EE 8215 S16 : Nonlinear Systems

Lecture 5 : Feb 02, 2016, Tue

- HW #2 Due Thu, Feb 11th
- Last time : - Pitchfork bifurcations
  - Phase portraits of 2^{nd} order linear systems.
- Today : - Hartman – Grobman Theorem
  - Conditions for absence (Bendixson) or presence (Poincare-Bendixson) of periodic orbits of 2^{nd} order systems
    - ↳ Not useful for higher-order systems

➢ Hartman - Grobman Theorem

- relates phase portraits of nonlinear systems w/ hyperbolic equ^m points to those of corresponding linearisation.

no eigenvalues on the imaginary axis.

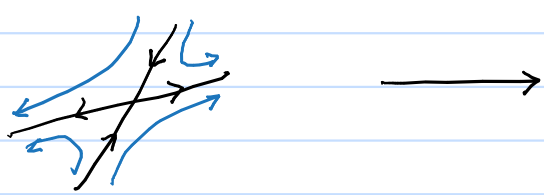
$\bar{x} \in \mathbb{R}^n$ w/ $f(\bar{x}) = 0$ is a hyperbolic equ^m point if linearisation $\left.\frac{\partial f}{\partial x}\right|_{x=\bar{x}}$ does not have eigenvalues on the $j\omega$-axis (i.e. zero real part).

→ HB Thm :

"If $\bar{x} \in \mathbb{R}^n$ is a hyperbolic equ^m point of $\dot{x} = f(x)$, then there is a homeomorphism from a neighbourhood of $\bar{x}$ to $\mathbb{R}^n$ that maps trajectories of $\dot{x} = f(x)$ to those of corresponding linearisation."

- homeomorphism : continuous map w/ continuous inverse

(see notes)



★ Note : Absence of eigenvalues on $j\omega$-axis is key!!

➢ Eg ① :

$$\dot{x}_1 = -x_2 + \boxed{a x_1 (x_1^2 + x_2^2)}$$
$$\dot{x}_2 = x_1 + \boxed{a x_2 (x_1^2 + x_2^2)}$$

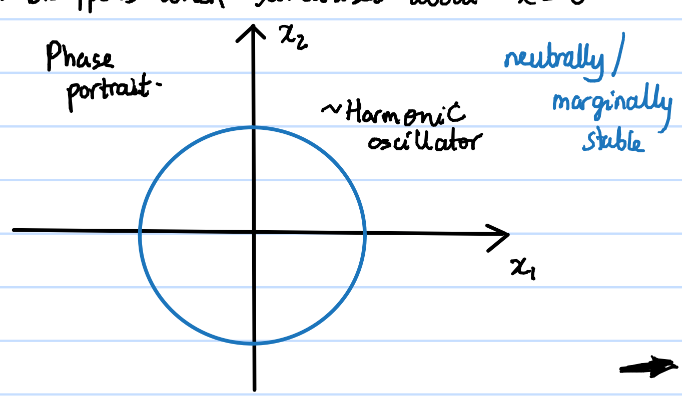
→ Disappears when linearised about $\bar{x} = 0$

Equ^m pt : $\bar{x} = 0$

Linearise around $\bar{x} = 0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Phase portrait :

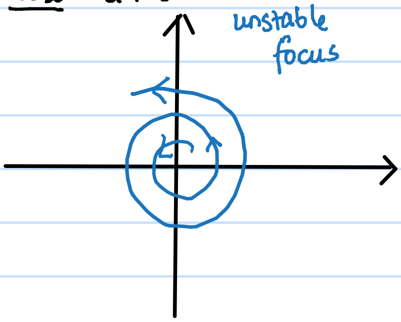


- In polar coordinates : $\dot{r} = ar^3$ → depends on a

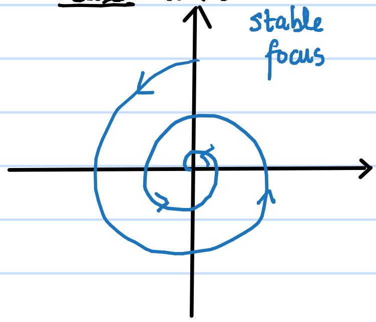
  $\dot{\theta} = 1$ → angle increasing all the time

  Eigenvalues are on the $j\omega$ axis → cannot use linearisation to conclude

<u>Case</u>: $a > 0$



<u>Case</u>: $a < 0$



<u>Periodic orbit</u>: closed trajectory ; starts somewhere & comes back after a certain time (neutrally stable)

➤ <u>Bendixon Criterion</u>:

- condition for the <u>absence</u> of periodic orbits → limit cycles (eg: van der Pol) → neutrally stable (eg: pendulum w/ no friction)
- For 2^{nd} order systems : $\dot{x}_1 = f_1(x_1, x_2)$, $\dot{x}_2 = f_2(x_1, x_2)$ $\quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2$

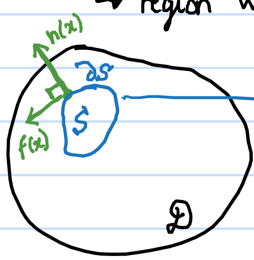
$$\text{div} f = \nabla \cdot f = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \end{bmatrix} \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

Essentially: check sign of $\text{div} f$

↳ Thm: If $\text{div } f$ is not identically equal to zero and does not change sign in a <u>simply connected region</u> $\mathcal{D} \subset \mathbb{R}^2$, then there are <u>no</u> periodic orbits in $\mathcal{D}$

↳ region w/o any holes

<u>Proof</u>:



Let there be a periodic orbit in $\mathcal{D}$ (closed trajectory in a plane)

→ closed trajectory of $\dot{x} = f(x)$

Green's Thm: $\displaystyle \int_{\partial S} \underbrace{f(x)\, n(x)}_{\perp \; = 0}\, dl = \iint_S \text{div } f(x)\, dS$

$$\iint_S \text{div } f(x)\, dS = 0$$

For the integral to be zero, either : $\text{div } f(x) = 0$

or : $\text{div } f(x)$ changes sign

Conditions of the Thm lead to contradiction

QED

Eg ②: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

For an LTI system:
a, b, c, d fixed $\Rightarrow$ cannot change sign

$$\text{div } f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = a + d$$

- If a, d are constants, $a+d$ cannot change sign

$\therefore$ If $a + d \neq 0 \Rightarrow$ no periodic orbits.

$\text{trace}(A) = a + d$

$$\text{trace}(A) = \lambda_1(A) + \lambda_2(A) = \sum_i \lambda_i$$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ Yes, harmonic oscillator

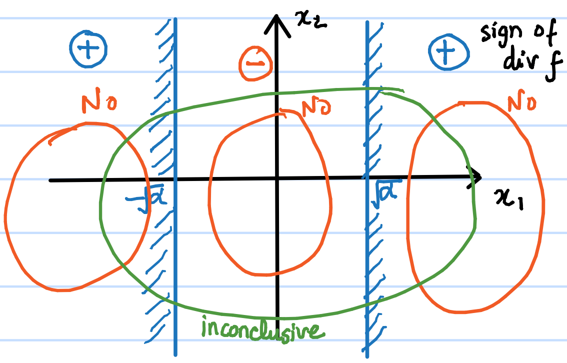
$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ No, Saddle.

But Bendixon does not allow to reach these conclusions.

Eg ③: 2nd order nonlinear system:

$$\dot{x}_1 = x_2 \qquad = f_1$$
$$\dot{x}_2 = -\alpha x_2 + x_1 - x_1^3 + x_1^2 x_2 \qquad = f_2$$

$$\text{div } f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 - \alpha + x_1^2 = x_1^2 - \alpha$$

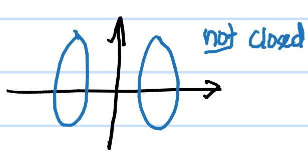


- Positively-invariant sets: start in the set; never leave it.

Poincare - Bendixon Thm:

- Given a 2nd order nonlinear system $\dot{x} = f(x)$, $x(t) \in \mathbb{R}^2$
  let M be a compact (closed & bounded) set
  If (a) there are no equm points in M, and
  (b) M is positively invariant
  then M contains a periodic orbit.

Closed set: connected set



$\dot{x} = f(x)$

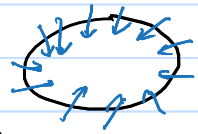
If trajectory that starts at x_0 is given by $\Phi(t, x_0)$

M is positively invariant if for each $x_0 \in M \Rightarrow \Phi(t, x_0) \in M$

Eg 4: Predator-prey model

Prey: $\dot{x} = (a - by)\,x$ $\quad a, b, c, d > 0$ | product: chance of predator-prey encounter

Predator: $\dot{y} = (cx - d)\,y$



(HW: show that 1^{st} quadrant is positively invariant

↳ Examine inner product @ boundary of set & normal