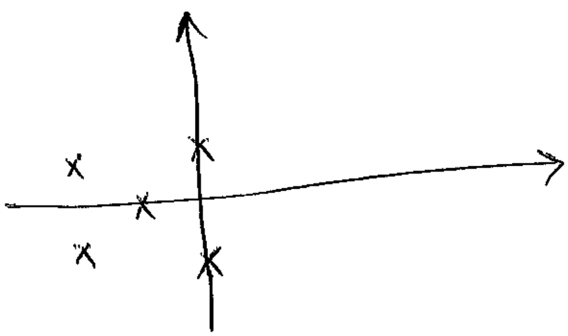
Lecture 10 — Feb. 27

Last time:

- Center Manifold Theory

$\dot{x} = f(x)$ — linearization around $\bar{x}=0$ that has
- K e_values on $j\omega$ axis
- $n-K$ e_values in the LHP



Stability of $\bar{x}=0$ of $\dot{x}=f(x)$

$\Updownarrow$

stability of $\dot{y} = A_1 y + g_1(y, h(y))$ → "center manifold"

→ $y(t) \in \mathbb{R}^K$

$\begin{bmatrix} y \\ z \end{bmatrix} = Tx$ → allows us to bring matrix A into block diagonal form.

$A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right)$ — A_1 → $j\omega$ axis; A_2 → e-values are in LHP

$\dot{z} = A_2 z + g_2(y, z)$

$z = h(y)$ → invariant manifold

Characterization of center manifold:

$$A_2 h(y) + g_2(y, h(y)) - \frac{\partial h}{\partial y}\left[A_1 y + g_1(y, h(y))\right] = 0 \quad (*)$$

Ex:
$$\dot{y} = 0.y + y.z$$
$$\dot{z} = (-1)z + ay^2$$

$$A_1 = 0 \;,\; g_1 = yz$$
$$A_2 = -1 \;,\; g_2 = ay^2$$

plugg in (*):

$$\begin{cases} -h + ay^2 - \frac{\partial h}{\partial y}(0.y + yh) = 0 \\ h(0) = 0 \;;\; \frac{\partial h}{\partial y}\Big|_0 = 0 \end{cases}$$

Taylor series of h around zero:

$$h(y) = \underbrace{h(0)}_{=0} + \underbrace{\frac{\partial h}{\partial y}\Big|_0}_{=0} y + \underbrace{h_2 y^2 + h_3 y^3 + \cdots}_{H.O.T} \quad = h_2 y^2 + h_3 y^3 + \cdots$$

plug this in $\dot{y} = y\,h(y) = h_2 y^3 + h_3 y^4 + \cdots$

near zero $y^3 \gg y^4 \rightarrow$ Sign of $h_2 \rightarrow$ stability

$h_2 < 0 \rightarrow$ stable

$$-h_2 y^2 + h_3 y^3 + \cdots + ay^2 = (2h_2 y + 3h_3 y^2)(h_2 y^3 + h_3 y^4 + \cdots)$$

$\rightarrow \boxed{h_2 = a}$, $\boxed{h_3 = 0}$ $\Big\} \rightarrow$ up to second order in y $\quad h(y) \simeq ay^2 + O(y^3)$

$\rightarrow \dot{y} \simeq ay^2 \times y = ay^3 \rightarrow$ local assymptotically stability for $a < 0$
instability for $a > 0$

$\rightarrow$ It's only valid near orgin!

Limit cycle cannot pass the origin! 

Can center manifold be closed?

Center manifold and limit cycles cannot have intersection! (uniqueness)

Existence & uniqueness of Solutions

$\dot{x} = f(x, \underline{t})$, $x(t_0) = x_0$

Recall : EE/AEM $\underline{\underline{5231}}$

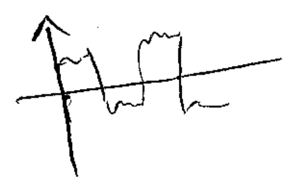
$f(x,t) = A(t)\, x \rightarrow$ Existence and uniqueness $\Leftrightarrow$ A(t) is a piecewise continuous function of time!

or

$f(x) = Ax$ (in time invariant case)

$\rightarrow$ Solutions always exist and it's unique!

$\quad \hookrightarrow e^{At} x_0$

each element of A(t) $\rightarrow$ 

Continuous

We will restrict our attention to piece-wise functions of time ($f(x, \cdot)$)

$\uparrow$ fixed

Q: How about properties w.r.t. x?

will Continuity cut it? (is enough?)

Ex: $\dot{x} = x^{\frac{1}{3}}$

Set $x(0)=0 \Rightarrow x(t)=0$ (one solution)

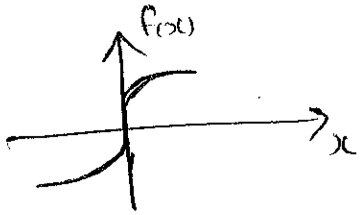
$$x^{-\frac{1}{3}}\,dx = dt \Rightarrow \int_{x_0}^{x} x^{-\frac{1}{3}}\,dx = \int_0^t dt$$

$$\Rightarrow \left.\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\right|_{\cancel{x_0}\,0}^{x} = t \rightarrow \frac{3}{2}x^{\frac{2}{3}} - \cancel{\frac{3}{2}x_0^{2/3}}\,0 = t$$

$$\rightarrow \boxed{x_{(t)} = \left(\frac{2}{3}\,t\right)^{\frac{3}{2}}}$$

we start at the origin but the slope is ∞! So, we'll not stay there!

$$\left.\frac{\partial f}{\partial x}\right|_0 = \left.\frac{1}{3}\,x^{-\frac{2}{3}}\right|_0 = \infty$$



So, continuity will Not cut it!

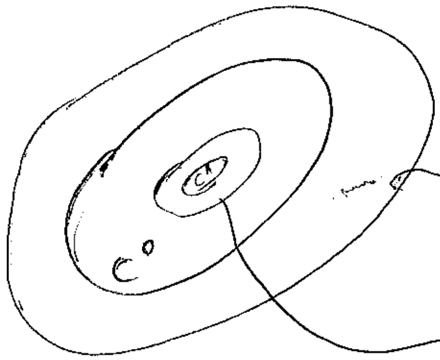
Fact: If f is a cts function of x (C^0)

$\Downarrow$

then, existence on a finite time horizon $(0, t_f)$

Not → necessarily uniqueness

| ex: $\dot{x} = x^{\frac{1}{3}}$ |



Lipschitz Continuity:

$$\| f(x,t) - f(y,t) \| \leq L \cdot \| x - y \|$$

t fixed

If this holds for all t and for all points x & y in a certain neighborhood of an arbitrary point $\bar{x} \in \mathbb{R}^n$ for some L (Lipschitz constant) then f is locally Lipschitz!

nbh