Leader selection in consensus networks

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joint work with:

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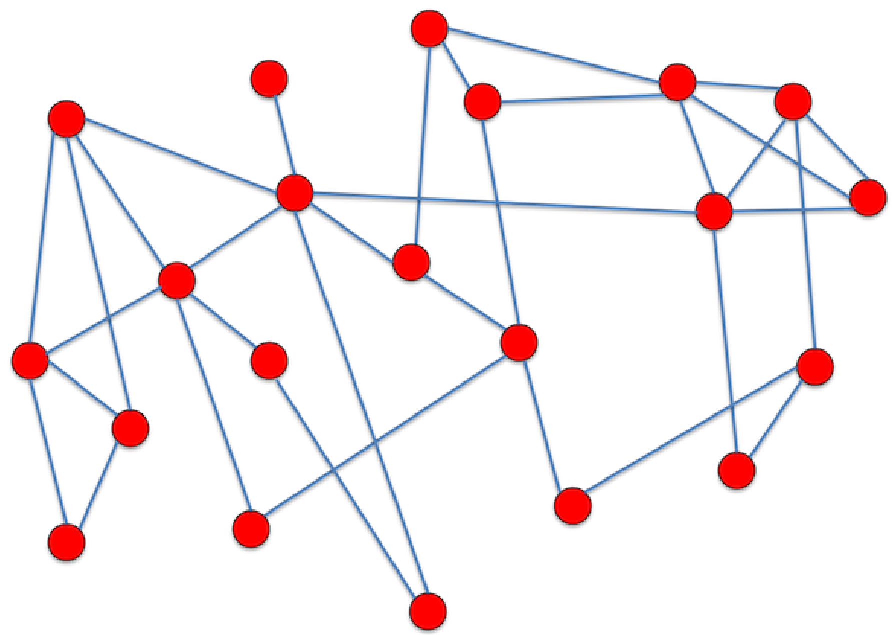
Consensus with stochastic disturbances



- RELATIVE INFORMATION EXCHANGE WITH NEIGHBORS
  - ⋆ **simplest distributed averaging algorithm**

$$\dot{\psi}_i(t) \;=\; -\sum_{j \,\in\, \mathcal{N}_i} \Big(\psi_i(t) \,-\, \psi_j(t)\Big) \;+\; \underbrace{w_i(t)}_{\text{white noise}}$$

$\downarrow$

white noise

Convergence

- UNFORCED NETWORK DYNAMICS

  ⋆ **diffusion on a graph with Laplacian** $L = L^T$

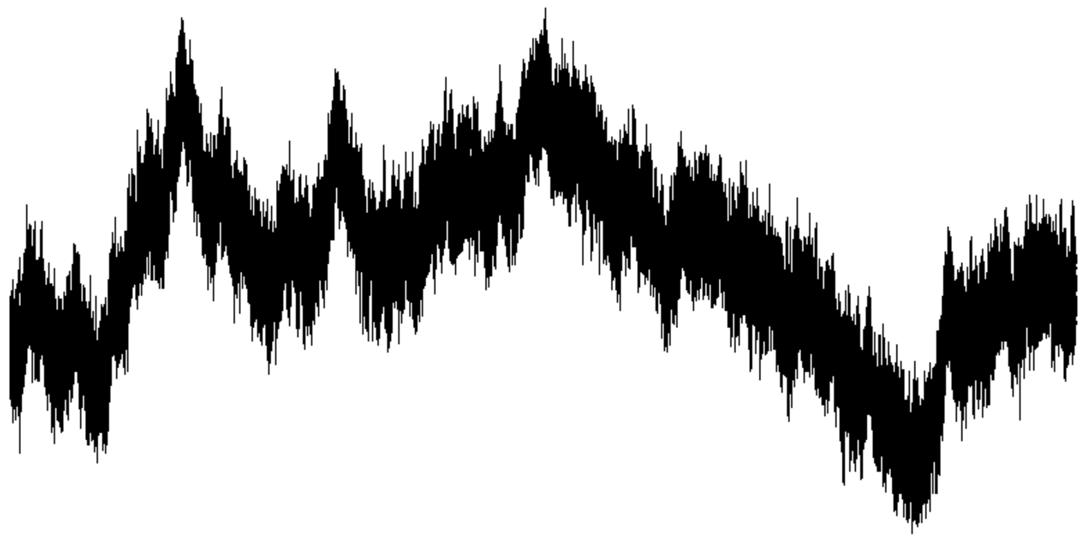
$$\begin{bmatrix} \dot\psi_1(t) \\ \vdots \\ \dot\psi_n(t) \end{bmatrix} = \begin{bmatrix} & & \\ & -L & \\ & & \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{bmatrix}$$

  ⋆ **e-values of** L**:** $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$

connected network		**convergence to the average**
$\lambda_2(L) > 0$	$\Rightarrow$	$\psi_i(t) \xrightarrow{t \to \infty} \bar\psi(t) := \dfrac{1}{n} \sum \psi_i(t)$

Consensus with stochastic disturbances

- NETWORK AVERAGE
  - ⋆ **undergoes random walk**



$$\lambda_2(L) > 0 \;\Rightarrow\; \begin{cases} \textbf{each } \psi_i(t) \textbf{ fluctuates around } \bar{\psi}(t) \\ \textbf{deviation from average: } \tilde{\psi}_i(t) := \psi_i(t) - \bar{\psi}(t) \end{cases}$$

steady-state variance: $$\lim_{t \to \infty} \sum \mathcal{E}\left(\tilde{\psi}_i^2(t)\right) = \sum_{i \neq 1} \frac{1}{2\,\lambda_i(L)}$$

Networks with leaders

- TWO GROUPS OF NODES
  - **Followers: relative information exchange**

$$\dot{\psi}_i \;=\; -\sum_{j \,\in \mathcal{N}_i} (\psi_i \,-\, \psi_j) \;+\; w_i$$

  - **Leaders: perfectly follow desired trajectory**

$$\psi_i \;\equiv\; 0 \quad\Rightarrow\quad \dot{\psi}_i \;\equiv\; 0$$

Network performance

$$\begin{bmatrix} \dot{\psi}_l(t) \\ \dot{\psi}_f(t) \end{bmatrix} = - \begin{bmatrix} 0 & 0 \\ * & L_x \end{bmatrix} \begin{bmatrix} \psi_l(t) \\ \psi_f(t) \end{bmatrix} + \begin{bmatrix} 0 \\ w(t) \end{bmatrix}$$

x – vector with components $\begin{cases} x_i = 1 & \text{node } i \text{ is a leader} \\ x_i = 0 & \text{otherwise} \end{cases}$

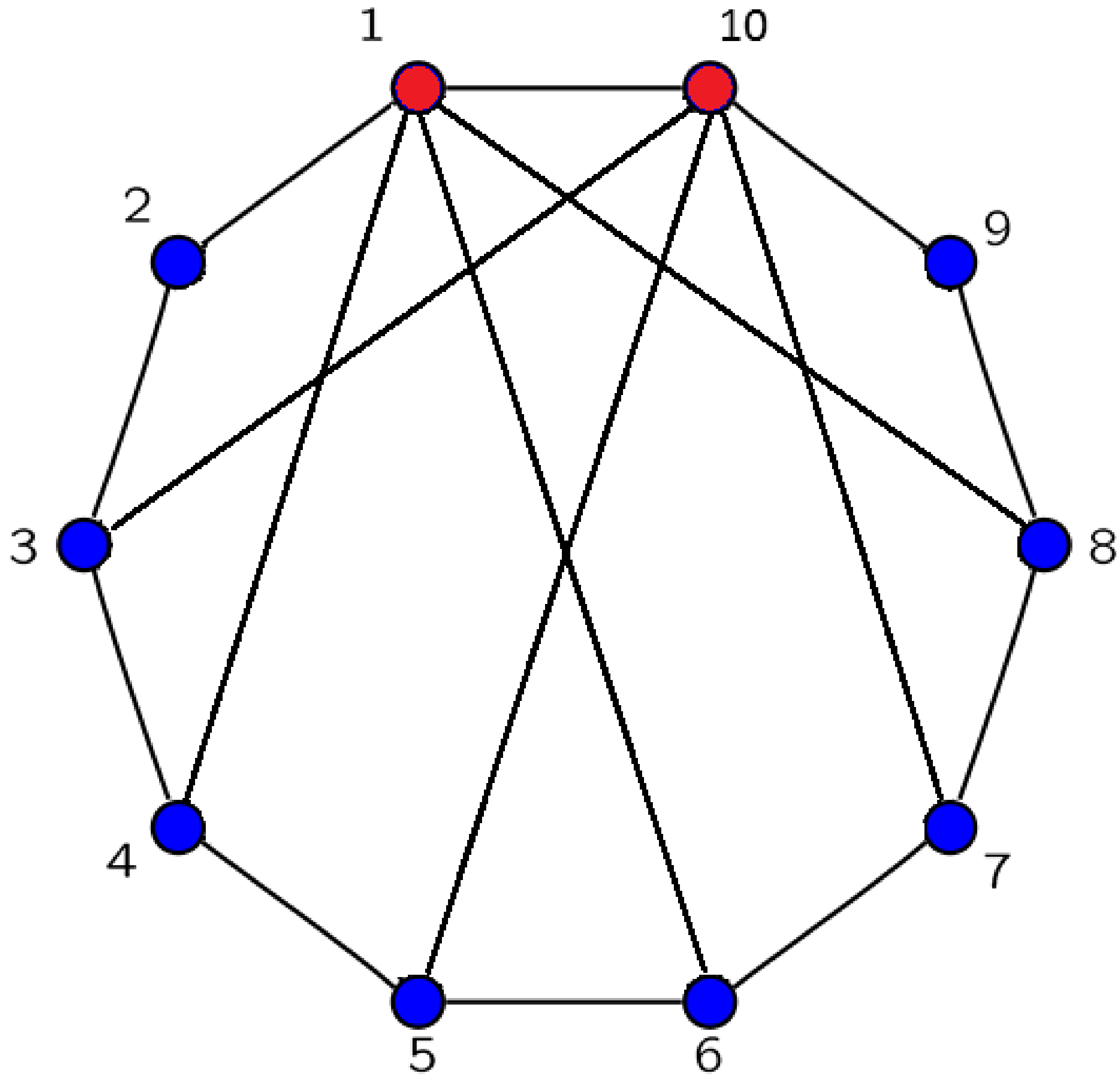
L_x – principal submatrix of L

remove rows & columns of L corresponding to leaders

- VARIANCE OF FOLLOWERS

⋆ **determined by** $\begin{cases} \textbf{network topology} \\ \textbf{locations of leaders} \end{cases}$

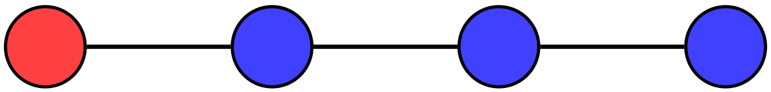
Examples



$$x = \begin{bmatrix} 1 & 0 & \cdots & 0 & 1 \end{bmatrix}^T$$

remove first and last rows & columns from L

- PATH GRAPH



$x_i \in \{0, 1\}$; 1 – leader, 0 – follower

$$x = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$$

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \Rightarrow L_x = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Leader selection problem

- Select N_l leaders to:
  - ⋆ **minimize variance of followers**

$$\begin{array}{rrcl} \underset{x}{\text{minimize}} & J_f(x) & = & \operatorname{trace}\left(L_x^{-1}\right) \\ \text{subject to} & x_i & \in & \{0,1\}, \quad i = 1,\ldots,n \\ & \mathbb{1}^T x & = & N_l \end{array}$$

Related work

- GREEDY ALGORITHMS WITH APPROXIMATIONS

  Patterson and Bamieh '10

- SUBMODULAR OPTIMIZATION WITH PERFORMANCE GUARANTEES

  Clark, Bushnell, and Poovendran '11, '12, '13

- SDP FOR SENSOR SELECTION PROBLEM

  Joshi and Boyd '09

- CONTROLLABILITY OF LEADER-FOLLOWER NETWORKS

  Tanner '04

  Liu, Chu, Wang, and Xie '08

  Rahmani, Ji, Mesbahi, and Egerstedt '09

  Kawashima and Egerstedt '12

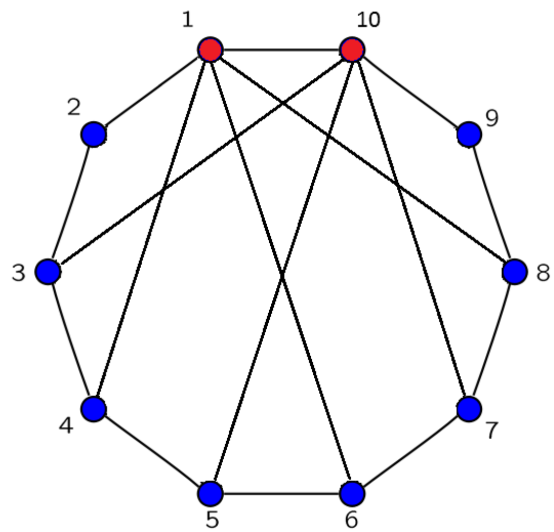
This talk

- SELECTION OF NOISE-CORRUPTED LEADERS
  - ⋆ common in applications
  - ⋆ provides insight into selection of noise-free leaders
  - ⋆ easier to solve

- EFFICIENT ALGORITHMS FOR BOUNDS ON GLOBAL OPTIMAL VALUE
  - ⋆ convex relaxations: lower bounds
  - ⋆ greedy algorithms: upper bounds

- EXAMPLES

Networks with noise-corrupted leaders



⋆ **Followers: relative information exchange**

$$\dot{\psi}_i \;=\; - \sum_{j \,\in\, \mathcal{N}_i} (\psi_i \,-\, \psi_j) \;+\; w_i, \qquad i \,\in\, \{2, \ldots, 9\}$$

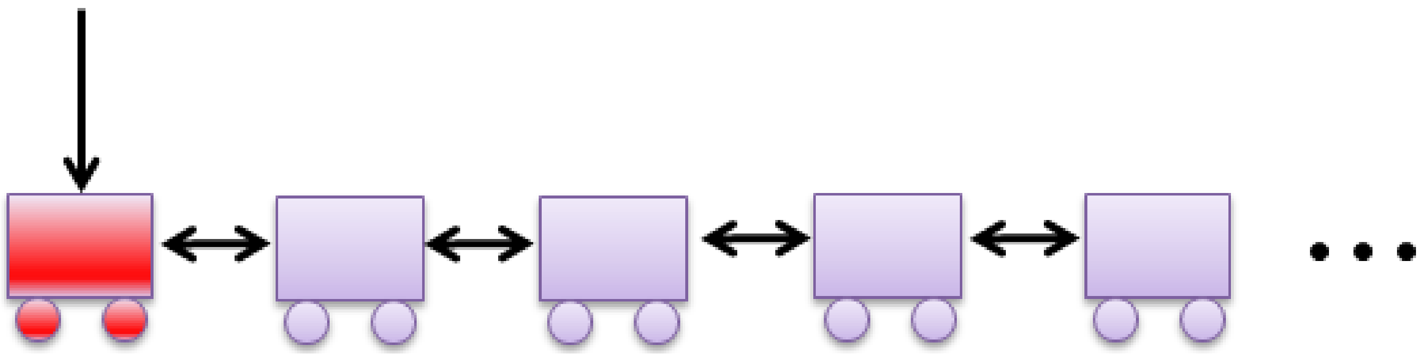
⋆ **Leaders: can also measure their own state**

$$\dot{\psi}_i \;=\; - \sum_{j \,\in\, \mathcal{N}_i} (\psi_i \,-\, \psi_j) \;+\; w_i \;-\; \alpha\, \psi_i, \qquad i \,\in\, \{1,\, 10\}$$

• NETWORK DYNAMICS

⋆ diagonally strengthened Laplacian

$$\begin{bmatrix} \dot{\psi}_1(t) \\ \vdots \\ \dot{\psi}_n(t) \end{bmatrix} = \begin{bmatrix} -(L + \alpha \, \mathrm{diag}(x)) \end{bmatrix} \begin{bmatrix} \psi_1(t) \\ \vdots \\ \psi_n(t) \end{bmatrix} + \begin{bmatrix} w_1(t) \\ \vdots \\ w_n(t) \end{bmatrix}$$



$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad \mathrm{diag}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_i \in \{0, 1\}$; 1 – leader, 0 – follower

Noise-corrupted formulation

- SELECT N_l LEADERS TO:

  ⋆ **minimize network variance**

$$\begin{array}{rcl} \underset{x}{\text{minimize}} \quad J(x) & = & \text{trace}\left((L + \alpha\,\text{diag}\,(x))^{-1}\right) \\ \text{subject to} \qquad x_i & \in & \{0,1\}, \qquad i = 1,\ldots,n \\ \mathbb{1}^T x & = & N_l \end{array}$$

- NOISE-FREE FORMULATION

$$\begin{array}{r} \text{leaders:} \\ \text{followers:} \end{array} \begin{bmatrix} L_l + \alpha I & L_0^T \\ L_0 & L_x \end{bmatrix}^{-1} \xrightarrow{\alpha \to \infty} \begin{bmatrix} 0 & 0 \\ 0 & L_x^{-1} \end{bmatrix}$$

Algorithms

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \text{trace}\left((L + \alpha\,\text{diag}\,(x))^{-1}\right) \\ \text{subject to} & x_i \in \{0, 1\}, \quad i = 1, \ldots, n \\ & \mathbb{1}^T x = N_l \end{array}$$

- FEATURES
  - ⋆ convex objective function
  - ⋆ nonconvex Boolean constraints
- APPROACH
  - ⋆ convex relaxations: lower bounds
  - ⋆ greedy algorithms: upper bounds

Convex relaxation

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \text{trace}\left((L + \alpha\,\text{diag}(x))^{-1}\right) \\ \text{subject to} & 0 \le x_i \le 1, \quad i = 1, \ldots, n \\ & \mathbb{1}^T x = N_l \end{array}$$

- Complexity of computing lower bound
  - ⋆ standard SDP solvers: $O\left(n^4\right)$
  - ⋆ customized interior point method: $O\left(n^3\right)$

Greedy algorithm

- **Select one-leader-at-a-time**

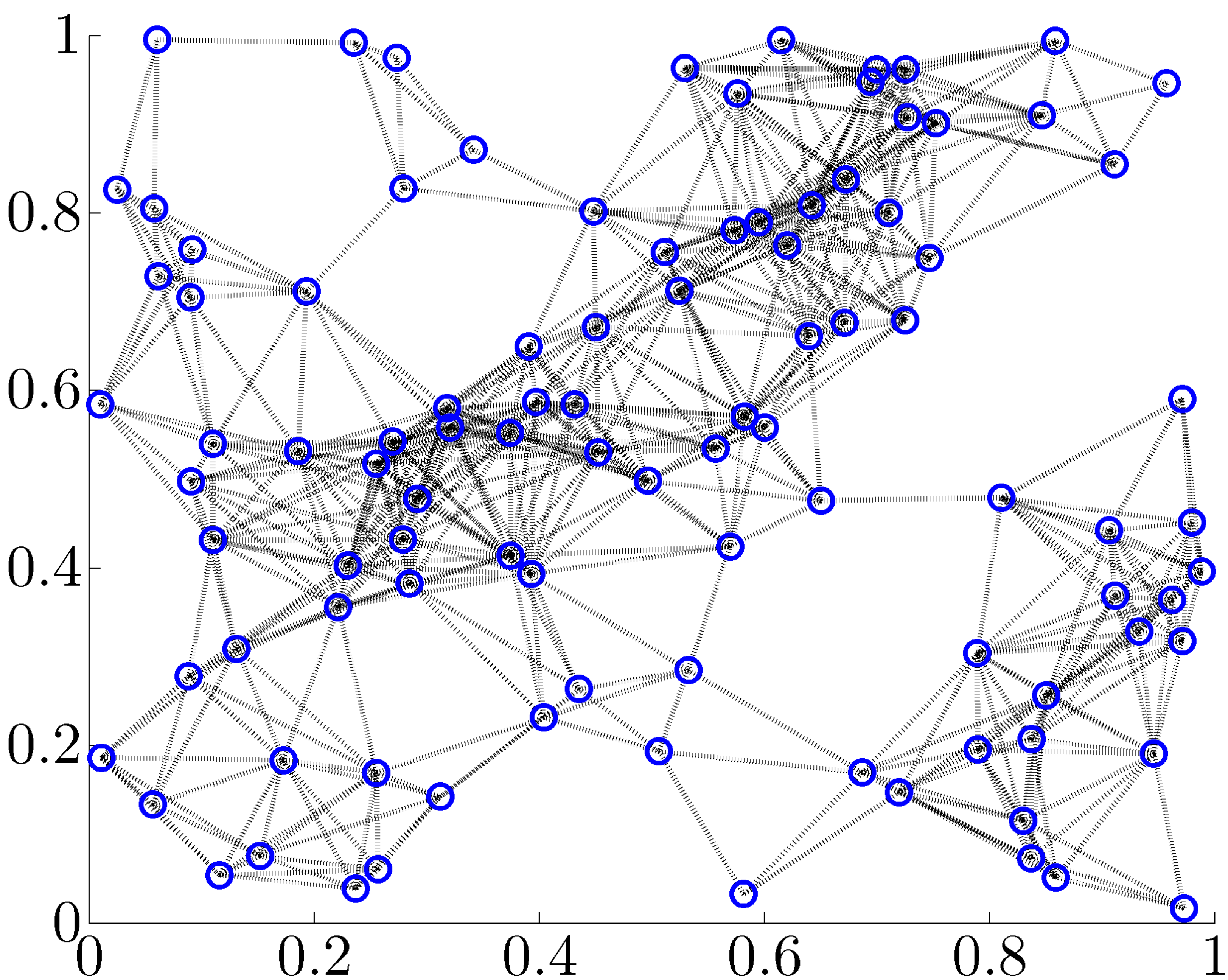
$$\left(L_{\rm s} \;+\; \alpha\, {\rm e}_i\, {\rm e}_i^T\right)^{-1}$$

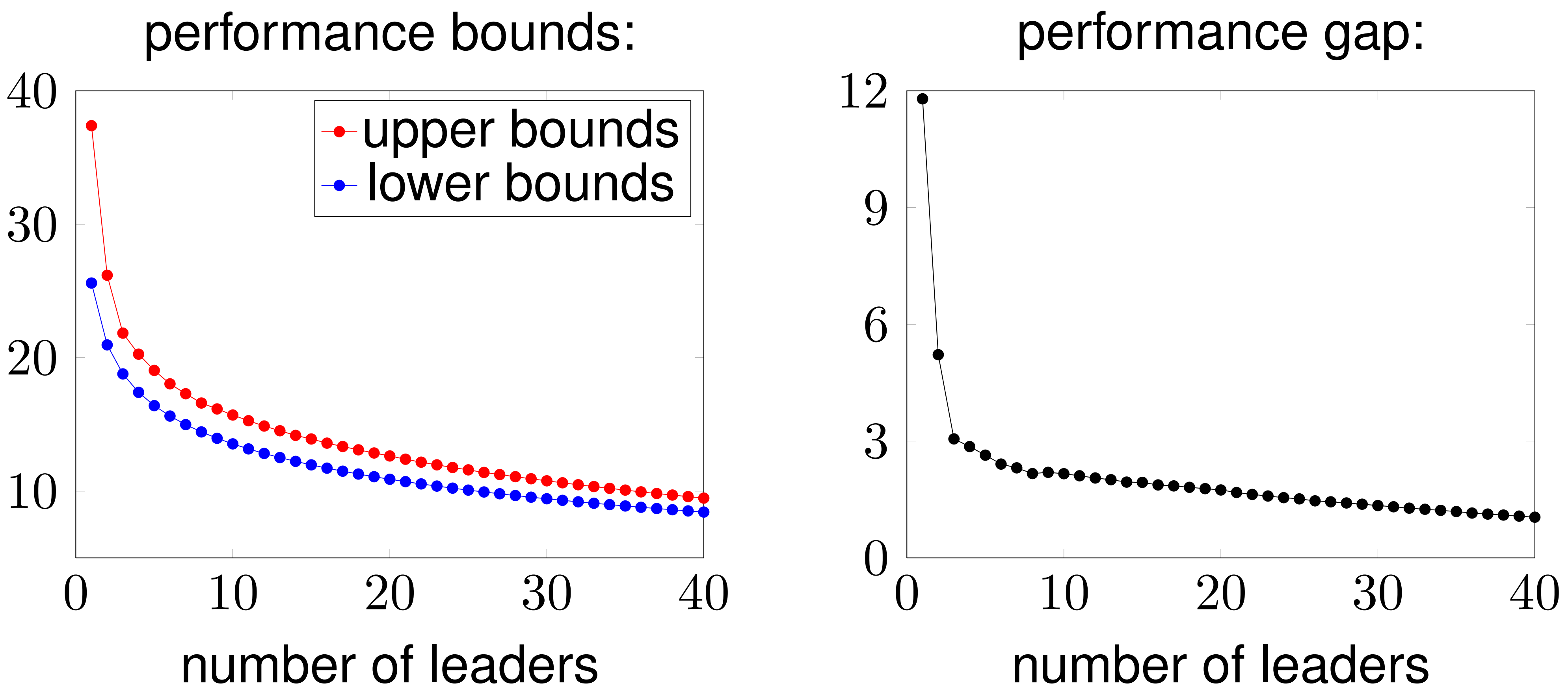
followed by swap between a leader i and a follower j

$$\left(\bar{L}_{\rm s} \;-\; \alpha\, {\rm e}_i\, {\rm e}_i^T \;+\; \alpha\, {\rm e}_j\, {\rm e}_j^T\right)^{-1}$$

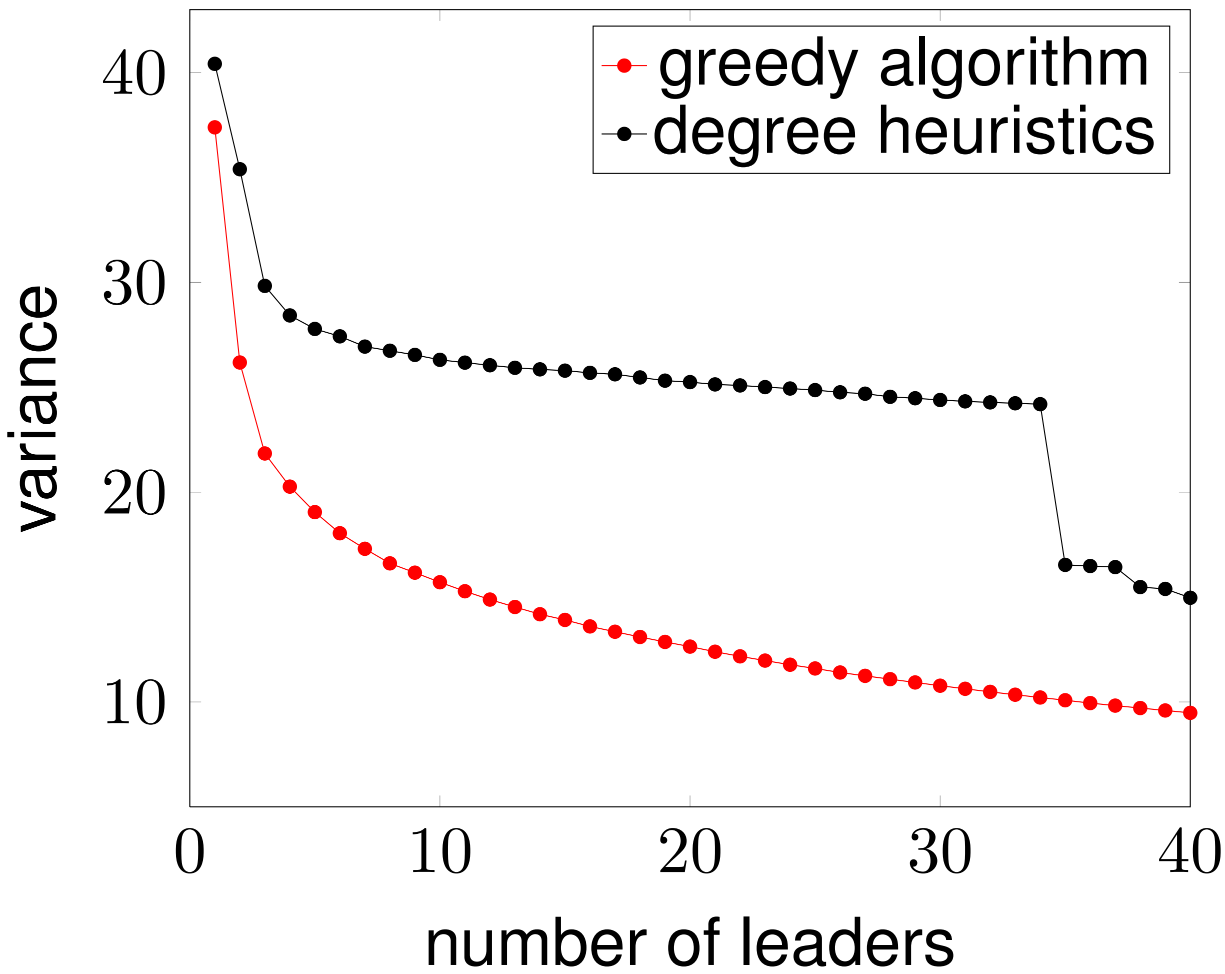
- COMPLEXITY
  - ⋆ rank-1 update: $O(n^2)$ per leader
  - ⋆ single matrix inversion: $O(n^3)$
  - ⋆ rank-2 update: $O(n^2)$ per swap

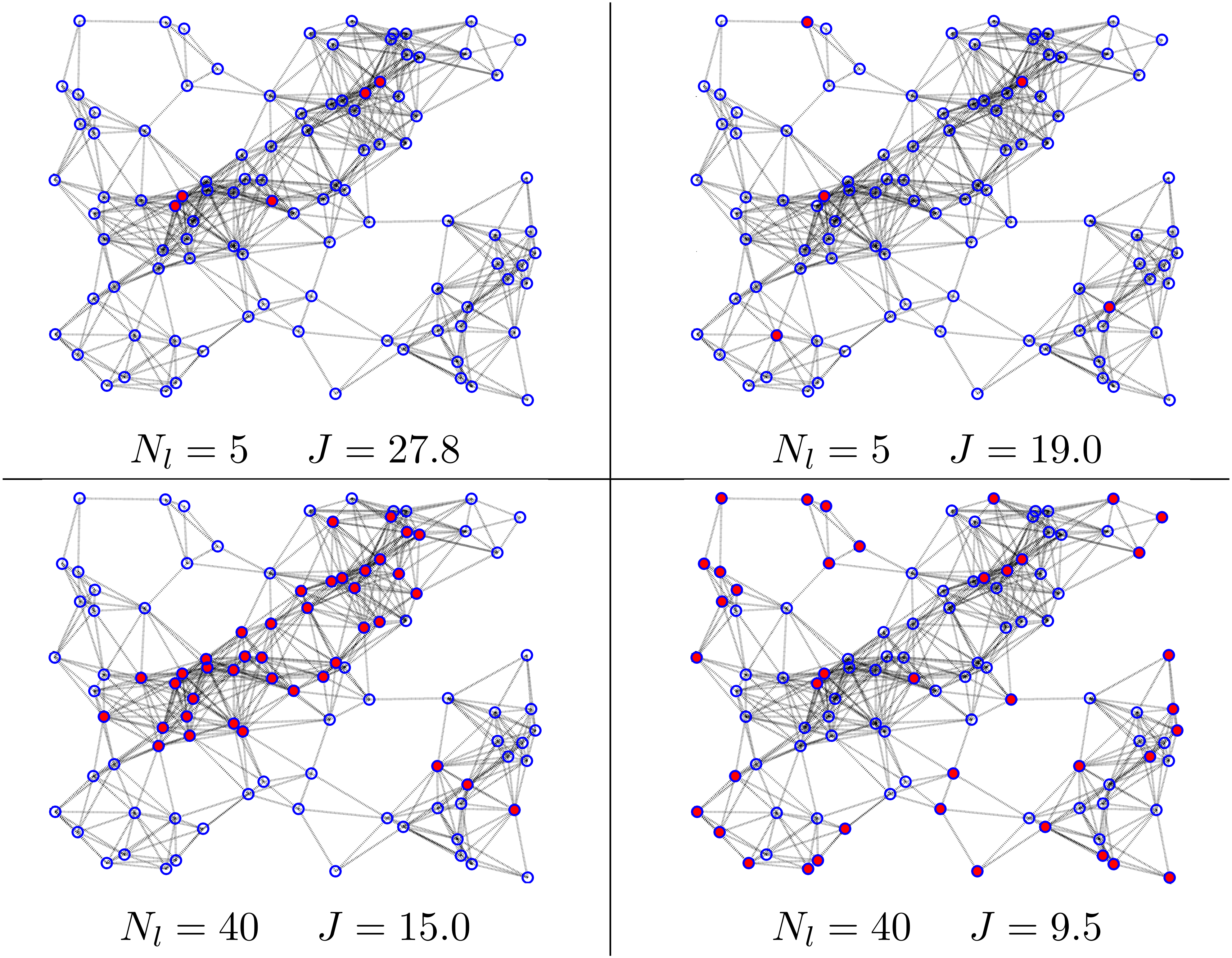
Example: a random network with 100 nodes




performance bounds:
upper bounds
lower bounds
40
30
20
10
0
10
20
30
40
number of leaders
performance gap:
12
9
6
3
0
0
10
20
30
40
number of leaders

Degree heuristics vs. greedy algorithm

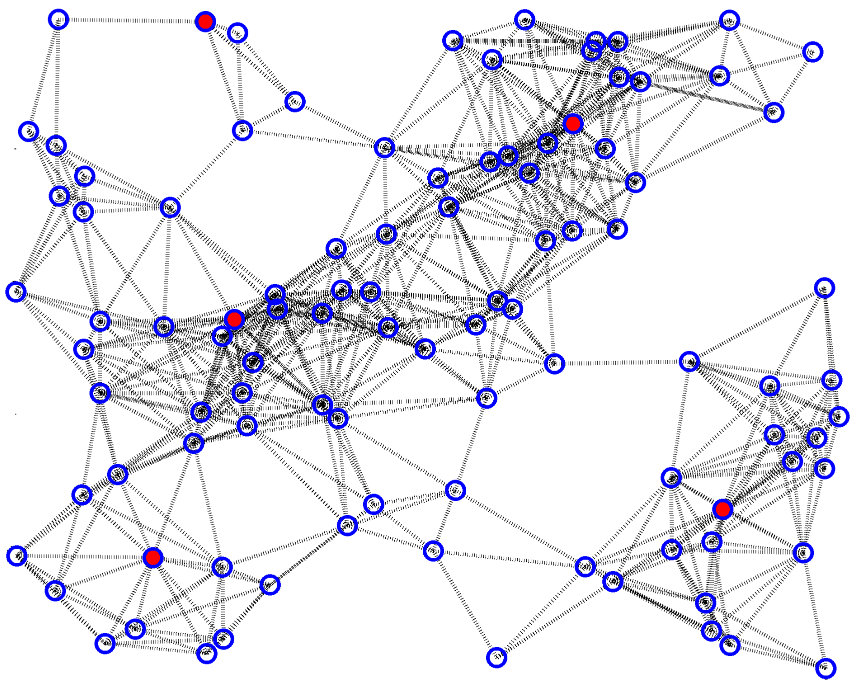




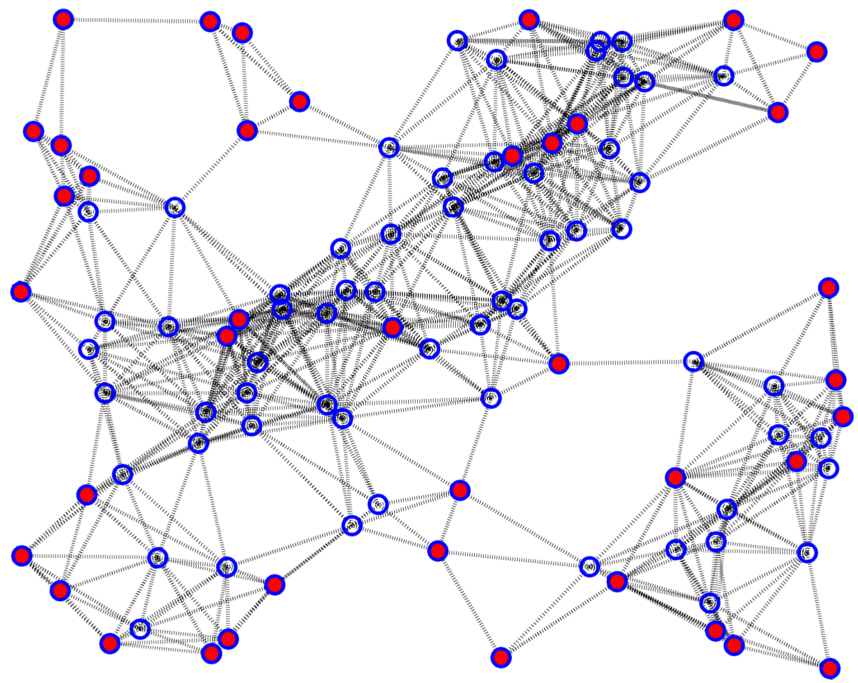
$N_l = 5 \quad J = 27.8$
$N_l = 5 \quad J = 19.0$
$N_l = 40 \quad J = 15.0$
$N_l = 40 \quad J = 9.5$

Few vs many leaders

partition graph and spread leaders:



boundary nodes with low-degree



Recap

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \text{trace}\left((L + \alpha \, \text{diag}\,(x))^{-1}\right) \\ \text{subject to} & x_i \in \{0,1\}, \quad i = 1,\ldots,n \\ & \mathbb{1}^T x = N_l \end{array}$$

- CONVEX RELAXATION: LOWER BOUND
  - ⋆ standard SDP solvers: $O\big(n^4\big)$
  - ⋆ customized interior point method: $O\big(n^3\big)$
- GREEDY ALGORITHM: UPPER BOUND
  - ⋆ without exploiting structure: $O\big(n^4 N_l\big)$
  - ⋆ low rank updates: $O\big(n^3\big)$
- PAPER/SOFTWARE
  - ⋆ arXiv:1302.0450
  - ⋆ www.umn.edu/∼mihailo/software/leaders/

Acknowledgments

TEAM:

Makan Fardad
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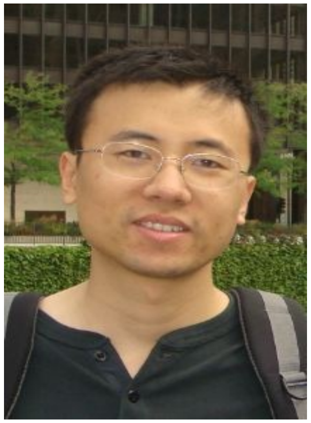

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