Lecture 17

April 3, 2014

Last time:

- Lyapunov Theory for LTI Systems
- LaSalles Inv. Principle for LTI sys.

Today:

- Stability via Linearization
- Integrator Backstepping

IF Setup: $\dot{x} = Ax$

$$V(x) = x^T P x$$

$$\dot{V}(x) = -x^T \underbrace{Q}_{C^T C} x \leq 0$$

$$= -y^T(t) \cdot y(t) \equiv \emptyset$$

$$y(t) = Cx(t) =$$

$$= Ce^{At} \cdot x_0 \equiv 0$$

$$\left.\begin{array}{c} CAe^{At}x_0 \\ \vdots \\ CA^k e^{At} x_0 \end{array}\right\} \Rightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} e^{At} x_0 = \emptyset$$

Linearization

$\dot{x} = f(x)$; $f(0) = 0$

$\Updownarrow$

$\dot{x} = Ax + g(x)$ ⊛

Assume A : Hurwitz

Show $\bar{x} = 0$ of ⊛ is ~~Locally~~ asymptotically stable

$$\left(\lim_{\|x\| \to 0} \frac{\|g(x)\|}{\|x\|} \Rightarrow 0 \right)$$

A : Hurwitz $\Rightarrow$ linear Lyapunov function will show local stability of Nonlinear

Prove: A : Hurwitz $\Rightarrow$ For any $Q = Q^T > 0$
there is $P = P^T > 0$
s.t.
$A^T P + PA = -Q$

$V(x) = x^T P x$ is a Lyapunov Function

for $\dot{x} = Ax$

Attempt to use $V(x) = x^T P x$ as a Lyapunov Function for ⊛

$$\dot V = \dot x^T P x + x^T P \dot x =$$

$$= [Ax + g(x)]^T P x + x^T P [Ax + g(x)] =$$

$$= x^T [A^T P + PA] x + 2 x^T P g(x)$$

$$= \underbrace{- x^T Q x + 2 x^T P\, g(x)} \le - x^T Q x + 2\|x\| \cdot \underbrace{\|P\|}_{\text{induced norm}} \|g\|$$

Fact: $\cancel{x^T Q x}$

$$\lambda_{min}(Q) \|x\|^2 \le x^T Q x \le \lambda_{max}(Q) \|x\|_2^2$$

$$\le - \lambda_{min}(Q) \|x\|^2 + 2 \|x\| \cdot \|P\| \cdot \|g\| =$$

$$= - \|x\|_2^2 \left(+ \lambda_{min}(Q) - \underbrace{2 \|P\| \frac{\|g(x)\|}{\|x\|}} \right)$$

we can define $\varepsilon + \delta$

s.t. $2\|P\| \frac{\|g(x)\|}{\|x\|} < \lambda_{min}(Q)$

Back Stepping

Consider $\dot{x} = f(x) + g(x)\,u$

x: State
u: Control ($u(t) \in \mathbb{R}$)

Assume there is $u = \alpha(x)$ — Assumption (1)

Such that $\dot{x} = f(x) + g(x)\,\alpha(x)$

and $\bar{x} = \emptyset$ is GAS.

Also let $\underbrace{V(x) > 0}_{\text{on } \mathbb{R}^n}$, radially unbounded

Such that $\dot{V} = \dfrac{\partial V}{\partial x} \cdot \dot{x} =$

$$= \frac{\partial V}{\partial x}\left[f(x) + g(x)\,\alpha(x)\right] =$$

$$= -W(x) < \emptyset \text{ on } \mathbb{R}^n$$

Ex

$\dot{x} = x^3 + u$

$V(x) = \frac{1}{2}x^2$

$\dot{V}(x) = x\dot{x} = x(x^3 + u)$

Choose $u = -x^3 - kx$; $k > 0$

$\hookrightarrow$ Any odd nonlinearity

$\dot{V} = -kx^2 < 0$

~~Ex~~ $\dot{x} = x^3 + \xi$

$\dot{\xi} = u \Rightarrow$ there is an integrator between original controller

Consider: $\dot{x} = f(x) + g(x) \cdot \xi$

and $\dot{\xi} = u$

(f) there exists $V(x)$ S.T. assumption (1) holds

Q: Can we find $u = \beta(x, \xi)$ S.t.

$$\left.\begin{aligned} \dot{x} &= f(x) + g(x)\xi \\ \dot{\xi} &= \beta(x, \xi) \end{aligned}\right\} \text{ has a G.A.S. Eq Pt. @ } (\bar{x}, \bar{\xi}) = (0,0) ?$$

~~Ex~~ $\dot{x} = \dot{x} + \xi$

$\dot{\xi} = u$

Step 1

$V_1(x) = \frac{1}{2} x^2$

$\dot{V}_1 = x\dot{x} = x(x^3 + \xi) =$

$= x(x^3 + \alpha(x)) = -W(x) < 0$

E.g. $\alpha(x) = -x^3 - K_1 x$

$K > 0$

Step 2

Augment $V_1(x)$ with $\underbrace{\frac{1}{2}(\xi - \alpha(x))^2}_{\frac{1}{2}z^2}$

$z := \xi - \alpha(x)$

$\dot{z} = \dot{\xi} - \underbrace{\frac{\partial \alpha}{\partial x}\dot{x}}_{\dot{\alpha}} =$

$= u - \frac{\partial \alpha}{\partial x}[x^3 + \xi]$

$$V_2(x, z) = V_1(x) + \frac{1}{2}z^2$$

$$\dot{V}_2 = \dot{V}_1 + z\dot{z}$$

$$= x(x^3 + \xi) + z(u - \dot{\alpha})$$

$$= x(x^3 + \alpha(x) + z) + z(u - \dot{\alpha})$$

$$= -W_1(x) + z(u - \dot{\alpha} + x)$$

"Cancelation" Control

$u = [\dot{\alpha} - x - k_2 z]$ gives:

$$\dot{V}_2 = -W_1(x) - k_2 z^2$$