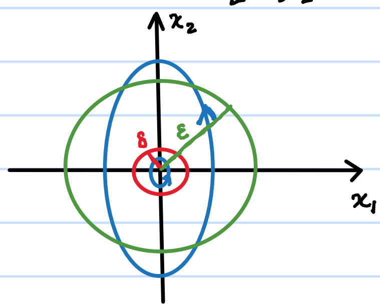
EE 8215 S15 : Nonlinear Systems
Lecture 10: Feb 23rd, 2016, Tue

- **Last time**: • Definition of stability of $\bar{x} = 0$
- **Today**: • Lyapunov direct method (Lyapunov's Method)

Eg ①: Harmonic oscillator : $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

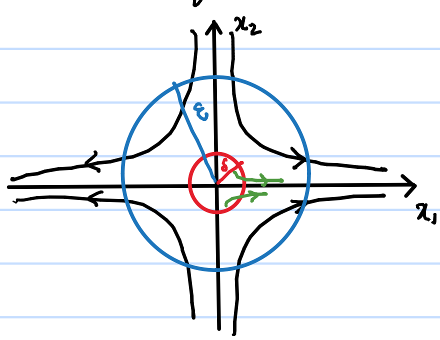
Phase portrait:



Equ^m pt ($\bar{x}=0$) stable in the sense of Lyapunov

$\Rightarrow$ System marginally stable.

Eg ②: Illustration of an unstable equ^m point

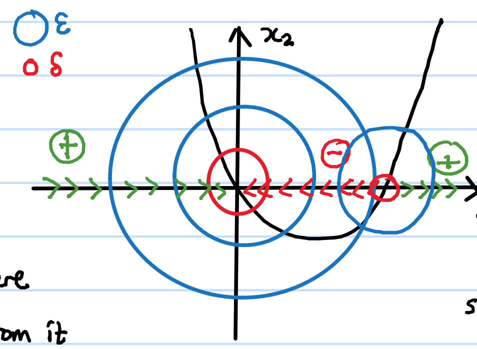


Eg ③: $\dot{x} = x(x-1)$

$\bar{x} = 0$ or $\bar{x} = 1$

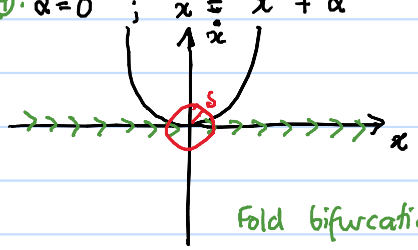
LAS : two equ^m pts, if starting at $x > 1$, $\longrightarrow \infty$

$\bar{x} = 1$ is unstable. $\because$ no matter where we start - always move away from it



Can choose any ε
↳ can always choose smaller δ.
so that it stays w/in δ.

Eg ④: $\alpha = 0$; $\dot{x} = x^2 + \alpha$

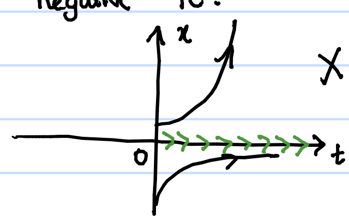


unstable

Fold bifurcation

Def^n: $\forall \varepsilon > 0$, $\exists\ \delta > 0$
s.t. $\|x_a\| < \delta \Rightarrow \|x(t)\| < \varepsilon$

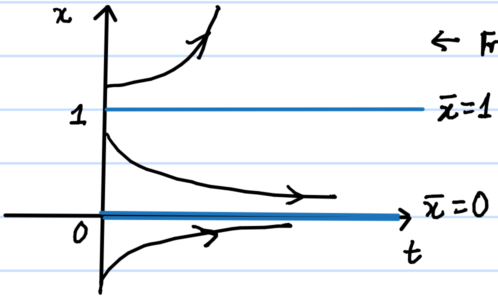
negative IC :



✗ if +ve ICs

✓ if −ve ICs

want $\forall \varepsilon$ i.e. for all ICs



← From eg. ③.

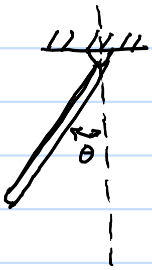
Physical system

Hamiltonian systems → energy conserved

↳ Lyapunov gets roots from this

Lyapunov Direct Method

Eg ⑤:



• State-space: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -a\sin(x_1) - b x_2 \end{bmatrix}$ ⊛ $a, b > 0$ ↳ viscous damping

$x_1 = \theta$ (angle)

• Energy: $V(x_1,x_2) = \underbrace{a\int_0^{x_1} \sin(\xi)\,d\xi}_{PE} + \underbrace{\tfrac{1}{2}x_2^2}_{KE}$ w/ normalized parameters

Q) What does energy do along the solutions of ⊛?

⇒ compute derivative w.r.t time:

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1}\cdot\frac{\partial x_1}{\partial t} + \frac{\partial V}{\partial x_2}\cdot\frac{\partial x_1}{\partial t} = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}\right]\cdot\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \frac{\partial V}{\partial x}\cdot f(x)$$

$$= \begin{bmatrix} a\sin(x_1) & x_2 \end{bmatrix}\begin{bmatrix} x_2 \\ -a\sin(x_1) - bx_2 \end{bmatrix} = a x_2 \cancel{\sin(x_1)} - a x_2 \cancel{\sin(x_1)} - b x_2^2$$

$= -bx_2^2$ (If no viscous damping i.e. $b=0$: energy is conserved)

$= -b x_2^2 - 0 x_1^2$

$\leq 0 \quad \forall\ \vec{x} \neq \vec{0}$ | i.e. stable in the sense of Lyapunov

↳ i.e. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Thm:

• Given: $V: \mathbb{R}^n \longrightarrow \mathbb{R}$

(scalar-valued function of the state vector: $x \in \mathbb{R}^n$

s.t. $V(0) = 0$ or $V(\text{equ}^m \text{ pt}) = 0$

$V(x) > 0 \quad \forall\ x \in \mathcal{D} \setminus \{0\}$

Then: ↑domain (i.e. some neighbourhood of 0 "ball")

1) If $\underbrace{\dot{V}(x)}_{\text{i.e. } \frac{dV}{dt}} \leq 0 \quad \forall\ x \in \mathcal{D}\setminus\{0\}$ then $\bar{x} = 0$ is stable (in the sense of Lyapunov)

↳ derivative along the solutions to the system

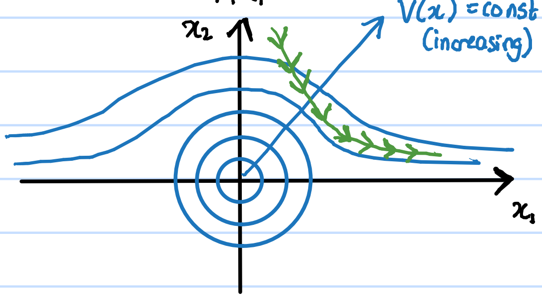
2) If $\frac{dV}{dt} < 0 \quad \forall\ x \in \mathcal{D}\setminus\{0\}$, then $\bar{x} = 0$ is LAS.

3) Note: for global asymptotic stability (GAS) → need $V(x) > 0 \quad \forall\ x \in \mathbb{R}^n$ (global +ve definiteness)

and $V(x) \longrightarrow \infty$ as $\|x\| \to \infty$ (radial unboundedness)

and $\frac{dV}{dt} < 0 \quad \forall\ x \in \mathbb{R}^n \setminus \{0\}$

Eg (6): $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$ $x(t) \in \mathbb{R}^2$



• level sets bring states away from ∞

> Linear Systems: $\dot{x} = Ax$

$\Rightarrow$ Need to examine quadratic Lyapunov functions – but cannot prove stability w/ them

$V(x) = x^T P x$, $P = P^T > 0$ (symmetric & positive-definite)

↳ overloaded notation

meaning: $\forall x \neq 0,\ x^T P x > 0$

if $\lambda_i(P) > 0 \quad \forall i \Rightarrow P = P^T > 0$

or check all principal minors → should be all > 0

– Can restrict to symmetric matrices ∵

Fact: $P = P_s + P_a$ (sum of its symmetric & non-symmetric part)

$P_s = \frac{1}{2}(P + P^T) \quad P_a = \frac{1}{2}(P - P^T)$

HW: show: $x^T P_a x = 0$

i.e. antisymmetric part does not contibute to quadratic ....

$$\frac{dV}{dt} = \dot{x}^T P x + x^T P \dot{x} = (Ax)^T P x + x^T P (Ax)$$

$$= x^T A^T P x + x^T P A x = x^T (A^T P + PA) x$$

$$Q = -(A^T P + PA)$$

If $Q = Q^T > 0 \Rightarrow \bar{x} = 0$ is G.A.S. in the sense of Lyapunov.

→ Quadratic Lyapunov functions always work for linear systems.