EE 8215 S16 : Nonlinear Systems
Lecture 2 — Jan 21st, 2016, Thu

- Last time : - warmup
- Today : - Range of phenomena in nonlinear system

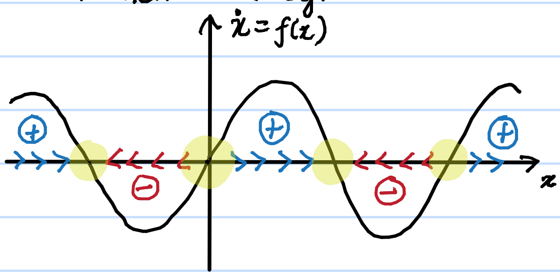
Eg ①: $\dot{x} = \sin(x)$; $x(t)$: scalar (real number)

- Equilibrium points : $\frac{d\bar{x}}{dt} = 0$, obtained by solving $f(\bar{x}) = 0$

$\sin(\bar{x}) = 0 \Rightarrow \bar{x} = k\pi$, $k \in \mathbb{Z}$

↳ In linear systems — equ^m points at zero
or at infinitely many points — subspace

In nonlinear systems — not the case



Linearisation :

$\dot{x} = f(x)$

$x = \bar{x} + \tilde{x}$

$\bar{x}$: Equ^m pt.

$\tilde{x}$: ↳ fluctuation or perturbation around equ^m point

$\bar{x}$: not necessarily equ^m point, but makes sense to use an equ^m point. (↱ can be any solution)

$$\dot{\bar{x}} + \dot{\tilde{x}} = f(\bar{x} + \tilde{x})$$

Can use Taylor series expansion around $\bar{x}$

$$\cancel{\dot{\bar{x}}} + \dot{\tilde{x}} = \cancel{f(\bar{x})} + \left.\frac{\partial f}{\partial x}\right|_{\bar{x}} \tilde{x} + \underbrace{\cancel{H.O.T}}_{\cancel{O(\|\tilde{x}\|^2)}} \to 0$$

↳ This eq^n holds for any solution $x(t)$ to $\dot{\bar{x}} = f(x)$

$$\dot{\tilde{x}} = \underbrace{\left.\frac{\partial f}{\partial x}\right|_{\bar{x}}}_{A} \tilde{x}$$

A : Jacobian (or dynamic matrix)

For $f(x) = \sin x$

$$A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}} = \cos(\bar{x}) = \begin{cases} \cos(2k\pi) = +1 ; & k \text{ even} \\ \cos((2k+1)\pi) = -1 ; & k \text{ odd} \end{cases}$$

Fact: from EE 5231 (to be formalised later)

→

(a) If $\mathrm{Re}\left(\lambda_i(A)\right) < 0$, for $i = 1, \dots, n$ $\Rightarrow$ $\bar{x}$ is locally asymptotically stable (LAS)

$Av = \lambda v$ → negative real part

↳ Any result due to linearisation is local → don't know about region of validity.

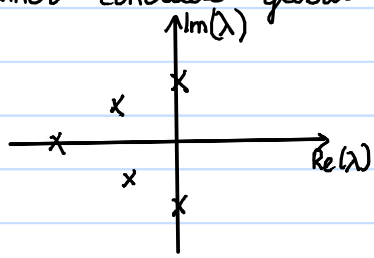
(b) If there is $i \in \{1, \dots, n\}$ s.t. $\mathrm{Re}\left(\lambda_i(A)\right) > 0$ $\Rightarrow$ $\bar{x}$ is unstable

positive real part

> Linearisation is useful, but there are challenges:

→ cannot conclude global properties

→



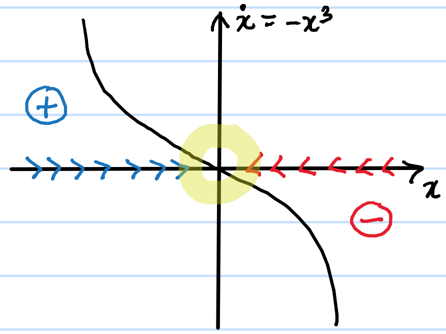
If there are eigenvalues w/ $\mathrm{Re}\ \lambda_i(A) \leq 0$ w/ some values having $\mathrm{Re}\,\lambda_i(A) = 0$

$\Rightarrow$ cannot conclude anything!

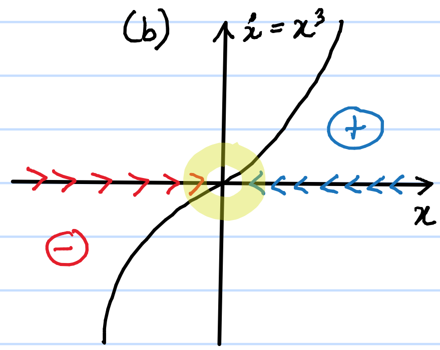
Eg 2: (a) $\dot{x} = -x^3$
(b) $\dot{x} = x^3$ } $\bar{x} = 0$ is the unique equ^m pt.

$$A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}} = \left.3x^2\right|_{\bar{x}=0} = 0$$

(a)



(b)

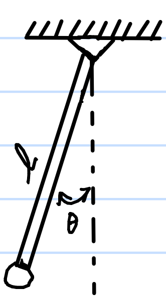


Globally stable; but linearisation does not show this in this case!

> Range of Phenomena for Nonlinear Systems:

1) Multiple isolated equilibrium points

Eg 3: Pendulum



mass: m, length: l

Newton's 2nd Law: $ml\ddot{\theta} + kl\dot{\theta} + mg\sin\theta = u$

State-space representation: $x_1 = \theta$, $x_2 = \dot{\theta}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2 + u \end{bmatrix}$$

Equ^m points: set $u = 0$ $\Rightarrow$ $\bar{x}_2 = 0$

$\sin(\bar{x}_1) = 0 \Rightarrow \bar{x}_1 = k\pi$

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}$$ ⟶ essentially two equ^m points: ↓ $\underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\bar{x}_{down}}$ and ↑ $\underbrace{\begin{bmatrix} \pi \\ 0 \end{bmatrix}}_{\bar{x}_{up}}$

Linearised matrix: $A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{x=\bar{x}} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos(\bar{x}_1) & -\frac{k}{m} \end{bmatrix}$

$A_{down} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & -\frac{k}{m} \end{bmatrix}$ $\qquad$ $A_{up} = \begin{bmatrix} 0 & 1 \\ +\frac{g}{\ell} & -\frac{k}{m} \end{bmatrix}$

good : -ve eigenvalue → LAS $\qquad$ bad : +ve eigenvalue → unstable

(from classical control : Routh – Hurwitz)

Cannot conclude anything for $k=0$ ∵ roots in the $j\omega$-axis

Eg ④: Logistic Equation : $\dot{x} = \alpha\left(1 - \frac{x}{k}\right)x$ ⇒ describes growth of some population

$\alpha, k > 0$, $x > 0$ → no negative pop^n!

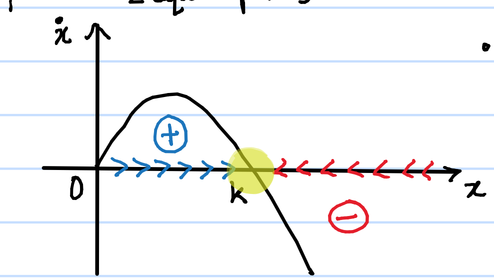
> Simpler model : (linear) $\qquad \dot{x} = \alpha x$

$$\Rightarrow x(t) = x(0)\,e^{\alpha t}$$

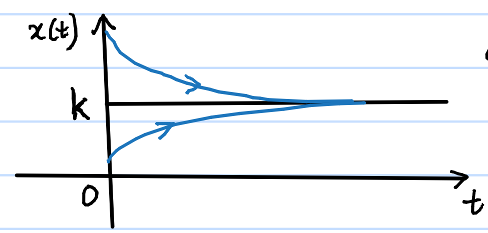
- Problem: unrealistic ∵ if $x(0) > 0$, pop^n grows exponentially w/ time.
  - ↳ leads to exponential pop^n growth

$\frac{\dot{x}}{x} = \alpha$ vs.	$\frac{\dot{x}}{x} = \alpha\left(1 - \frac{x}{k}\right)$
constant	Linear decay
1 equ^m point	2 equ^m points

k : carrying capacity



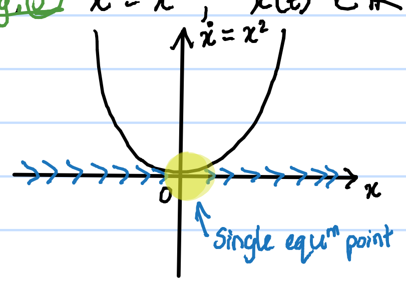
• Population goes to k no matter what the starting point



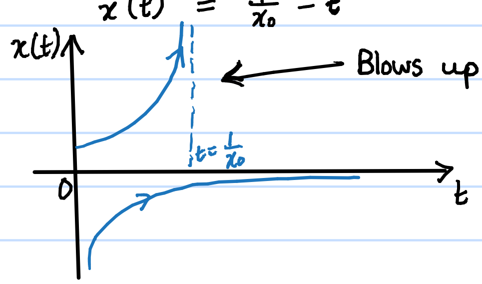
cannot cross equilibrium points

2) Finite Escape Time

Eg ⑤: $\dot{x} = x^2$; $x(t) \in \mathbb{R}$



$$x(t) = \frac{1}{\frac{1}{x_0} - t}$$

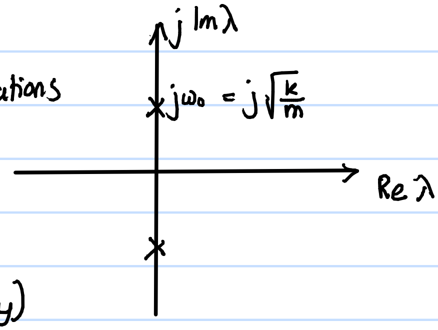


← Blows up in finite time

or converges to zero in finite time (different from linear systems)
↳ convergence always exponential

3) Limit Cycles

- Sustained oscillations / Structurally robust oscillations

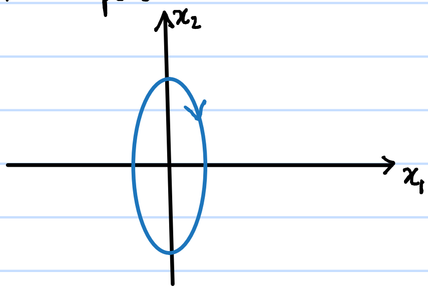


Eg ⑥: Harmonic oscillator : $m\ddot{y} + ky = 0$

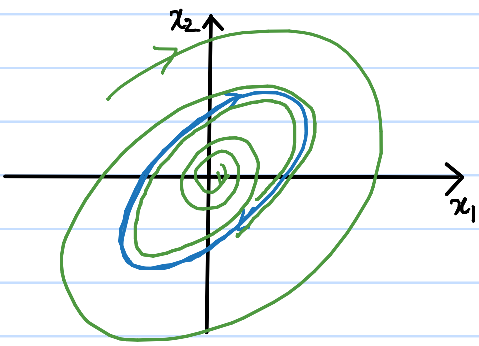
$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \rightarrow \text{eigenvalues on the } j\omega \text{ axis}$$

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{(fundamental frequency)}$$

Phase portrait:

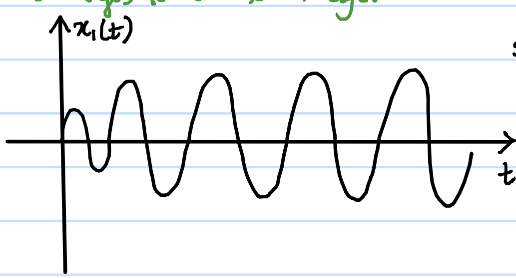


These are non-robust — amplitude depends on IC
↳ Oscillations in linear systems are structurally non-robust i.e. a bit of noise can destroy ...



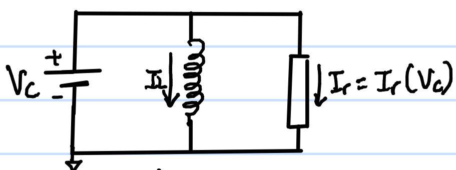
Limit cycle

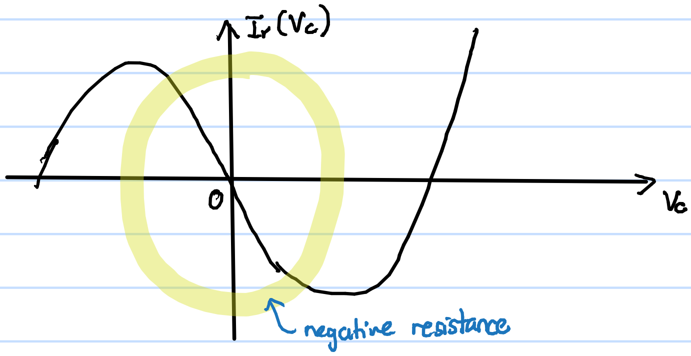
Converges to the limit cycle



sustained oscillations

Eg ⑦: Van der Pol oscillator





$$\dot{I}_L = \frac{1}{L} V_c$$
$$\dot{V}_c = -\frac{1}{C} I_L + \frac{1}{C}(V_c - V_c^3)$$

$\bar{x} = 0$ unique equ^m point

Linearise: $A = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & +\frac{1}{C} \end{bmatrix}$

↳ this causes solution to blow up → unstable

(Structurally robust oscillations)

4) Chaotic dynamics (Chaos

Eg 8: Lorenz attractor.

$$\dot{x} = a(x-y)$$
$$\dot{y} = x(b-z)-y$$
$$\dot{z} = xy - \tau z$$

a, b, τ : parameters

x, y, z : states

Consider $a = 10$, $b = 28$, $\tau = \frac{8}{3}$

$\Rightarrow$ Chaos

↳ No simple characterisation of asymptotic behaviour)

↳ huge sensitivity to ICs