Lecture 14

March 13 / 2014

Last time

- Lyapunov stability $\begin{cases} \text{stability} \\ \left.\begin{array}{l}\text{local} \\ \text{global}\end{array}\right\} \text{asymptotic stability} \\ \text{of } \bar{x} = 0 \end{cases}$
- Lyapunov functions

Today

- Lyapunov functions (cont'd)

$V(x)$: $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$

$V(x)$: locally positive definite $\left.\right\}$ $\rightarrow \dfrac{dV}{dt} = \dfrac{\partial V}{\partial x} \dot{x} = \dfrac{\partial V}{\partial x} f(x)$

↳ on D that contain $\bar{x} = 0$

V: globally positive Definite ⊕ radially unbounded:

$\left(V \xrightarrow[\|X\| \to \infty]{} \infty \right)$

⊕

$\dfrac{dV}{dt} < 0$, for all $x \setminus \{0\}$

$\rightarrow$ Global asymptotic stability!

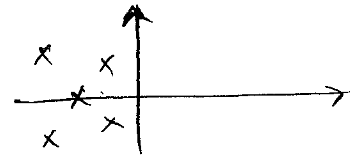
$$\rightarrow \boxed{\begin{bmatrix} \dfrac{\partial V}{\partial x_1} & \cdots & \dfrac{\partial V}{\partial x_n} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \leq 0}$$

plan of action:

→ a few examples

→ sketch of a proof

Ex1: $\dot{x} = Ax$ with A: Hurwitz $\rightarrow$



$(\mathrm{Re}\,\lambda_i(A) < 0,\ i = 1, \dots, n)$

$V(x) = x^T P x \quad ; \quad P = P^T > 0 \quad (\lambda_i(P) > 0)$

$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T \underbrace{(A^T P + P A)}_{-Q} x < 0$ for all $x \neq 0$

$\Rightarrow = -x^T Q x < 0 \iff -Q = Q^T < 0 \rightarrow \boxed{Q = Q^T > 0}$

$\rightarrow \dot{x} = Ax$ is stable (A Hurwitz or $\bar{x} = 0$ is GAS)

$\iff \forall Q = Q^T > 0$, there is $P = P^T > 0$ s.t.

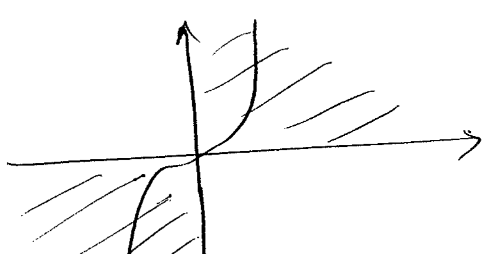
$$\boxed{A^T P + P A = -Q}$$

unique Solution given by: $\boxed{P = \int_0^\infty e^{A^T t} Q e^{At} dt}$ $\Rightarrow$ we don't have such a thing for nonlinear Systems!

DT: $A^T P A - P = 0 \rightarrow P = \sum_{k=0}^{\infty} (A^k)^T Q A^k$

Ex2: (a)

$\dot{x} = -g(x) \quad ; \quad x(t) \in \mathbb{R}$



$g(x) = ax, \quad a > 0$

$g(x) = a x^{2n+1} \quad ; \quad a > 0$

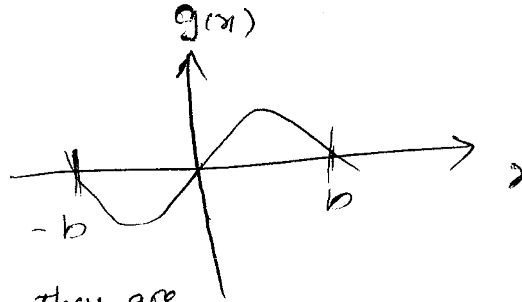
$n = \{0, 1, 2, \dots\}$

(odd nonlinearity)

Ex 2 : (b)

$$\dot{x} = -(x - x^3)$$

$$g(x) = x - x^3$$



$g(x) = \sin(x) \rightarrow$ locally they are like each other!

$$g(x) = \text{sat}(x)$$

Eg. $\dot{x} = a x^{2n+1}$, $a>0$

$V(x) = \frac{1}{2} x^2 \rightarrow \dot{V}(x) = \underbrace{\frac{1}{2} \cdot 2x}_{\frac{\partial V}{\partial x}} \dot{x} = \boxed{-a x^{2(n+1)} < 0}$ globally negative Definite

globally positive Definite (gpd) + radially unbounded $\rightarrow$ GAS (global asymptotic Stability)

Alternative Choice of V :

$V(x) = \int_0^x g(\xi)\, d\xi$ (from 0 to $-a$, $a>0$ $\rightarrow \int_0^{-a} > 0$)

$\dot{V} = \frac{\partial V}{\partial x} \dot{x} = g(x)\dot{x} = g(x)(-g(x)) = \boxed{-g^2(x) < 0}$

(where g belongs to I and III quadrant at least locally)

Ex 3 :

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a g(x_1) - b x_2 \end{cases}$$

$a>0$, $b \geq 0$

last time: $g(x_1) = \sin x_1$

$$V(x) = a\int_0^{x_1} g(\xi)\,d\xi + \frac{1}{2}x_2^2$$

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} = [a g(x_1) \;\; x_2]\begin{bmatrix} x_2 \\ -ag(x_1) - bx_2 \end{bmatrix} =$$

$$\boxed{a x_2 \cancel{g(x_1)} - a x_2 \cancel{g(x_1)} - bx_2^2} = -bx_2^2 \leq 0$$

$$\Rightarrow [\dot{V}] = -0 \cdot x_1^2 - bx_2^2 \leq 0$$

Note: using this Lyapunov function (which is locally positive definite) $\Rightarrow$ $\dot{V}$: locally negative semi-definite, $\dot{V} \leq 0$ on D ($x_1 = 0.2, x_2 = 0$)

$\Rightarrow$ Can only conclude stability (in the sense of Lyapunov) (of $\bar{x} = 0$)

but not local asymptotic stability!

possible fixes:

1°) Choose different lyapunov function

2°) use la Salle's invariance principle (After the spring break)

(we'll sketch the proof after the spring break)