Lecture 13

March 11/2014

Last time:

- Sensitivity w.r.t. parameters
  (Sensitivity Eq.)

Today:

- Lyapunov Stability

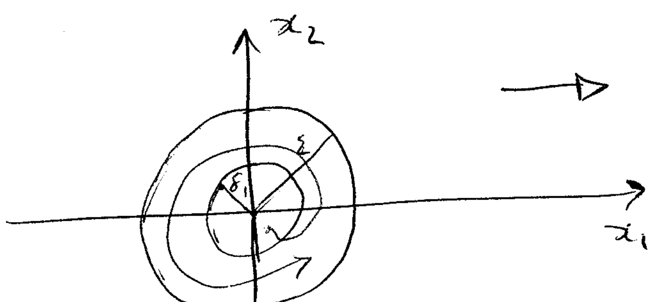
Note: . midterm : Thursday 03/27/14

$\dot{x} = f(x)$ with an eq. point @ the origin! $\begin{cases} \bar{x} = 0 \\ f(\bar{x}) = 0 \end{cases}$

1°) Stability of $\bar{x} = 0$

$\bar{x} = 0$ is stable iff for any $\varepsilon > 0$ there is $\delta_1 > 0$ s.t.
for all ICs with
$\|x_0\| < \delta_1 \Rightarrow \|x(t)\| < \varepsilon$ for all $t \geq t_0$

$\| \; \|$: norm of a vector e.g. a 2-norm :
$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$



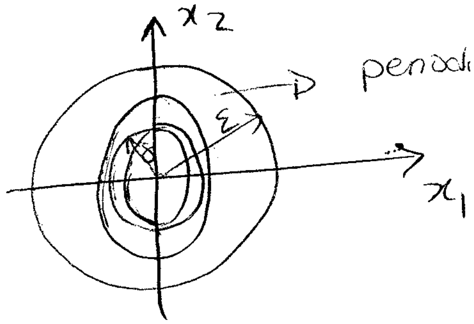
$\longrightarrow$ Start close to $\bar{x} = 0$, stay close to it!

No Info about asymptotic behavior! (i.e. : $t \to \infty$)

Ex Harmonic oscillator:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

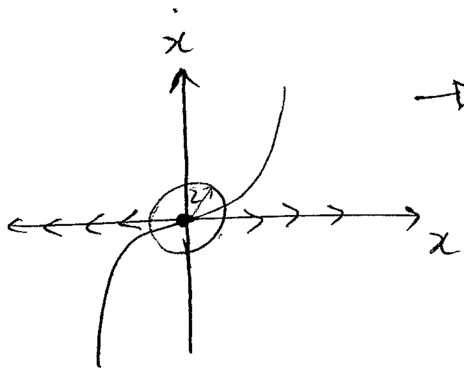
MS system
LC circuit
pendulum (down)



2°) Instability

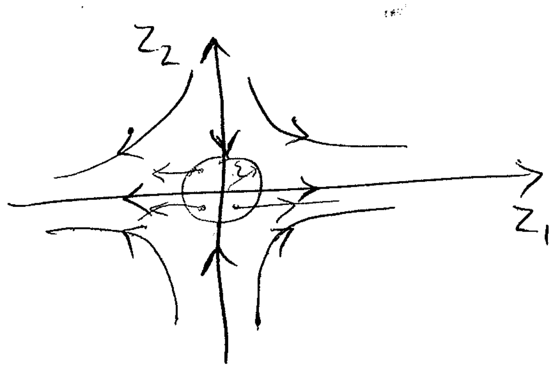
$x = 0$ is unstable if it's not stable!

Ex: $\dot{x} = x^3$



→ no matter where we start we'll go far from the origin!

Ex: saddle



3°) Local asymptotic Stability (of $\bar{x}=0$)

If $\underline{1°}$ holds ⊕ there is $\delta_2>0$ s.t. $\boxed{\|x_0\|<\delta_2 \Rightarrow \lim_{t\to\infty}\|x(t)\|=0}$

attractiveness

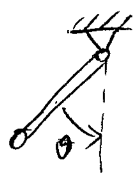
4°) Global asymptotic Stability (of $\bar{x}=0$)

If $\underline{3°}$ holds on $\mathbb{R}^n$ (i.e. no restrictions on δ_2) $\Rightarrow$ GAS

→ Checking stability properties using definitions can be cumbersome!!

alternative way of doing it:

Ex:



$\dot{x}_1 = x_2$, $x_1=\theta$: angle

$\dot{x}_2 = -a\sin x_1 - b x_2$, $x_2=\dot{\theta}$: speed

$a>0$; $b\geq 0$

Energy:

$$E(t) = \underbrace{a\int_0^{x_1}\sin\xi \, d\xi}_{\text{potential}} + \underbrace{\frac{1}{2}x_2^2}_{\text{Kinetic}}$$

$$\frac{dE(t)}{dt} = \frac{\partial E}{\partial x_1}\dot{x}_1 + \frac{\partial E}{\partial x_2}\dot{x}_2 = a\sin(x_1)\dot{x}_1 + x_2\dot{x}_2 \Longrightarrow$$

from this point on we'll evaluate what energy is doing along solutions of the system!

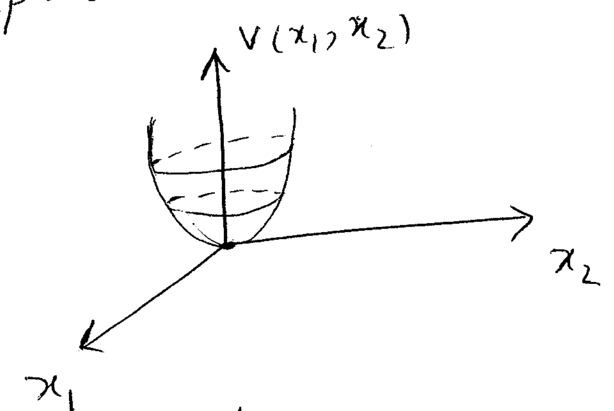
$$= a\sin(x_1)x_2 + x_2(-a\sin x_1 - bx_2)$$

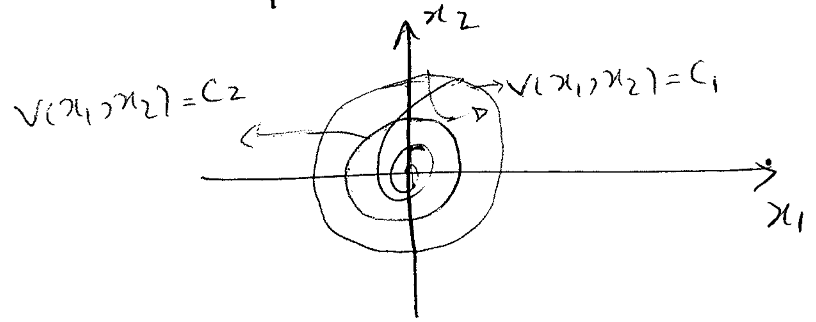
$$\Rightarrow \frac{dE(t)}{dt} = \boxed{-bx_2^2 \leq 0}$$

Summary: $\frac{dE}{dt} = -bx_2^2 \leq 0 \Rightarrow$ which means that $E(t)$ is a non-increasing function of time. If $b=0$ (no viscous damping) then

$$\frac{dE}{dt} = 0 \Rightarrow E(t) = \text{Const}$$ [(Energy conserved) (along solutions of our System!)]

Lyapunov Direct Method
(Lyapunov functions)





$C_2 < C_1$

Function:

$V: D \subset \mathbb{R}^n \to \mathbb{R}_+$

$V(0) = 0$

$V(x) > 0$ for all $x \in D \setminus \{0\}$

positive definite function on D

If $D = \mathbb{R}^n \rightarrow$ globally positive definite!

Ex: $n = 2 \rightarrow V(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$

is globally positive definite!

For the same function but $\boxed{n=3}$:

is not PD it's PSD! because for $(x_1 = 0, x_2 = 0, x_3 \neq 0)$ it's zero but we ~~should~~ must have $V(0) = 0$ and $V(x) > 0$ for other points! (not $=0$)

$$V(x) = \frac{1}{2}[x_1 \ x_2 \ x_3] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^T P x$$

$P = \frac{1}{2}\,\mathrm{diag}\,[1, 1, 0]$

$\rightarrow$ So, it PSD!

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x} f(x) = \left[\frac{\partial V}{\partial x_1} \ \cdots \ \frac{\partial V}{\partial x_n}\right] \begin{bmatrix} f_1(x_1, \ldots, x_n) \\ \vdots \\ f_n(x_1, \ldots, x_n) \end{bmatrix}$$

$$= \boxed{\sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i}$$

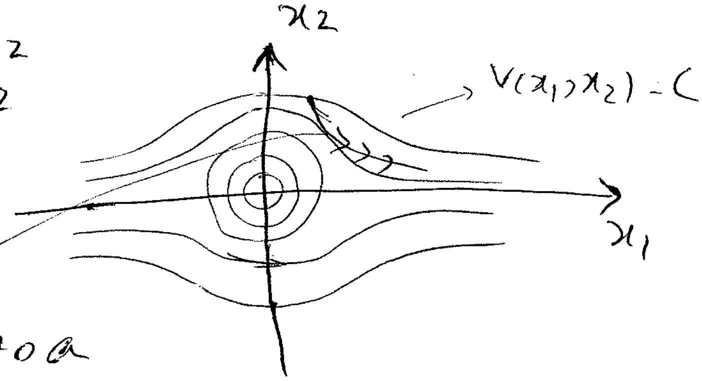
(a) If $\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x} f \leq 0$ on D then

$\bar{x} = 0$ is stable!

(b) If $\frac{dV}{dt} < 0$ on $D \setminus \{0\}$ then $\bar{x} = 0$ locally asymptotically stable!

(C) we need extra condition :, . .

Ex : $V(x) = \dfrac{x_1^2}{1+x_1^2} + x_2^2$



decreasing level sets, but we are going to a very big value of x_1 !

(C) $V(x)$ is globally PD + radially unbounded

[$V(x) \to \infty$ when $\|x\| \to \infty$]

$\dfrac{dV}{dt} < 0$ for all $x \in \mathbb{R}^n \setminus \{0\} \Rightarrow$

$\bar{x} = 0$ is global asymptotically stability