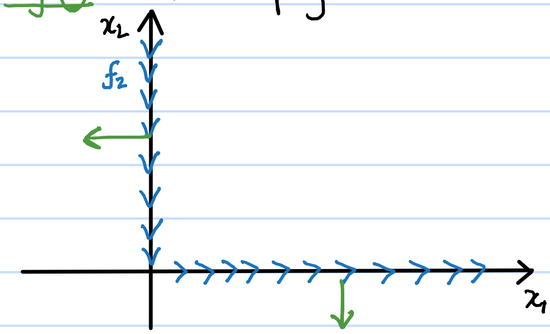
EE 8215 S'16 : Nonlinear Systems
Lecture 6 : Feb 9, 2016, Tue

> **Last time:** - Bendixson Thm (Absence of periodic orbits)
> - Poincaré - Bendixson Thm (Presence of periodic orbits)
> } for 2^{nd} order systems

M: positively invariant set if for all $x_0 \in M \Rightarrow \underset{\text{trajectory}}{\phi(t, x_0)} \in M \quad \forall t \geq 0$

Eg ①: Predator-prey model



$$\dot{x}_1 = (a - b x_2) x_1$$
$$\dot{x}_2 = (c x_1 - d) x_2$$

- $x_1 = 0 \Rightarrow f_1 = 0, \quad f_2 = -d x_2$
- $x_2 = 0 \Rightarrow f_2 = 0, \quad f_1 = \quad x_1$

- Direction tangential to boundary of the set → +vely invariant
- At the Boundary field: inner product ≤ 0 → points into the set (< 0) or tangential ($= 0$)

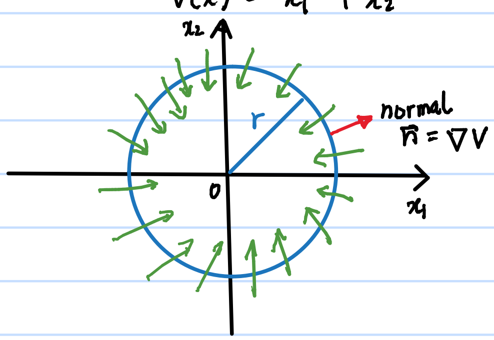
Eg ②:
$$\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2)$$

→ Show that the ball of radius r:

$B_r = \{ x \in \mathbb{R}^2 ; \; x_1^2 + x_2^2 \leq r \}$ is +vely invariant for large enough r (TBD)

⇒ Study the level sets of the ball : $V(x)$ const.

$V(x) = x_1^2 + x_2^2$



want to find r s.t. this holds

$$(\nabla V)^T \cdot f(x) \leq 0$$

$$\nabla V = \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$(\nabla V)^T \cdot f(x) = \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2$$

$$= \left[2x_1^2 + 2x_1 x_2 \underbrace{- 2x_1 (x_1^2 + x_2^2)}_{\text{good}} \right] + \left[-4x_1 x_2 \underbrace{+ 2x_2^2}_{\text{bad}} - 2x_2^2 (x_1^2 + x_2^2) \right]$$

$$= -2(x_1^2 + x_2^2)^2 + 2(x_1^2 + x_2^2) - 2x_1 x_2$$

want to bound "badness" of this term

$(a \pm b)^2 \geq 0$

$a^2 + b^2 \geq \mp 2ab$

$$\leq -2(x_1^2 + x_2^2)^2 + 2(x_1^2 + x_2^2) + (x_1^2 + x_2^2)$$
$$\leq -2(x_1^2 + x_2^2)^2 + 3(x_1^2 + x_2^2)$$

$$(\nabla V)^T \cdot f(x) \leq -2(x_1^2+x_2^2)\underbrace{\left[x_1^2+x_2^2-\frac{3}{2}\right]}_{\text{This terms needs to be } \geq 0}$$

$$(\nabla V)^T \cdot f(x) \leq 0 \quad \text{if} \quad x_1^2+x_2^2-\frac{3}{2} \geq 0$$

$$r^2 \geq \frac{3}{2} \quad \Rightarrow \quad r \geq \sqrt{\frac{3}{2}}$$

> **Poincaré – Bendixson Thm:**

- 2nd order system : $\dot{x} = f(x)$ $\quad x(t) \in \mathbb{R}^2$
- M : closed and bounded set
- If (a) No equ^m pt.
 (b) M +vely invariant } M contains a periodic orbit

Note: (a) can be relaxed: if M contains a single equ^m pt. which is either an unstable node or an unstable focus. → then M contains a periodic orbit.

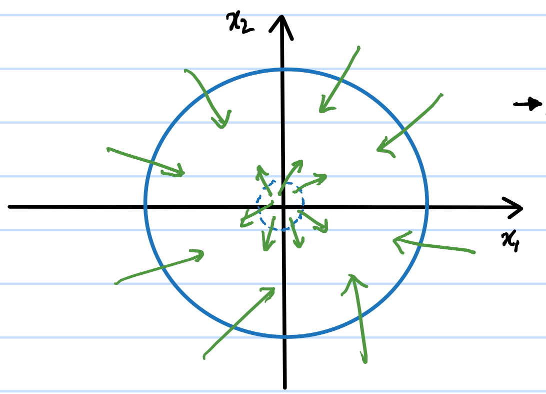
For eg ②: linearisation at (0,0) :

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \qquad sI - A = \begin{bmatrix} s-1 & -1 \\ 2 & s-1 \end{bmatrix}$$

$$\det(sI-A) = (s-1)^2 - 2(-1) = s^2 - 2s + 1 + 2 = s^2 - 2s + 3$$

Complex conjugate eigenvalues: $s_{1,2} = \dfrac{2 \pm \sqrt{4-12}}{2} = 1 \pm j2\sqrt{2}$

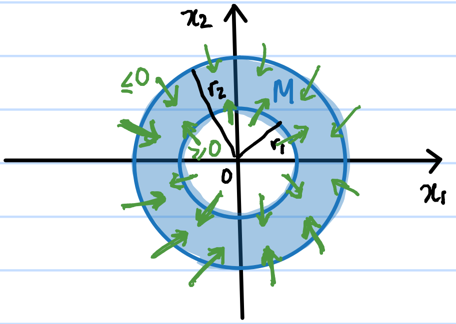
unstable focus



→ Small neighbourhood around (0,0) → goes outside the neighbourhood.

Eg ③: $\dot{x}_1 = -x_2 \qquad \dot{x}_2 = x_1$ $\qquad$ (Linear system) | Harmonic oscillator

Unique equ^m point : $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$



Candidate for M: $M := \{x \in \mathbb{R}^2 ; \ r_1^2 \leq x_1^2 + x_2^2 \leq r_2^2$

$V(x) = x_1^2 + x_2^2$

$(\nabla V)^T \cdot f = 2x_1(-x_2) + 2x_2(x_1) = 0$

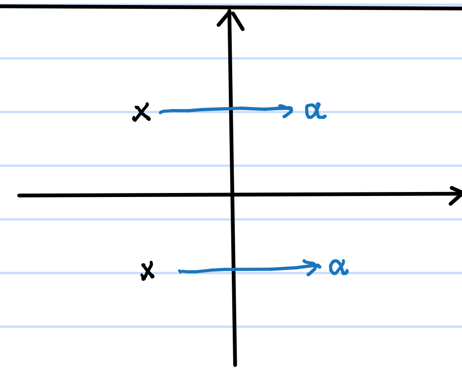
⇒ Any circle is a periodic orbit.

If $(\nabla V)^T \cdot f \neq 0$ then $(\nabla V)^T \cdot f \leq 0$ outer boundary
≥ 0 inner boundary

So far: Bifurcations
1) Fold $\quad \dot{x} = \alpha \pm x^2$
2) Transcritical $\quad \dot{x} = \alpha x \mp x^2$
3) Pitchfork $\quad \dot{x} = \alpha x \mp x^3$

$\Big\} \; \alpha \in \mathbb{R}$

Two common features:
- essentially 1^{st} order phenomena (can occur in higher dimensions but typically restricted to 1D subspace
- when $\alpha = \alpha_{crit} \Rightarrow A = \frac{\partial f}{\partial x}\Big|_{\alpha = \alpha_{crit}}$ disappears $\rightarrow$ uninformative



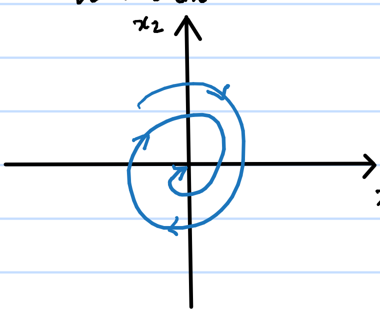
▷ Hopf Bifurcations

- cannot be seen in 1^{st} order systems
- involve limit cycles
- Two types:
  - supercritical 🙂
  - subcritical 🙁

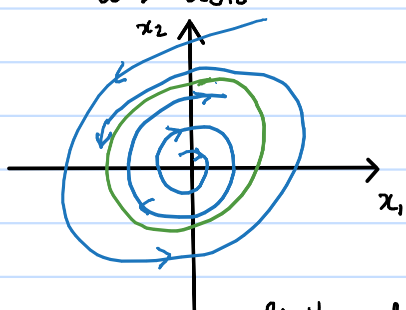
1) Supercritical

- involves loss of stability of equ^m point
- stable focus → complex conjugate eigenvalues; as α grows, eigenvalues pushed to RHP

$\alpha < \alpha_{crit}$

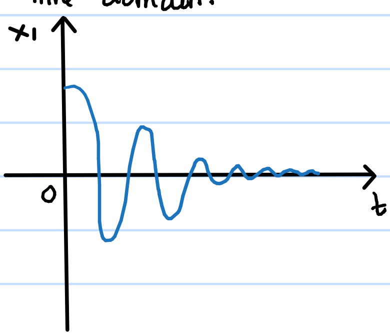


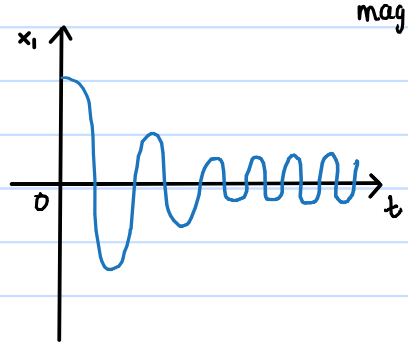
$\alpha > \alpha_{crit}$



limit cycle appears
magnitude of limit cycle $\propto \sqrt{\alpha}$

Time domain:





Eg ④: Supercritical Hopf Bifurcation

$$\left.\begin{aligned} \dot{x}_1 &= x_1(\alpha - x_1^2 - x_2^2) - x_2 \\ \dot{x}_2 &= x_2(\alpha - x_1^2 - x_2^2) + x_1 \end{aligned}\right\}$$

In polar coordinates: $\quad \dot{r} = \alpha r - r^3$, $\quad \dot{\theta} = 1$

Equilibrium points:
- none for θ
- $\dot{r} = 0 \Rightarrow \bar{r}(\alpha - \bar{r}^2) = 0 \Rightarrow$ $(r > 0)$

Critical points:
$\bar{r} = 0$ (Equ^m pt.)
$\bar{r} = \sqrt{\alpha}$ if $\alpha > 0$ (Limit cycle)