EE 8215 **HW 6** **Spring 2016**

Due Tuesday 05/10/16 (noon, Sourav's office)

1. Khalil, Problem 13.27. (In part (b), do simulations but skip the performance comparison question.)
2. In class, we used the PR Lemma to show that a positive real linear system,

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

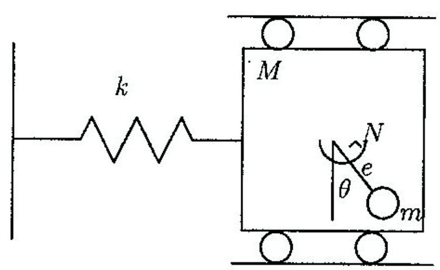
has relative degree one, that is, $CB \neq 0$. Show that positive realness also implies a minimum phase property. (Hint: Write the system equations in *normal form* and apply Positive Real Lemma.)

3. The dynamics of the *translational oscillator with rotating actuator* (TORA) are described by:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{-x_1 + \epsilon {x_4}^2 \sin x_3}{1 - \epsilon^2 \cos^2 x_3} + \frac{-\epsilon \cos x_3}{1 - \epsilon^2 \cos^2 x_3} u \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{1}{1 - \epsilon^2 \cos^2 x_3} \left(\epsilon \cos x_3 (x_1 - \epsilon x_4^2 \sin x_3) + u \right) \end{aligned}$$

where x_1 and x_2 are the displacement and the velocity of the platform, x_3 and x_4 are the angle and angular velocity of the rotor carrying the mass m, and u is the control torque applied to the rotor. The parameter $\epsilon < 1$ depends on the eccentricity e and the masses m and M.

With $y = x_3$ as the output, determine the relative degree and the zero dynamics. Provide a physical interpretation of the zero dynamics.



4. Show that the parallel connection of two passive dynamical systems is passive. Can you claim the same for the series connection of two passive systems? Either provide a proof or a counterexample.
5. Let

$$H(s) = \frac{s + \lambda}{s^2 + as + b}$$

with $a > 0$, $b > 0$.

   (a) For which values of λ is $H(s)$ Positive Real (PR)?

   (b) Using your answer to (a), select two values, λ_1 and λ_2, such that

$$\begin{aligned} H_1(s) &= \frac{s + \lambda_1}{s^2 + s + 1} \text{ is PR,} \\ H_2(s) &= \frac{s + \lambda_2}{s^2 + s + 1} \text{ is not.} \end{aligned}$$

   Verify the PR property or its absence from the Nyquist plots of $H_1(s)$ and $H_2(s)$. (You can use the MATLAB nyquist command.)

(c) For $H_1(s)$ and $H_2(s)$ write a state-space realization and solve for $P = P^T > 0$ in the PR lemma. Explain why your attempt fails for $H_2(s)$.

6. Consider the following model for a three-stage ring oscillator, discussed in class:

$$\begin{aligned} \tau_1 \dot{x}_1 &= -x_1 - \alpha_1 \tanh(\beta_1 x_3) \\ \tau_2 \dot{x}_2 &= -x_2 - \alpha_2 \tanh(\beta_2 x_1) \\ \tau_3 \dot{x}_3 &= -x_3 - \alpha_3 \tanh(\beta_3 x_2) \end{aligned}$$

where $\tau_i, \alpha_i, \beta_i$ are positive constants and x_i represent voltages, $i = 1, 2, 3$.

(a) Suppose $\alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_3\beta_3 =: \mu$, and prove that the origin is globally asymptotically stable when $\mu < 2$.

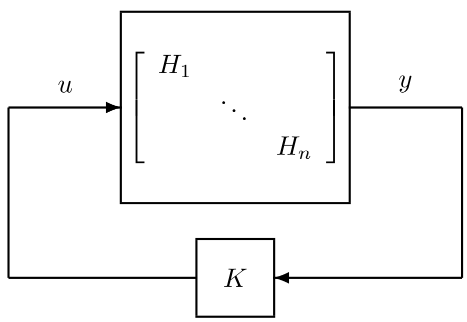
(b) Show that, if $\tau_1 = \tau_2 = \tau_3 =: \tau$, then $\mu < 2$ is also necessary for asymptotic stability. What type of bifurcation occurs at $\mu = 2$?

(c) Investigate the dynamical behavior of this system for $\mu > 2$ with numerical simulations. (You can take $\tau = 1$ for simplicity. Note that changing τ simply scales the time variable: If $x(t)$ is a solution for $\tau = 1$, then $x(t/\tau)$ is a solution for $\tau \neq 1$.)

7. Consider the systems $H_i,\ i = 1, \cdots, n$, whose inputs u_i and outputs y_i are coupled according to:

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = K \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

as in the figure below, where K is an $n \times n$ matrix.



Suppose each H_i satisfies the dissipation inequality:

$$\dot{V}_i(x_i) \leq -y_i^2 + \gamma_i^2 u_i^2$$

with a positive definite storage function of its state vector x_i.

(a) Determine a matrix inequality that restricts the matrices:

$$D := \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}, \quad \Gamma := \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_n \end{bmatrix}$$

and K, such that $V(x) = \sum_{i=1}^n d_i V_i(x_i)$ is a Lyapunov function for the interconnected system.

(b) Investigate when an appropriate matrix D satisfying this inequality exists for $K \in \mathbb{R}^{2\times 2}$ given by

$$K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

13.24 Consider the system (13.44)–(13.45), where $A - BK$ is Hurwitz, the origin of $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable with a Lyapunov function $V_0(\eta)$ such that $[\partial V_0/\partial \eta] f_0(\eta, 0) \leq -W(\eta)$ for some positive definite function $W(\eta)$. Suppose $\|\delta\| \leq k[\|\xi\| + W(\eta)]$. Using a composite Lyapunov function of the form $V = V_0(\eta) + \lambda\sqrt{\xi^T P \xi}$, where P is the solution of $P(A - BK) + (A - BK)^T P = -I$, show that, for sufficiently small k, the origin $z = 0$ is asymptotically stable

13.25 Consider the system

$$\dot{x}_1 = x_2 + 2x_1^2, \quad \dot{x}_2 = x_3 + u, \quad \dot{x}_3 = x_1 - x_3, \quad y = x_1$$

Design a state feedback control law such that the output y asymptotically tracks the reference signal $r(t) = \sin t$.

13.26 Repeat the previous exercise for the system

$$\dot{x}_1 = x_2 + x_1 \sin x_1, \quad \dot{x}_2 = x_1 x_2 + u, \quad y = x_1$$

13.27 The magnetic suspension system of Exercise 1.18 is modeled by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= g - \frac{k}{m}x_2 - \frac{L_0 a x_3^2}{2m(a + x_1)^2} \\
\dot{x}_3 &= \frac{1}{L(x_1)}\left[-Rx_3 + \frac{L_0 a x_2 x_3}{(a + x_1)^2} + u\right]
\end{aligned}$$

where $x_1 = y$, $x_2 = \dot{y}$, $x_3 = i$, and $u = v$. Use the following numerical data: $m = 0.1$ kg, $k = 0.001$ N/m/sec, $g = 9.81$ m/sec^2, $a = 0.05$ m, $L_0 = 0.01$ H, $L_1 = 0.02$ H, and $R = 1\ \Omega$.

(a) Show that the system is feedback linearizable.

(b) Using feedback linearization, design a state feedback control law to stabilize the ball at $y = 0.05$ m. Repeat parts (d) and (e) of Exercise 12.8 and compare the performance of this controller with the one designed in part (c) of that exercise.

(c) Show that, with the ball position y as the output, the system is input–output linearizable.

(d) Using feedback linearization, design a state feedback control law so that the output y asymptotically tracks $r(t) = 0.05 + 0.01 \sin t$. Simulate the closed-loop system.

third equation is a torque equation for the shaft, with J as the rotor inertia and c_3 as a damping coefficient. The term $c_1 i_f \omega$ is the back e.m.f. induced in the armature circuit, and $c_2 i_f i_a$ is the torque produced by the interaction of the armature current with the field circuit flux.

(a) For a separately excited DC motor, the voltages v_a and v_f are independent control inputs. Choose appropriate state variables and find the state equation.

(b) Specialize the state equation of part(a) to the field controlled DC motor, where v_f is the control input, while v_a is held constant.

(c) Specialize the state equation of part(a) to the armature controlled DC motor, where v_a is the control input, while v_f is held constant. Can you reduce the order of the model in this case?

(d) In a shunt wound DC motor, the field and armature windings are connected in parallel and an external resistance R_x is connected in series with the field winding to limit the field flux; that is, $v = v_a = v_f + R_x i_f$. With v as the control input, write down the state equation.

1.18 Figure 1.26 shows a schematic diagram of a magnetic suspension system, where a ball of magnetic material is suspended by means of an electromagnet whose current is controlled by feedback from the, optically measured, ball position [211, pp. 192–200]. This system has the basic ingredients of systems constructed to levitate mass, used in gyroscopes, accelerometers, and fast trains. The equation of motion of the ball is

$$m\ddot{y} = -k\dot{y} + mg + F(y, i)$$

where m is the mass of the ball, $y \geq 0$ is the vertical (downward) position of the ball measured from a reference point ($y = 0$ when the ball is next to the coil), k is a viscous friction coefficient, g is the acceleration due to gravity, $F(y, i)$ is the force generated by the electromagnet, and i is its electric current. The inductance of the electromagnet depends on the position of the ball and can be modeled as

$$L(y) = L_1 + \frac{L_0}{1 + y/a}$$

where L_1, L_0, and a are positive constants. This model represents the case that the inductance has its highest value when the ball is next to the coil and decreases to a constant value as the ball is removed to $y = \infty$. With $E(y, i) = \frac{1}{2}L(y)i^2$ as the energy stored in the electromagnet, the force $F(y, i)$ is given by

$$F(y, i) = \frac{\partial E}{\partial y} = -\frac{L_0 i^2}{2a(1 + y/a)^2}$$

When the electric circuit of the coil is driven by a voltage source with voltage v, Kirchhoff's voltage law gives the relationship $v = \dot{\phi} + Ri$, where R is the series resistance of the circuit and $\phi = L(y)i$ is the magnetic flux linkage.

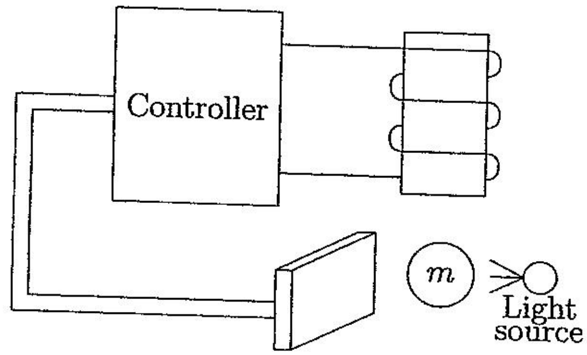


Figure 1.26: Magnetic suspension system of Exercise 1.18.

(a) Using $x_1 = y$, $x_2 = \dot{y}$, and $x_3 = i$ as state variables and $u = v$ as control input, find the state equation.

(b) Suppose it is desired to balance the ball at a certain position $r > 0$. Find the steady-state values I_{ss} and V_{ss} of i and v, respectively, which are necessary to maintain such balance.

The next three exercises give examples of hydraulic systems [41].

1.19 Figure 1.27 shows a hydraulic system where liquid is stored in an open tank. The cross-sectional area of the tank, $A(h)$, is a function of h, the height of the liquid level above the bottom of the tank. The liquid volume v is given by $v = \int_0^h A(\lambda)\, d\lambda$. For a liquid of density ρ, the absolute pressure p is given by $p = \rho g h + p_a$, where p_a is the atmospheric pressure (assumed constant) and g is the acceleration due to gravity. The tank receives liquid at a flow rate w_i and loses liquid through a valve that obeys the flow-pressure relationship $w_o = k\sqrt{\Delta p}$. In the current case, $\Delta p = p - p_a$. Take $u = w_i$ to be the control input and $y = h$ to be the output.

(a) Using h as the state variable, determine the state model.

(b) Using $p - p_a$ as the state variable, determine the state model.

(c) Find u_{ss} that is needed to maintain the output at a constant value r.

1.20 The hydraulic system shown in Figure 1.28 consists of a constant speed centrifugal pump feeding a tank from which liquid flows through a pipe and a valve that obeys the relationship $w_o = k\sqrt{p - p_a}$. The pump characteristic for the specified pump speed is shown in Figure 1.29. Let us denote this relationship by $\Delta p = \phi(w_i)$ and denote its inverse, whenever defined, by $w_i = \phi^{-1}(\Delta p)$. For the current pump, $\Delta p = p - p_a$. The cross-sectional area of the tank is uniform; therefore, $v = Ah$ and $p = p_a + \rho g v / A$, where the variables are defined in the previous exercise.