Lecture 12 — March 06 / 2014

Last time:

- Existence & uniqueness
- Continuity w.r.t
  - initial conditions
  - parameters
- Sensitivity w.r.t.
  - parameters

$$\dot{x} = f(x, t, \mu)$$

$$x(t,\mu) = x_0 + \int_{t_0}^{t} f(x,\tau,\mu)\,d\tau$$

differentiate w.r.t. $\underline{\mu}$ and $\underline{t}$ to obtain:

$$x_\mu(t,\mu) := \frac{\partial x(t,\mu)}{\partial \mu}$$

$$\boxed{\dot{x}_\mu(t,\mu) = \boxed{\frac{\partial f}{\partial x}(x,t,\mu)}\, x_\mu(t,\mu) + \boxed{\frac{\partial f}{\partial \mu}(x,t,\mu)}}$$

PDE in (t,μ)

we'll study $S(t) := \frac{\partial x}{\partial \mu}(t,\mu)\Big|_{\mu=\bar{\mu}}$ (Sensitivity of solutions @ $\mu = \bar{\mu} = \text{Const}$)

$$\underset{\text{unknown}}{x(t,\mu)} = \underset{\text{fixed}}{x(t,\bar{\mu})} + \underbrace{\frac{\partial x}{\partial \mu}\Big|_{\bar{\mu}}}_{S(t)} \underset{\text{Small}}{(\mu-\bar{\mu})} + \text{H.O.T} \qquad (*)$$

$$x(t,\mu) \simeq x(t,\bar\mu) + S(t)\,(\mu-\bar\mu) + O(\|\mu-\bar\mu\|^2)$$

Note:

$\circledast$ @ $\mu=\bar\mu$ becomes:

$$\boxed{\dot S(t) = F(t)\,S(t) + G(t)}$$

where: $S(t) = \left.\dfrac{\partial x}{\partial \mu}\right|_{\bar\mu}$

$$F(t) := \left.\frac{\partial f}{\partial x}\right|_{\bar\mu} = \frac{\partial f}{\partial x}(x(t,\bar\mu), t, \bar\mu)$$

it's not function of x anymore and it's a function of $\bar\mu$! (evaluated at $\bar\mu$)

it's just function of t!

$$G(t) := \left.\frac{\partial f}{\partial \mu}\right|_{\bar\mu} = \frac{\partial f}{\partial \mu}(x(t,\bar\mu), t, \bar\mu)$$

Key: one way coupling between

$\dot x = f(x,t,\bar\mu) \Rightarrow$ Solution $x(t,\bar\mu)$

and sensitivity Eq: $\dot S(t) = F(t)\,S(t) + G(t)$

EX: Fold Bifurcation

$\dot x = x^2 + \mu$

$x(t,\bar\mu)$: known

Study sensitivity of solutions around $\bar\mu$

$$\left.\begin{array}{l} \dfrac{\partial f}{\partial x} = 2x \\[2ex] \dfrac{\partial f}{\partial \mu} = 1 \end{array}\right\} \rightarrow \quad \dot{S}(t) = 2x(t,\bar{\mu})\, S(t) + 1$$

$$\text{I.C. : } S(0) = 0$$

$\boxed{\text{Ex 3.7 in khalil}}$

Ex

$$\begin{array}{l} x(t) \in \mathbb{R}^2 \\ \mu \in \mathbb{R}^3 \end{array} \rightarrow \quad S(t) = \underbrace{\begin{bmatrix} x_{1\mu_1} & x_{1\mu_2} & x_{1\mu_3} \\ x_{2\mu_1} & x_{2\mu_2} & x_{2\mu_3} \end{bmatrix}}_{} \leftarrow \mathbb{R}^{2\times 3}$$

Stability (In the sense of Lyapunov)

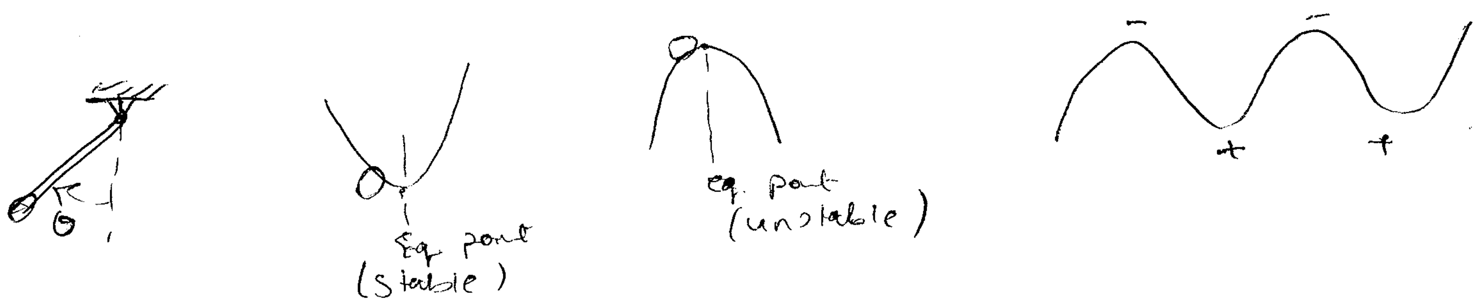
unforced $\quad \dot{x} = f(x)$

Time-Invariant

Eq. point @ $\bar{x} = 0$ $\quad$ (If not, shift coordinates)

$$z(t) := x(t) - \bar{x} \rightarrow \quad \dot{z}(t) = \dot{x}(t) = f(z(t) + \bar{x})$$

$$\dot{z} = f(z + \bar{x}) \rightarrow \quad \bar{z} = 0 \text{ is an (equilibrium point)}$$



Wed, March 12 $\quad$ 3pm $\quad$ 3-180

Thurs, March 13 $\quad$ 4pm, ECE Col.

3.115

3.125