{z_2}{z_1} = \frac{z_2(0)}{z_1(0)} e^{(\lambda_2 - \lambda_1)t}$$

$$z_i(t) = e^{\lambda_i t} z_i(0) \rightarrow \frac{z_1(t)}{z_1(0)} = e^{\lambda_1 t} \rightarrow t = \frac{1}{\lambda_1} \log\left(\frac{z_1(t)}{z_1(0)}\right)$$

. . . can show that $\boxed{C = \frac{z_2(0)}{(z_1(0))^{\frac{\lambda_2}{\lambda_1}}}}$

Ex: $\lambda_1 < \lambda_2 < 0$

(stable node)

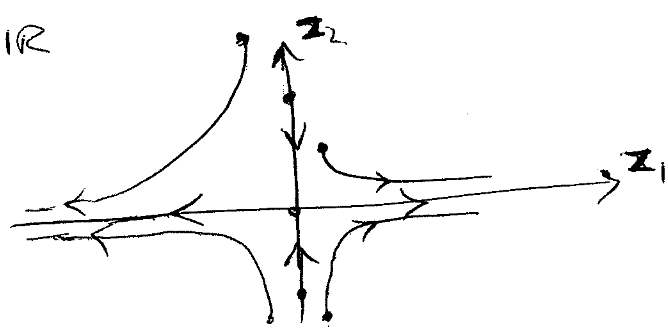


another way of finding $z_2(z_1) \rightarrow$

$$\frac{\frac{dz_2}{dt}}{\frac{dz_1}{dt}} = \frac{\lambda_2}{\lambda_1}\frac{z_2}{z_1} \rightarrow \boxed{\left(\frac{\dot{z}_2}{\dot{z}_1}\right) = \frac{\lambda_2}{\lambda_1}\left(\frac{z_2}{z_1}\right)}$$

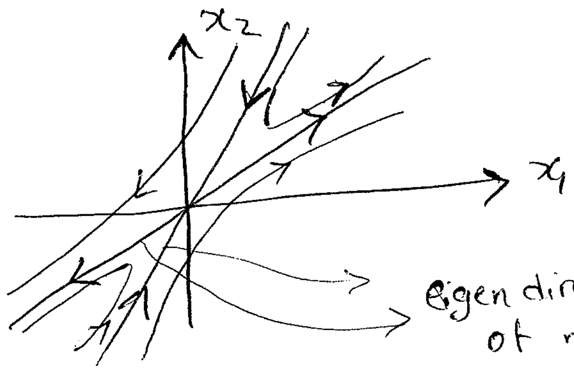
Ex: $\lambda_2 < 0 < \lambda_1 \in \mathbb{R}$

(saddle node)



$$Z = Tx \Longrightarrow \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In the x-Coordinates: (for saddle node)



→ these directions are eigenvectors!

[for z-coordinate: e_1 & e_2 are eigenvectors]

eigen directions of matrix A (Separatrices)

Aside:

Solutions of linear systems:

$$x(t) = e^{At} x(0)$$

$$Z = Tx = T e^{At} T^{-1} z(0)$$

Fact: If A has a complete set of linearly independent eigenvectors, $[v_1, \dots, v_n]$ then we can choose: $\boxed{T = V = [v_1 \cdots v_n]}$

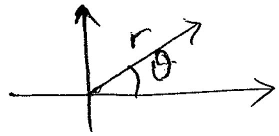
$$\Longrightarrow z(t) = e^{\Lambda t} z(0), \quad \Lambda = \mathrm{diag}(\lambda_1, \dots \lambda_n)$$

Case (c):

$$\bar{A} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \rightarrow \lambda_{12} = \alpha \pm j\beta$$

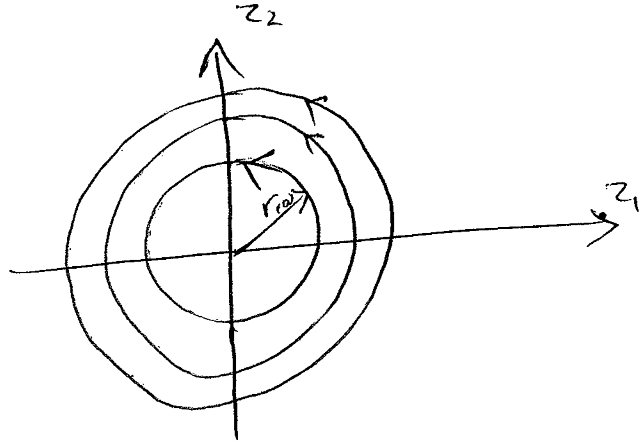
$$\boxed{\begin{aligned} \dot{z}_1 &= \alpha z_1 - \beta z_2 \\ \dot{z}_2 &= \alpha z_2 + \beta z_1 \end{aligned}}$$

Polar Coordinates



$$\boxed{\begin{aligned} \dot{r} &= \alpha r \\ \dot{\theta} &= \beta \end{aligned}} \quad \begin{aligned} &\rightarrow r(t) = e^{\alpha t} r(0) \\ &\rightarrow \theta(t) = \beta t + \theta(0) \end{aligned}$$

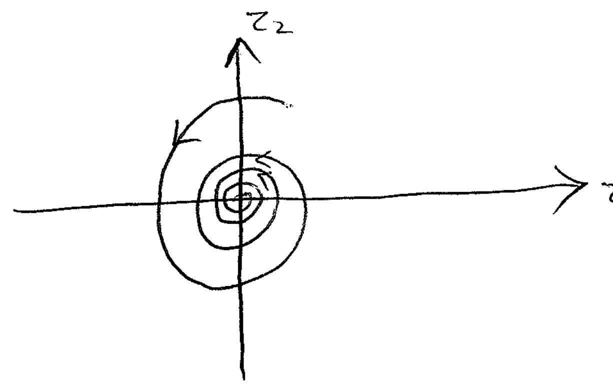
(The change of coordinate brings system to the even simpler one in comparison with z coordinates)



"Center"

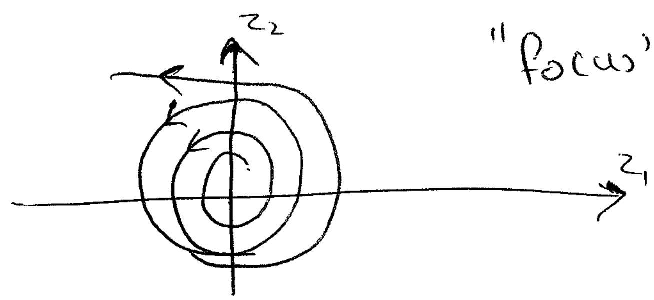
$\boxed{\text{If } \alpha = 0}$

$\boxed{\text{If } \alpha < 0}$



"focus"

$\boxed{\text{If } \alpha > 0}$



Case (b):

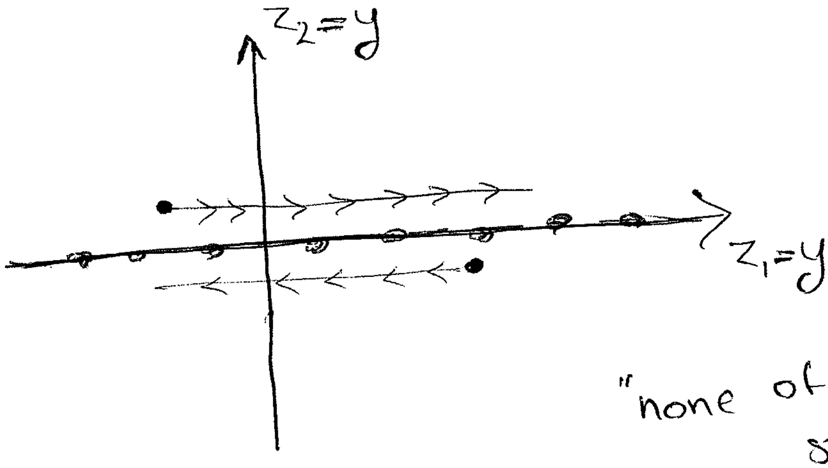
$$\bar{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \ddot{y} = 0$$ (moving mass on frictionless surface)

$$\rightarrow \begin{cases} \dot{z}_1 = z_2 \;, & z_1 = y \\ \dot{z}_2 = 0 \;, & z_2 = \dot{y} \end{cases}$$

$z_2(t) = z_2(0)$

$z_1(t) = z_1(0) + z_2(0)\, t$

$y \to \begin{cases} y(t) = y(0) + \dot{y}(0)\, t \\ \dot{y}(t) = \dot{y}(0) \end{cases}$



"none of the eq. points are stable!"

Hartman - Grobman Thm

Note: If eq. point $\bar{x}$ of $\dot{x} = f(x)$ is hyperbolic ($\tilde{A} = \frac{\partial f}{\partial x}\big|_{x=\bar{x}}$ doesn't have purely imaginary eigenvalues) then phase portraits of $\dot{\tilde{x}} = A\tilde{x}$ can be related to phase portrait of $\dot{x} = f(x)$

Thm: If $\bar{x}$ is a hyperbolic eq. point of $\dot{x} = f(x)$ then there is a homeomorphism from a neighborhood of $\bar{x} \in \mathbb{R}^n$ that maps trajectories of $\dot{x} = f(x)$ to those of corresponding linearization.

homeomorphism: continuous map with continuous inverse!

Note: absence of e-values on $j\omega$-axis of essence here; (i.e. node, saddle, focus)

Ex:

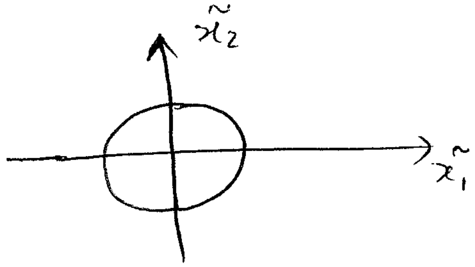
$$\dot{x}_1 = -x_2 + a x_1 (x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + a x_2 (x_1^2 + x_2^2)$$

Linearization around the origin: $\rightarrow \bar{x} = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

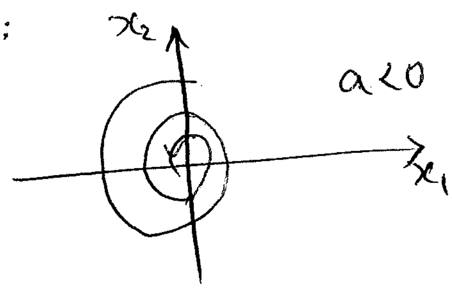
$$\rightarrow \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \rightarrow \lambda_{1,2} = \pm j$$

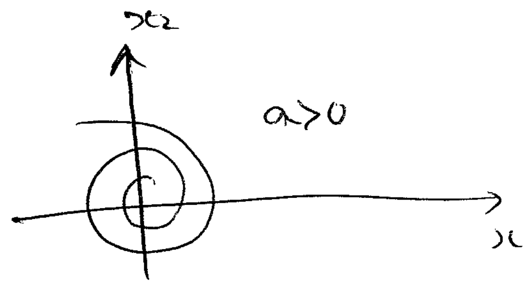
(center)

Linearization phase portrait:



Nonlinear:





polar coordinates: $\begin{cases} \dot{r} = a r^3 \\ \dot{\theta} = 1 \end{cases}$

If $a = 0 \rightarrow$ two systems are equal! but if $a \neq 0$

further calculations are required

So, the theorem works if we don't have any eigenvalues on imaginary axis.