Lecture 16 04|01|14

Before the Midterm

- La Salle's Invariance principle
- Lyapunov Theory for LTI Systems

Today

- more Lyapunov Theory
- Stability via Linearization (proof)

$\dot{x} = Ax$

$V(x) = x^T P x \quad \Rightarrow A^T P + PA = -Q \quad \text{(ALE)}$

$\dot{x} = Ax$ Stable (ie. A: Hurwitz $\Rightarrow Re(\lambda_i(A)) < 0$ $\forall i = 1, \cdots, n$)

$\Updownarrow$

For any $Q = Q^T > 0$ there is $P = P^T > 0$ Such That:

$$A^T P + PA = -Q$$

$\Longleftarrow$ (Start w/ Q)

Unique Solution of Algebraic Lyapunov Equation:

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$

*if A is not hurwitz, the integral will not Converge

MATLAB:
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>>P= lyap(A,Q)
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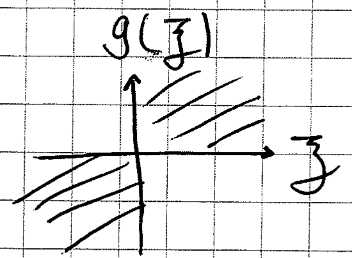
<u>Recall:</u> $\dot{x}_1 = x_2$

$\dot{x}_2 = -a g(x_1) - b x_2 \quad ; \; a > 0, \; b \geq 0$

Where:



$\Rightarrow g(\xi)$ in 1^{st} + 3^{rd} quadrant

$$V(x) = a \int_0^{x_1} g(\xi)\, d\xi + \frac{1}{2} x_2^2$$

$$\dot{V} = -b x_2^2 \leq 0$$

In Linear Case: $g(\xi) = \xi$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \iff \left\{ \begin{array}{l} \ddot{y} + b\dot{y} + a y = 0 \\ y = x_1 \\ \dot{y} = x_2 \end{array} \right.$$

$\Rightarrow$ Stable for $a > 0, \; b > 0$

$$\det(sI - A) = s^2 + bs + a$$

$a, b > 0 \Rightarrow A$ is hurwitz

in example:

$$V(x) = a \int_0^{x_1} \xi \, d\xi + \frac{1}{2} x_2^2$$

$$= \frac{1}{2} a x_1^2 + \frac{1}{2} x_2^2 = [x_1 \;\; x_2] \underbrace{\begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}}_{P} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

now: $a > 0 \Rightarrow P = P^T > 0$

where $P = \begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

with $P = \begin{bmatrix} \frac{a}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$$\Rightarrow Q = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \geq 0 \text{ if } b > 0$$

Recap of what we want to do:

given $\dot{x} = Ax$

propose $V(x) = x^T P x$

with $P = P^T > 0$

Such that:

$$A^T P + PA = -Q$$

$$Q = Q^T \geq 0 \text{ (\underline{Not} } > 0)$$

Pos. semi-definite

$$\dot{V}(x) = x^T (A^T P + PA) x = -x^T Q x \leq 0 \text{ (\underline{Not} } < 0)$$

Factor Q into $Q = C^T C$

where $\operatorname{rank}(Q) = r$

$$Q = C^T \cdot C \; ; \; C \in \mathbb{R}^{r \times n}$$

In the example:

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} 0 \\ \sqrt{b} \end{bmatrix} \begin{bmatrix} 0 & \sqrt{b} \end{bmatrix}$$

In other words:

$$\mathring{V}(x) = -x^T C^T \underbrace{Cx}_{y}$$

$$\mathring{V}(x) = -y^T y \leq 0$$

$$\mathring{V} = 0 \iff y = 0$$
$$(Cx = 0)$$

$$Cx = 0 \iff \underbrace{\begin{bmatrix} C \text{ (fat Matrix)} \end{bmatrix}}_{r \times n} \underbrace{\begin{bmatrix} x \end{bmatrix}}_{n \times 1} = \underbrace{\begin{bmatrix} \emptyset \end{bmatrix}}_{r \times 1}$$

<u>Question</u>: Under what Conditions

$$y = 0 \iff x = 0 \ ?$$

Recall: $\left.\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}\right\}$ Observability is there an initial condition such that $x(\emptyset) = x_0 \neq \emptyset$ and $y \equiv \emptyset$?

if the answer to this question is no, then the system is observable

Equivalence between $y = \emptyset$ and $x = \emptyset$

iff pair (A, C) observable

i.e. $\cancel{\text{rank}\begin{pmatrix} C^* \\ C^* A \\ \vdots \\ C^* A^{n-1} \end{pmatrix} = n}$ $\quad \text{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$

Back to Example:

$$A = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}; \quad C = \begin{bmatrix} 0 & \sqrt{b} \end{bmatrix}$$

Observability Matrix:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{b} \\ -a\sqrt{b} & -b\sqrt{b} \end{bmatrix}$$

$$\det(\mathcal{O}) = a \cdot b \neq \emptyset$$

$$\Rightarrow \begin{cases} a \neq \emptyset & (a > 0 \text{ to begin with}) \\ b \neq \emptyset \end{cases}$$

$b \neq \emptyset \Rightarrow$ damping needs to be present.