Linear State-Space Equations

$$\begin{bmatrix} \dot{x}_1(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \underbrace{\begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix}}_{B} \begin{bmatrix} u_1(t) \\ \vdots \\ u_p(t) \end{bmatrix}$$

$x_1(t), \dots, x_n(t)$ are state-variables

$u_1(t), \dots, u_p(t)$ are system inputs

We can define $\underline{x}(t) := \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ $\quad \underline{u}(t) := \begin{bmatrix} u_1(t) \\ \vdots \\ u_p(t) \end{bmatrix}$

$\dot{\underline{x}}(t) = A\,\underline{x}(t) + B\,\underline{u}(t)$ $\qquad$ (State-Equations)

State equations are <u>n linear coupled 1st order</u> diff. eq'ns

The outputs are linear combinations of the state-variables and the inputs.

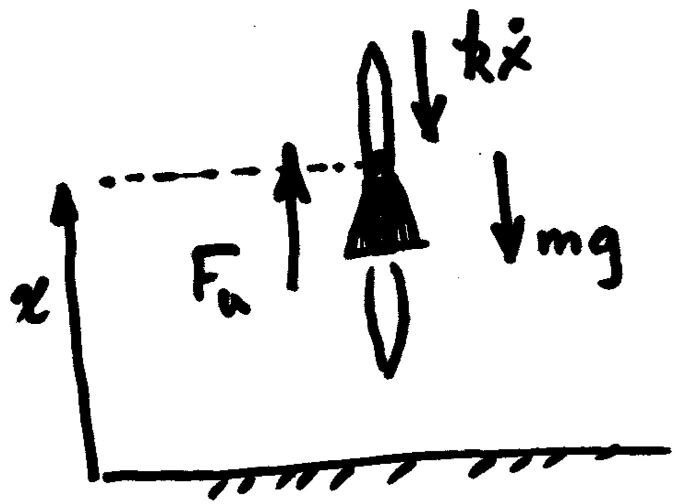
$$\underbrace{\begin{bmatrix} y_1(t) \\ \vdots \\ y_q(t) \end{bmatrix}}_{\underline{y}(t)} = \underbrace{\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{q1} & \cdots & c_{qn} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}}_{\underline{x}} + \underbrace{\begin{bmatrix} d_{11} & \cdots & d_{1p} \\ \vdots & & \vdots \\ d_{q1} & \cdots & d_{qp} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} u_1(t) \\ \vdots \\ u_p(t) \end{bmatrix}}_{\underline{u}}$$

$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t) \qquad \text{(Output- Equations)}$$

Together, state equations & output equations are the state-space description of the system

$$\boxed{\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t) \end{aligned}}$$

Example 1 (Rocket)



F_u Thrust force

mg Weight of rocket

$k\dot{x}$ Air friction force

Newton's law: $F_u - mg - k\dot{x} = m\ddot{x}$

or $m\ddot{x} + k\dot{x} = (F_u - mg)$ 2nd order linear diff. eq'n

dividing by m $\ddot{x} + \frac{k}{m}\dot{x} = \frac{F_u}{m} - g$

Let $x_1 := x$ (Rocket position)

$x_2 := \dot{x}$ (Rocket velocity)

$$\dot{x}_1(t) = x_2(t)$$

$$\begin{aligned}\dot{x}_2(t) &= \ddot{x}(t) \\ &= -\frac{k}{m}\dot{x}(t) + \frac{F_u}{m} - g \\ &= -\frac{k}{m}x_2(t) + \frac{F_u}{m} - g\end{aligned}$$

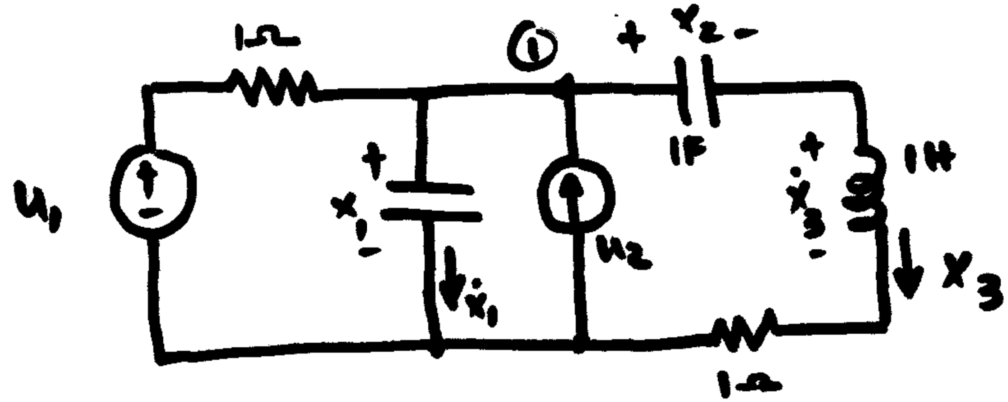
In matrix form:

$$\underbrace{\begin{bmatrix}\dot{x}_1(t) \\ \dot{x}_2(t)\end{bmatrix}}_{\dot{\underline{x}}(t)} = \underbrace{\begin{bmatrix}0 & 1 \\ 0 & -\frac{k}{m}\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1(t) \\ x_2(t)\end{bmatrix}}_{\underline{x}} + \underbrace{\begin{bmatrix}0 \\ 1\end{bmatrix}}_{B}\underbrace{\left(\frac{F_u}{m} - g\right)}_{\underline{u}(t)}$$

$$y(t) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1(t) \\ x_2(t)\end{bmatrix}$$ if <u>velocity</u> is output of interest

or $$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1(t) \\ x_2(t)\end{bmatrix}$$ if position is output of interest

Example 2 (Electric Circuit)



Use: Inductor currents
Capacitor voltages
as state-variables

Summing current at node 1

$$(u_1 - x_1) - \dot{x}_1 + u_2 - x_3 = 0 \quad \cdots \quad (\mathrm{I})$$

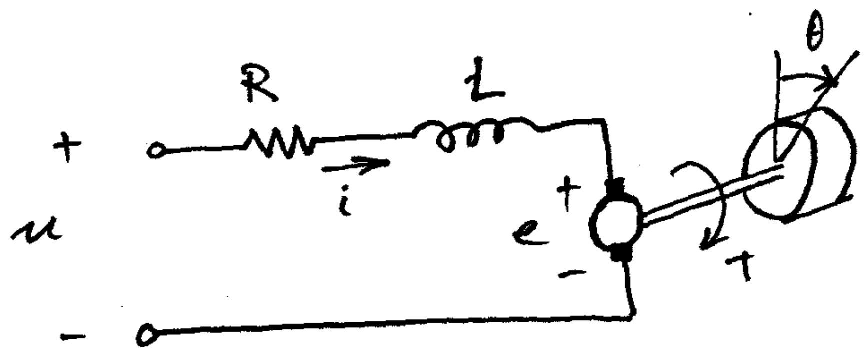
$$\dot{x}_2 = x_3 \quad \cdots \quad (\mathrm{II})$$

$$\dot{x}_3 + x_3 - x_1 + x_2 = 0 \quad \cdots \quad (\mathrm{III})$$

In matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Example 3 (Electric Motor)



$$u = Ri + L\frac{di}{dt} + e \implies \frac{di}{dt} = -\frac{R}{L}i - \frac{e}{L} + \frac{u}{L}$$

$$\left.\begin{aligned} T &= I\ddot{\theta} \\ T &= k_1 i \end{aligned}\right\} \implies \ddot{\theta} = \frac{k_1 i}{I}$$

$$e = k_2\dot{\theta}$$

Let $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = i$. Then

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{k_1}{I}x_3$$

$$\dot{x}_3 = -\frac{R}{L}x_3 - \frac{1}{L}k_2 x_2 + \frac{u}{L}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & k_1/I \\ 0 & -\frac{k_2}{L} & -\frac{R}{L} \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix}}_{B} u$$

If $y = \dot{\theta}$

$$y = \underbrace{[0 \quad 1 \quad 0]}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underset{\sim}{0}\, u$$

Example 4

$$\dddot{y}(t) + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = \tilde{u}(t)$$

Let

$$x_1 := y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$

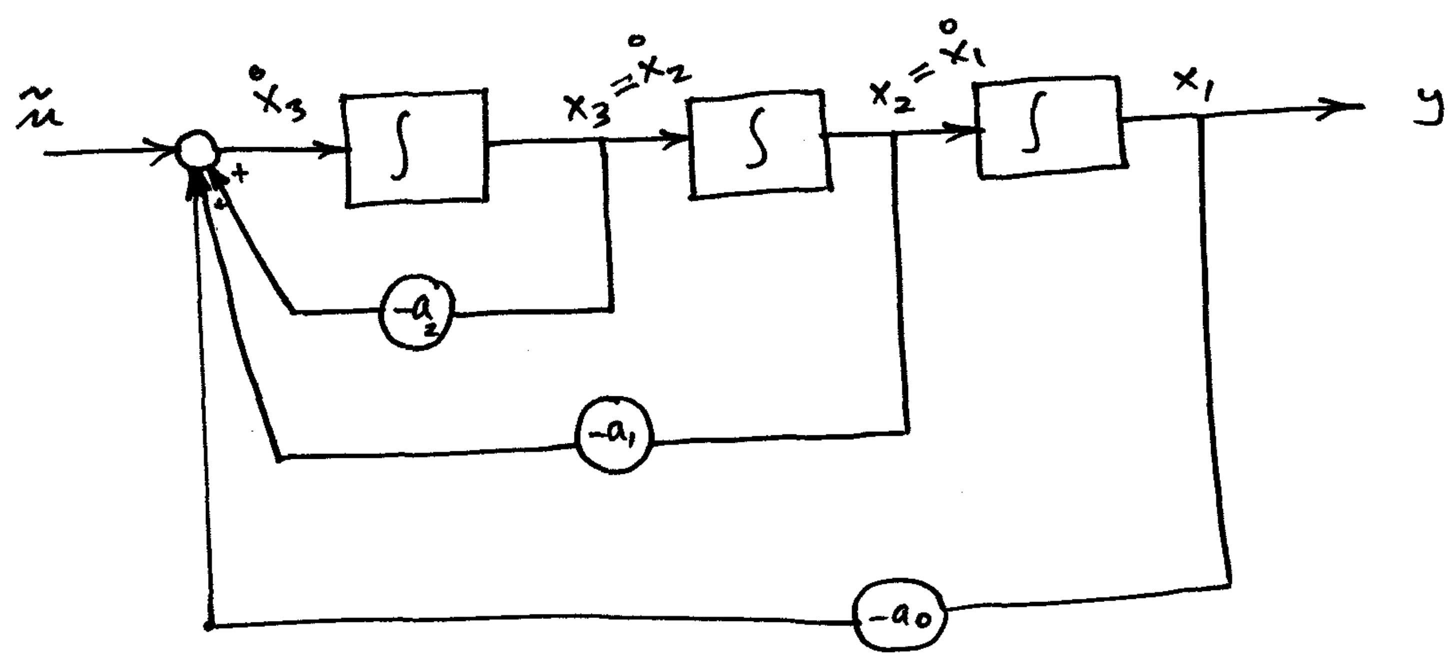
Then

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = \dddot{y}(t) = -a_2 \ddot{y}(t) - a_1 \dot{y}(t) - a_0 y(t) + \tilde{u}(t)$$
$$= -a_2 x_3 - a_1 x_2 - a_0 x_1 + \tilde{u}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{B} \tilde{u}$$

$$y = \overbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}^{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \overbrace{0}^{D} \tilde{u}$$

Analog Computer Simulation



Example 5 :

$$y(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} \, u(s)$$

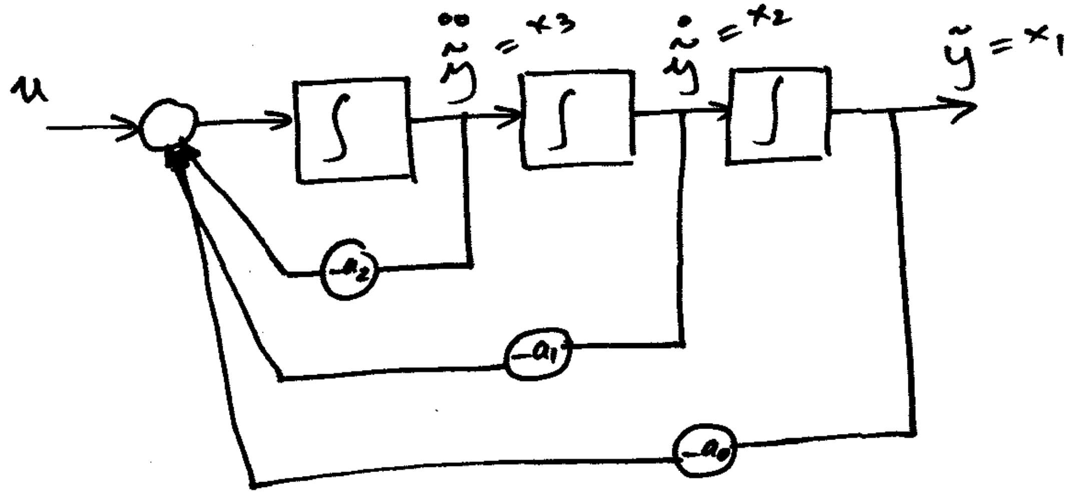
Or equivalently, $\dddot{y}(t) + a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_2 \ddot{u}(t) + b_1 \dot{u}(t) + b_0 u(t)$

To get a state-space representation, we solve the problem in 2 steps:

1. Letting $\tilde{y}(s) := \dfrac{u(s)}{s^3 + a_2 s^2 + a_1 s + a_0}$, we can obtain a state-space description with $\tilde{y}(s)$ as the output:

$$(s^3 + a_2 s^2 + a_1 s + a_0)\tilde{y}(s) = u(s) \qquad \text{or}$$

$$\dddot{\tilde{y}}(t) + a_2 \ddot{\tilde{y}}(t) + a_1 \dot{\tilde{y}}(t) + a_0 \tilde{y}(t) = u(t)$$



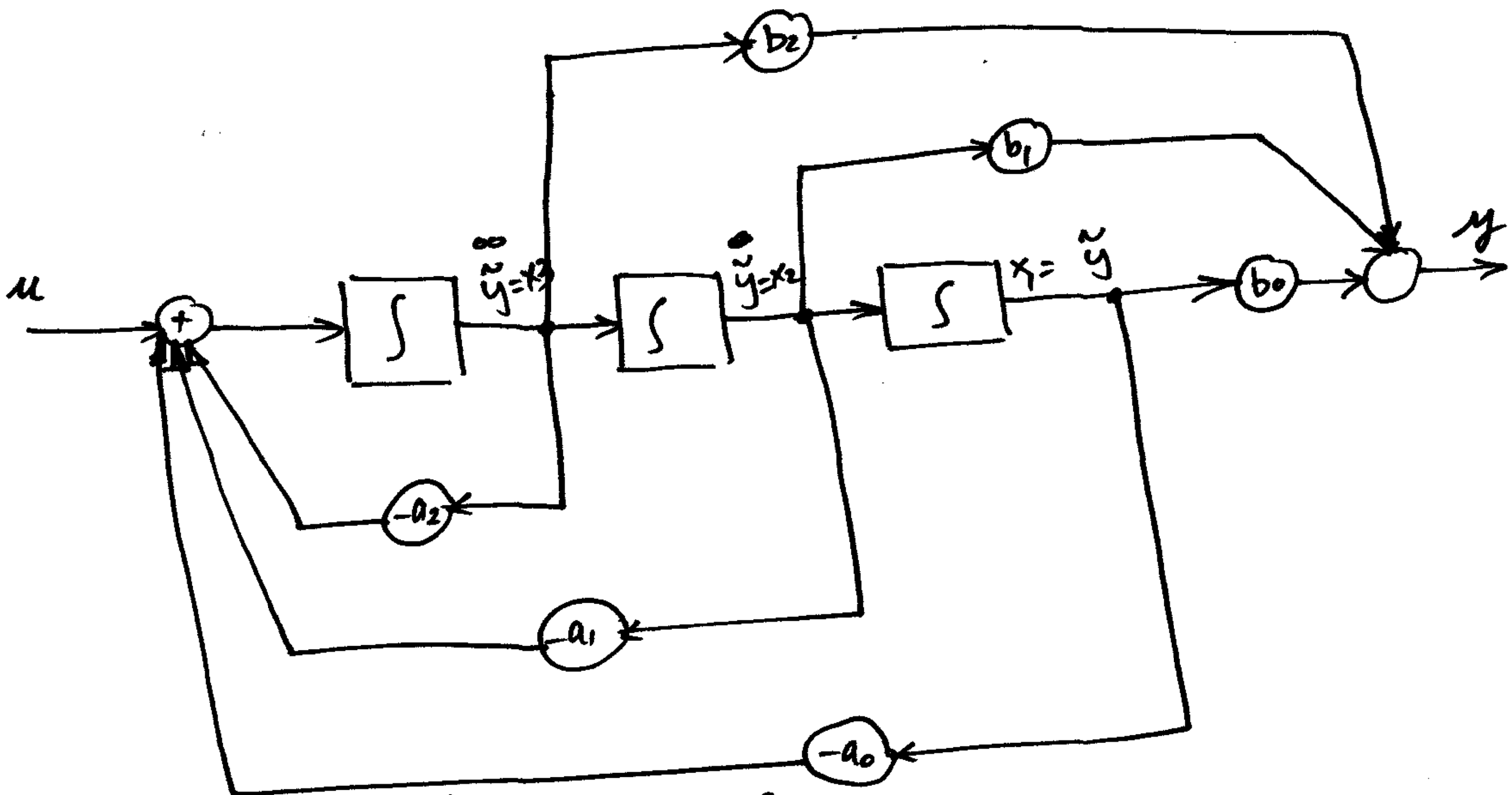
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

(2) $y(s) = (b_2 s^2 + b_1 s + b_0)\tilde{y}(s)$ or

$$y(t) = b_2 \ddot{\tilde{y}}(t) + b_1 \dot{\tilde{y}}(t) + b_0 \tilde{y}(t)$$

$$= b_2 x_3 + b_1 x_2 + b_0 x_1$$

$$y(t) = \underbrace{\begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}^{B} u$$

$$y = \underbrace{\begin{bmatrix} b_0 & b_1 & b_2 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{[0]}_{D} u$$

Linear Systems as approximations to nonlinear Systems

Given:

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(x_1, \dots, x_n, u_1, \dots, u_p) \\ \vdots \\ f_n(x_1, \dots, x_n, u_1, \dots, u_p) \end{bmatrix} \quad \text{(State eq's)}$$

$$\begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix} = \begin{bmatrix} g_1(x_1, \dots, x_n, u_1, \dots, u_p) \\ \vdots \\ g_q = (x_1, \dots, x_n, u_1, \dots, u_p) \end{bmatrix} \quad \text{(Output eq's)}$$

Or

$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{u}) \qquad \text{(State eq'n)}$$

$$\underline{y} = \underline{g}(\underline{x}, \underline{u}) \qquad \text{(Output eq'n)}$$

$(\underline{x}_0^{\in \mathbb{R}}, \underline{u}_0^{\in \mathbb{R}})$ is an equilibrium solution if

$$\underline{f}(\underline{x}_0, \underline{u}_0) = 0$$

In this case $\underline{x}(t) \equiv \underline{x}_0$ and $\underline{u}(t) \equiv u_0$ solve the differential equation (Constant Solution)

We can look at solutions close the equilibrium

Let $x(t) := x_0 + \delta x(t)$

$u(t) := u_0 + \delta u(t)$

$$\mathring{x}_i(t) = \delta\dot{x}_i = f_i(\underline{x}, \underline{u})$$

$$= \underbrace{\cancel{f_i(\underline{x}_0, \underline{u}_0)}}_{0} + \left.\frac{\partial f_i}{\partial x_1}\right|_{\substack{x_0 \\ u_0}} \delta x_1 + \cdots + \left.\frac{\partial f_i}{\partial x_n}\right|_{\substack{x_0 \\ u_0}} \delta x_n$$

$$+ \left.\frac{\partial f_i}{\partial u_1}\right|_{\substack{x_0 \\ u_0}} \delta u_1 + \cdots + \left.\frac{\partial f_i}{\partial u_p}\right|_{\substack{x_0 \\ u_0}} \delta u_p$$

$+$ higher order terms

Similarly for each output y_i . . .

In matrix form:

$$\begin{bmatrix} \delta\dot{x}_1 \\ \vdots \\ \delta\dot{x}_n \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}}_{A} \Bigg|_{\substack{x_0 \\ u_0}} \begin{bmatrix} \delta x_1 \\ \vdots \\ \delta x_n \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \vdots & & \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_p} \end{bmatrix}}_{B} \Bigg|_{\substack{x_0 \\ u_0}} \begin{bmatrix} \delta u_1 \\ \vdots \\ \delta u_p \end{bmatrix}$$

+ higher order terms

Ignoring H.O.T. we have

$$\boxed{\begin{array}{ll} \delta\dot{\underline{x}} \cong A\,\underline{\delta x} + B\,\underline{\delta u} & \text{(Linearized Input eq's)} \\ \underline{\delta y} \cong C\,\underline{\delta x} + D\,\underline{\delta u} & \text{(Linearized Output eq's)} \end{array}}$$

Input - Output System Description

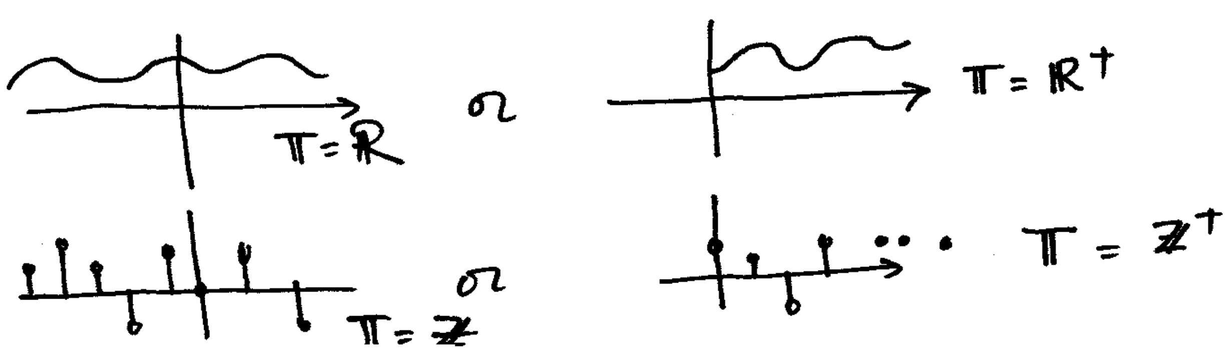
$$\underline{u} \longrightarrow \boxed{H} \longrightarrow \underline{y}$$

A system can also be described by its input - output map

$$\underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix} , \quad \underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix}$$

$$u_i : \mathbb{T} \longrightarrow \mathbb{R}$$

where $\mathbb{T}$ is typically $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, or $\mathbb{Z}^+$



Let us consider $\mathbb{T} = \mathbb{R}$ i.e. the time interval is $(-\infty, \infty)$

<u>Definition</u>: A system described by the mapping H is said to be <u>linear</u> if $H(\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2) = \alpha_1 H(\underline{u}_1) + \alpha_2 H(\underline{u}_2)$

for all $\underline{u}_1$, $\underline{u}_2$, α_1, and α_2.

$\boxed{\therefore \text{ The superposition principle applies}}$

The linearity condition:

$$H(\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2) = \alpha_1 H(\underline{u}_1) + \alpha_2 H(\underline{u}_2) \qquad \forall\, \underline{u}_1, \underline{u}_2 \quad \forall\, \alpha_1, \alpha_2 \in \mathbb{R}$$

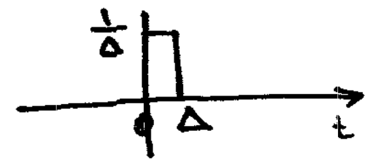
is equivalent to:

1. $H(\alpha \underline{u}) = \alpha H(\underline{u}) \qquad \forall\, \underline{u},\ \forall\, \alpha \in \mathbb{R}$ (Homogeneity)

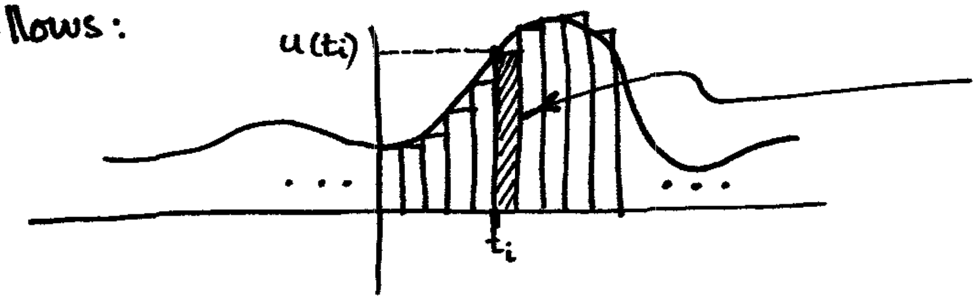
and 2. $H(\underline{u}_1 + \underline{u}_2) = H(\underline{u}_1) + H(\underline{u}_2) \qquad \forall\, \underline{u}_1, \underline{u}_2$ (Additivity)

We can take advantage of the linearity property to <u>calculate</u> <u>the</u> <u>output</u> of a given linear system <u>for</u> <u>any</u> <u>input</u>.

- First, define $\delta_\Delta(t)$ to be the pulse:



- A given input can be approximated by a sequence of pulses as follows:



$\delta_\Delta(t-t_i) \cdot \Delta \cdot u(t_i)$

$$u(t) \simeq \sum_i u(t_i)\, \delta_\Delta(t-t_i)\, \Delta$$

$$y = Hu \simeq H\Big(\sum_i u(t_i)\,\delta_\Delta(\bullet - t_i)\,\Delta\Big)$$

$$\overset{\text{by}}{\underset{\text{linearity}}{=}} \sum_i u(t_i)\,\underbrace{H(\delta_\Delta(\bullet - t_i))}_{g_\Delta(\bullet,\,t_i)}\,\Delta$$

$$y(t) \simeq \sum_i u(t_i)\, g_\Delta(t, t_i)\,\Delta$$

Taking the limit as $\Delta \to 0$

$$\boxed{y(t) = \int_{-\infty}^{+\infty} u(\tau)\, g(t,\tau)\, d\tau}$$

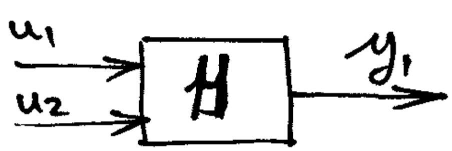
where $g(t,\tau) := \big(H\,\delta(\bullet - \tau)\big)(t)$

i.e. $g(t,\tau) =$ the output at time t when the input is δ function applied at time τ.

For a linear system H, $\boxed{y(t) = \int_{-\infty}^{\infty} g(t,\tau)u(\tau)\,d\tau}$

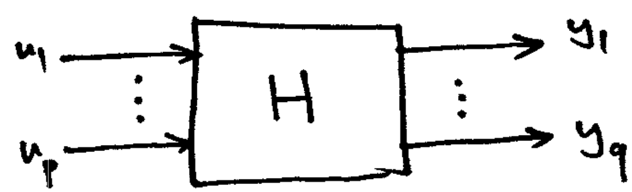
What about the output for <u>MIMO</u> linear system?

2 input, 1 output



By linearity,
$$y_1 = H\left(\begin{bmatrix} u_1 \\ 0 \end{bmatrix}\right) + H\left(\begin{bmatrix} 0 \\ u_2 \end{bmatrix}\right)$$
$$= H_{11}u_1 + H_{12}u_2$$
$$= \int_{-\infty}^{\infty} g_{11}(t,\tau)u_1(\tau)\,d\tau + \int_{-\infty}^{\infty} g_{12}(t,\tau)u_2(\tau)\,d\tau$$

For a general MIMO system



$$\left.\begin{aligned} y_1(t) &= \int_{-\infty}^{\infty} g_{11}(t,\tau)u_1(\tau)\,d\tau + \cdots + \int_{-\infty}^{\infty} g_{1p}(t,\tau)u_p(\tau)\,d\tau \\ &\ \vdots \\ y_q(t) &= \int_{-\infty}^{\infty} g_{q1}(t,\tau)u_1(\tau)\,d\tau + \cdots + \int_{-\infty}^{\infty} g_{qp}(t,\tau)u_p(\tau)\,d\tau \end{aligned}\right\}$$

In matrix form:

$$\boxed{\underset{q\times 1}{\underline{y}(t)} = \int_{-\infty}^{\infty} \underset{\cancel{p\times q}\ q\times p}{G(t,\tau)}\ \underset{p\times 1}{\underline{u}(\tau)}\,d\tau}$$

"Input–Output Description for the linear system"

where $$G(t,\tau) = \begin{bmatrix} g_{11}(t,\tau) & \cdots & g_{1p}(t,\tau) \\ \vdots & & \\ g_{q1}(t,\tau) & \cdots & g_{qp}(t,\tau) \end{bmatrix}$$

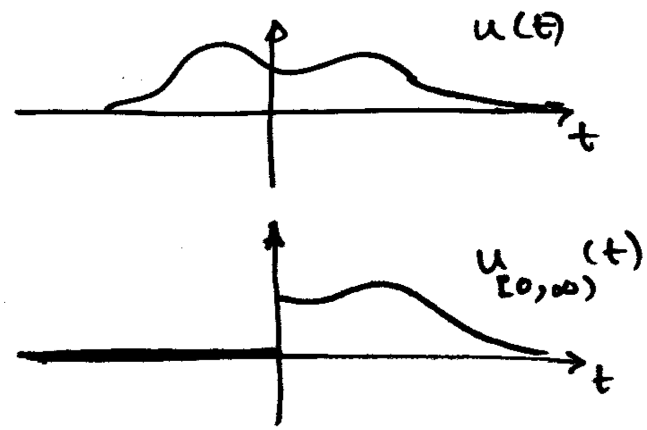
Causality

To introduce causality we first make a definition:

Def: Given a signal $u(t)$ defined over $(-\infty, \infty)$ the signal $u_{[a,b]}(t)$ is defined as follows

$$u_{[a,b]}(t) := \begin{cases} u(t) & a \le t \le b \\ 0 & \text{elsewhere} \end{cases}$$

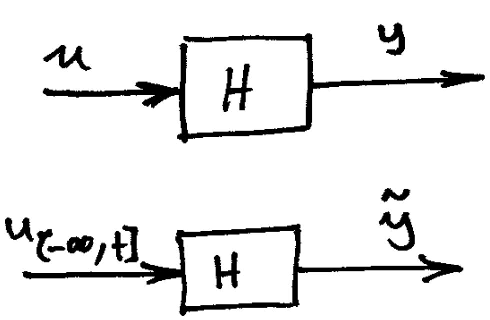
Example:



Definition (Causality)

A system described by the mapping H is said to be causal if
$$\left(H\, u_{(-\infty,t]} \right)_{(-\infty,t]} = \left(H u \right)_{(-\infty,t]}$$
for all u and for all $t \in \mathbb{R}$.

Suppose



If the system is causal; then
$$y_{(-\infty,t]} = \tilde{y}_{(-\infty,t]}$$

i.e. system cannot look a-head!

For a linear system, $y(t) = \int_{-\infty}^{+\infty} g(t,\tau)\, u(\tau)\, d\tau$

where $g(t,\tau)$ is the response to an impulse applied at time τ.

If the linear system is <u>causal</u>, $g(t,\tau) = 0$ for $t < \tau$.

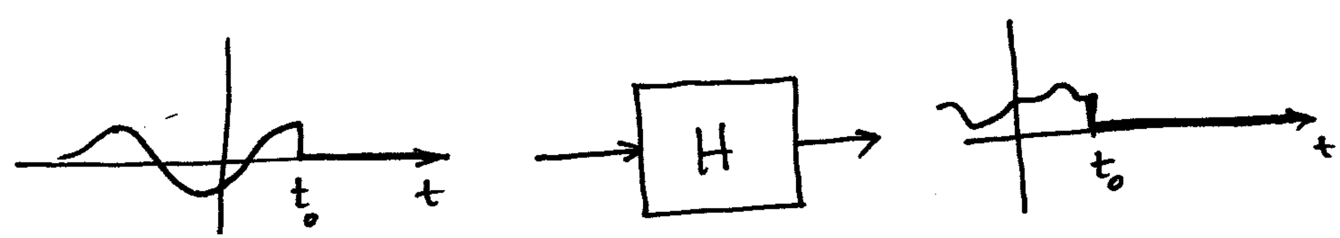
Thus, for a ① <u>causal</u> ② <u>linear system</u>, the output is related to the input by:

$$\boxed{y(t) = \int_{-\infty}^{t} g(t,\tau)\, u(\tau)\, d\tau}$$

Relaxedness:

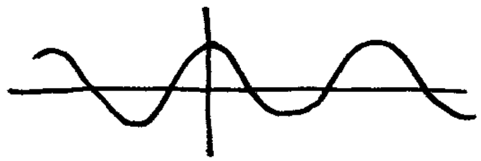
Definition: Let u be an input applied to a linear system modeled by H. The system is said to be relaxed at t_0 if $\left(H u_{(-\infty, t_0]}\right)(t) = 0 \quad \forall\, t \geq t_0$.

e.g.

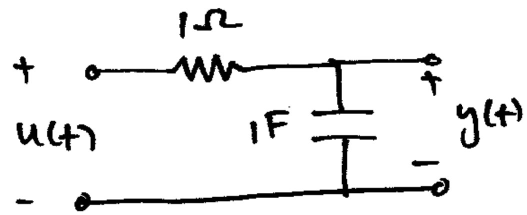


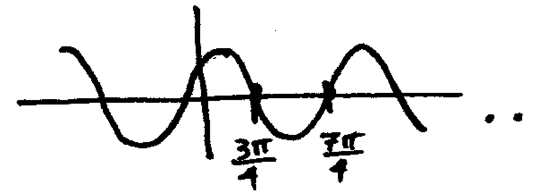
Remark: Relaxedness at time t_0 is a function of the input that had been applied prior to time t_0.

Example



$u(t) = \cos t$





$y(t) = \cos(t - \frac{\pi}{4}) \cdot \frac{1}{\sqrt{2}}$

The circuit is relaxed at $t = 3\pi/4$, $\frac{7\pi}{4}$, ...

Reason: A these times the output is zero. The capacitor has no charge at these times and if the input is set to zero (and maintained at zero) at the exact time the output is zero, the output will remain zero.

For a linear system

$$\underline{y}(t) = \int_{-\infty}^{\infty} G(t,\tau)\,\underline{u}(\tau)\,d\tau$$

$$= \underbrace{\int_{-\infty}^{t_0} G(t,\tau)\,\underline{u}(\tau)\,d\tau}_{\substack{\text{If the system is relaxed} \\ \text{at } t=t_0\text{, this term is} \\ \text{zero for } t \ge t_0}} + \int_{t_0}^{\infty} G(t,\tau)\,\underline{u}(\tau)\,d\tau$$

Therefore, for a linear system which is <u>relaxed at t_0</u>

$$\underline{y}(t) = \int_{t_0}^{\infty} G(t,\tau)\,\underline{u}(\tau)\,d\tau \qquad t \ge t_0$$

$$\underline{y}(t) = \int_{t_0}^{t} G(t,\tau)\,\underline{u}(\tau)\,d\tau$$ causal linear system which is relaxed at time t_0

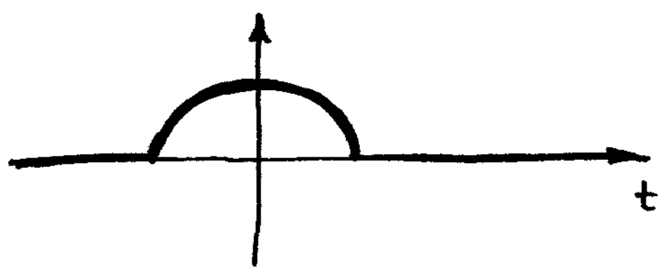
Time Invariance

We start by defining the shift-by-α operator (mapping), denoted by Q_α.

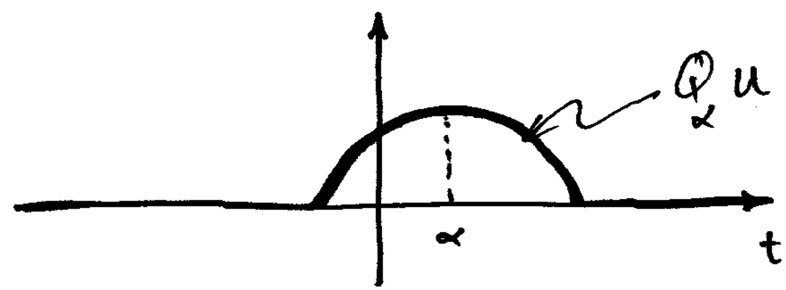
<u>Definition</u>: The shift-by-α map ($\alpha \in \mathbb{R}$) is defined by

$$(Q_\alpha u)(t) = u(t-\alpha) \qquad \forall t.$$

Example:



$u(\cdot)$



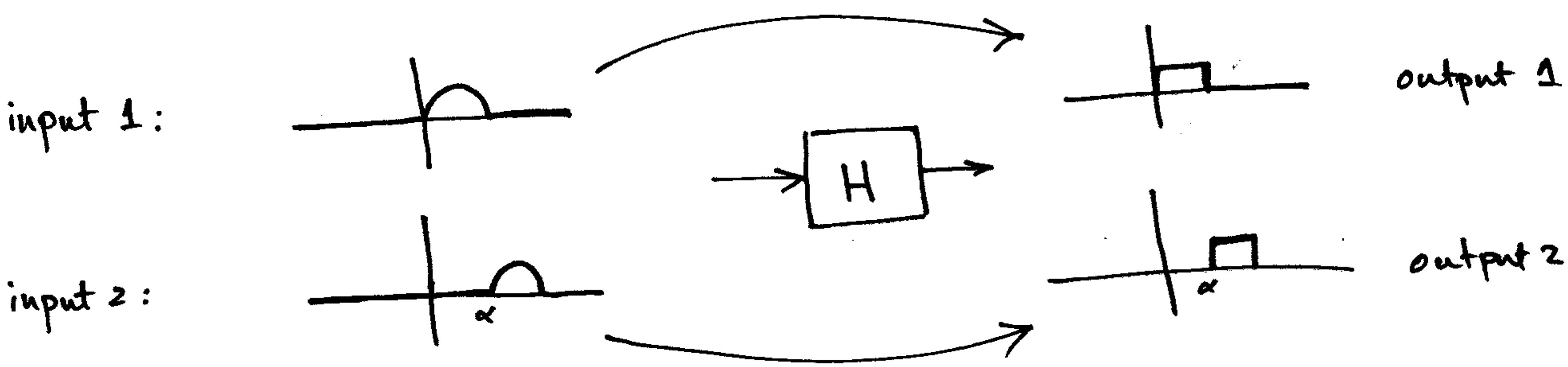
$u(\cdot-\alpha)$

<u>Definition</u> (Time-Invariance):

A system modeled by the input output mapping H is said to be <u>time-invariant</u> if

$$Q_\alpha(Hu) = H(Q_\alpha u) \qquad \forall\ \alpha \in \mathbb{R},\ \forall\ u$$

<u>Example:</u>



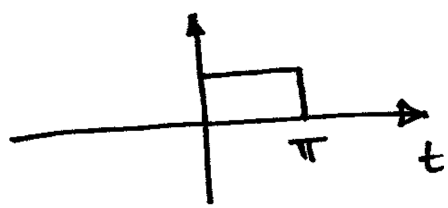
$$Q_\alpha H = H Q_\alpha$$

<u>Def:</u> A system which is not time-invariant is said to be <u>time-varying</u>.

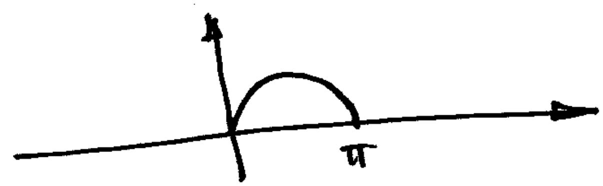
<u>Example</u>: A System's I/O mapping is given by

$$y(t) = \sin(t) \cdot u(t) \qquad \forall t$$

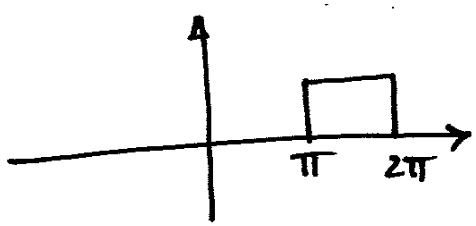
This system is time-varying. In fact



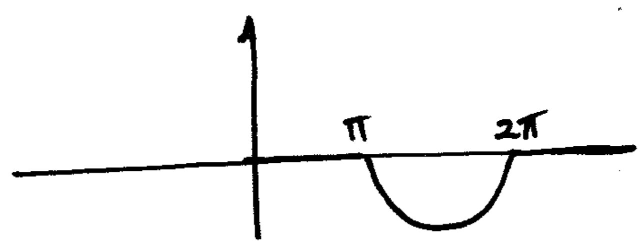
$u(\cdot)$



$y(\cdot)$



$Q_\pi u$



$\neq \; Q_\pi y$

Given a <u>linear</u> system modeled by H. Then

$$(Hu)(t) = \int_{-\infty}^{\infty} g(t,\tau)\, u(\tau)\, d\tau$$

<u>Question:</u> If H is also assumed to be time-invariant (in addition to linearity), what structure will time-invariance impose on $g(t,\tau)$?

<u>Answer:</u>
By time invariance, $g(t+\alpha, \tau+\alpha) = g(t,\tau)$ for all α, for all t,τ

Equivalently, $Q_\alpha\, g(\cdot,\tau) = g(\cdot, \tau+\alpha)$

As a result, for any given t, τ letting $\alpha = -\tau$ we have $g(t,\tau) = g(t+\alpha, \tau+\alpha) = g(t-\tau, 0)$

<u>Summarizing:</u>

For a linear time invariant system

$$(Hu)(t) = \int_{-\infty}^{\infty} \tilde{g}(t-\tau)u(\tau)\,d\tau \qquad \forall t$$

where $\tilde{g}(\cdot) := g(\cdot, 0)$ = impulse response

If the system is, in addition <u>causal</u> and is <u>relaxed at $t = t_0$</u>, then

$$(Hu)(t) = \int_{t_0}^{t} \tilde{g}(t-\tau)u(\tau)\,d\tau \qquad t \geq t_0$$

Linear Time-Invariant Systems in the Frequency Domain

The output of an Linear Time-Invariant System (LTI) which is relaxed at $t=0$ is given by

$$y(t) = \int_0^\infty \tilde{g}(t-\tau)\, u(\tau)\, d\tau \qquad t \geq 0$$

Taking the Laplace Transform of both sides, we have:

$$\mathcal{L}(y) = \hat{y} \quad \text{where} \quad \hat{y}(s) = \int_0^\infty y(t)\, e^{-st}\, dt$$

$$= \int_0^\infty \underbrace{\int_0^\infty \tilde{g}(t-\tau)\, u(\tau)\, d\tau}_{y(t)}\; e^{-st}\, dt$$

$$= \int_0^\infty \int_0^\infty \tilde{g}(t-\tau)\, e^{-st}\, u(\tau)\, d\tau\, dt$$

$$= \int_0^\infty \int_0^\infty \tilde{g}(t-\tau)\, e^{-st}\, dt \quad u(\tau)\, d\tau$$

$$= \int_0^\infty \int_0^\infty \tilde{g}(t-\tau)\, e^{-s(t-\tau)}\, dt \; e^{-s\tau}\, u(\tau)\, d\tau$$

$$= \int_0^\infty \underbrace{\int_0^\infty \tilde{g}(v)\, e^{-sv}\, dv}_{\hat{\tilde{g}}(s)} \; e^{-s\tau}\, u(\tau)\, d\tau$$

$$= \hat{\tilde{g}}(s) \int_0^\infty e^{-s\tau}\, u(\tau)\, d\tau$$

$$= \hat{\tilde{g}}(s)\, \hat{u}(s)$$

<u>SISO case:</u>

$$\boxed{\hat{y}(s) = \hat{\tilde{g}}(s)\,\hat{u}(s)}$$

$\hat{\tilde{g}}(s)$ is the system <u>transfer</u> <u>function</u>. It has two interpretations:

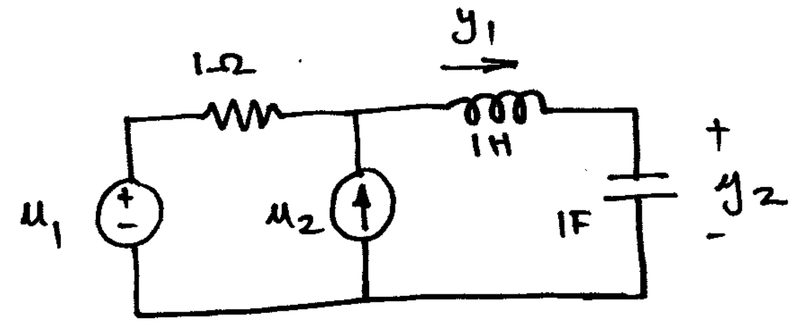
1. $\hat{\tilde{g}}(s)$ is the Laplace transform of the system's impulse response.
2. $\hat{\tilde{g}}(s) = \hat{y}(s)/\hat{u}(s)$ where y is the output corresponding to the input u when the system is relaxed at $t=0$

<u>MIMO case:</u>

$$\boxed{\underline{\hat{y}}(s) = \hat{\tilde{G}}(s)\cdot\underline{\hat{u}}(s)}$$

$\hat{\tilde{G}}(s)$ is the <u>transfer</u> <u>function</u> <u>matrix</u>. It is the Laplace transform of the impulse response matrix.

<u>Example:</u>



$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}$

<u>y_1 component due to u_1 $(u_2=0)$</u> : $\hat{y}_1\Big|_{\hat{u}_2=0} = \dfrac{s}{s^2+s+1}\,\hat{u}_1$

<u>y_1 component due to u_2 $(u_1=0)$</u> : $\hat{y}_1\Big|_{\hat{u}_1=0} = \dfrac{s}{s^2+s+1}\,\hat{u}_2$

<u>y_2 component due to u_1 $(u_2=0)$</u> : $\hat{y}_2\Big|_{\hat{u}_2=0} = \dfrac{1}{s^2+s+1}\,\hat{u}_1$

<u>y_2 component due to u_2 $(u_1=0)$</u> : $\hat{y}_2\Big|_{\hat{u}_1=0} = \dfrac{1}{s^2+s+1}\,u_2$

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{s}{s^2+s+1} & \frac{s}{s^2+s+1} \\ \frac{1}{s^2+s+1} & \frac{1}{s^2+s+1} \end{bmatrix}}_{G(s)} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}$$

The notion of a "state":

Input – Output description described a system only in terms of the relation between the possible inputs and corresponding outputs.

No Internal Describtion was Required!

An internal description requires the notion of a system "state".

Definition: (State of a system)
The state of given system at time t_0 is the amount of information at t_0 which together with the system input for $t \geq t_0$ determines uniquely the response of the system in every aspect.

A very broad class of systems can be modeled by

$$\left.\begin{aligned} \mathring{x}_1 &= f_1(x_1,\dots,x_n,u_1,\dots,u_p,t) \\ &\vdots \\ \mathring{x}_n &= f_n(x_1,\dots,x_n,u_1,\dots,u_p,t) \end{aligned}\right] \Rightarrow \quad \underline{\mathring{x}} = \underline{f}(\underline{x},\underline{u},t)$$

together with

$$\left.\begin{aligned} y_1 &= g_1(x_1,\dots,x_n,u_1,\dots,u_p,t) \\ &\vdots \\ y_q &= g_q(x_1,\dots,x_n,u_1,\dots,u_p,t) \end{aligned}\right] \Rightarrow \quad \underline{y} = \underline{g}(\underline{x},\underline{u},t)$$

where

$$\underline{x} := \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \underline{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix}, \quad \text{and} \quad \underline{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix}$$

We have seen an important special case, where

$$\underline{f}(\underline{x}, \underline{u}, t) = A(t).\underline{x} + B(t).\underline{u}$$

and
$$\underline{g}(\underline{x}, \underline{u}, t) = C(t).\underline{x} + D(t).\underline{u}$$

<u>Fact</u> (Existence & Uniqueness)

Under some mild conditions on $\underline{f}(\cdot,\cdot,\cdot)$, the value of $x(\cdot)$ at t_0 qualifies as the "state" of the system at time t_0, i.e. knowledge of $x(t_0)$ and $u(t)$ for $t \geq t_0$ gives a unique $\{y(t) : t \geq t_0\}$ & $\{x(t) : t \geq t_0\}$ which solves the equations:

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t)$$
$$\underline{y} = \underline{g}(\underline{x}, \underline{u}, t)$$

For the special case:
$$\mathring{x} = A(t)x + B(t)u$$
$$y = C(t)x + D(t)u$$

a sufficient condition for the existence of ~~a~~ unique solutions $x(t)$, $y(t)$ for $t \geq t_0$ given $x(t_0)$ and $u(t)$, $t \geq 0$ is that $A(\cdot)$ be a continuous function.

<u>We will make this assumption throughout the course.</u>

<u>Note:</u> The above condition is always satisfied when $A(\cdot)$ is a <u>constant</u> matrix.

Consider the model
$$\dot{\underline{x}} = A(t)\,\underline{x} + B(t)\,\underline{u} \qquad \underline{x}(t_0) = \underline{x}_0$$
$$\underline{y} = C(t)\,\underline{x} + D(t)\,\underline{u}$$

It defines an input-output mapping H where $\underline{y} = H\underline{u}$, and $\underline{u}$ and $\underline{y}$ take values over $(-\infty, \infty)$.

<u>Question</u>: Is H a linear I/O map ?

As a mapping, H will be linear whenever $\underline{x}_0 = 0$.

We need to prove this:

Let $u_1(\cdot)$ and $u_2(\cdot)$ be two inputs defined on $(-\infty, \infty)$.
Let $x_1(\cdot)$ and $x_2(\cdot)$ be the corresponding solutions and $y_1(\cdot)$ and $y_2(\cdot)$ be the corresponding outputs

Therefore,
$$\mathring{\underline{x}}_1 = A(t)\,\underline{x}_1 + B(t)\,\underline{u}_1 \qquad \underline{x}_1(t_0) = 0$$
$$\underline{y}_1 = C(t)\,x_1 + D(t)\,\underline{u}_1$$

and
$$\mathring{\underline{x}}_2 = A(t)\,\underline{x}_2 + B(t)\,\underline{u}_2 \quad . \quad x_2(t_0) = 0$$
$$\underline{y}_2 = C(t)\,\underline{x}_2 + D(t)\,\underline{u}_2$$

We must show that if $\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2$ is applied as an input, then the resulting output will be $\alpha_1 \underline{y}_1 + \alpha_2 \underline{y}_2$.

<u>Claim</u>: if $\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2$ is applied at the input, then the unique state solution is $\alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2$.

This is easy to show.

$$\frac{d}{dt}\left(\alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2\right) = \alpha_1 \frac{d}{dt}\underline{x}_1 + \alpha_2 \frac{d}{dt}\underline{x}_2$$

$$= \alpha_1 \left(A(t)\,\underline{x}_1 + \underline{B}(t)\underline{u}_1\right) + \alpha_2 \left(A(t)\underline{x}_2 + B(t)\underline{u}_2\right)$$

$$= A(t)\left(\alpha_1 x_1 + \alpha_2 x_2\right) + B(t)\left(\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2\right)$$

Hence $\alpha_1 x_1 + \alpha_2 x_2$ satisfies the diff. equation with $\alpha_1 \underline{u}_1 + \alpha_2 \underline{u}_2$ as input.

We must also show it satisfies the initial condition:

$$\left(\alpha_1 x_1 + \alpha_2 x_2\right)(t_0) = \alpha_1 x_1(t_0) + \alpha_2 x_2(t_0) = 0$$

This proves the claim.

<u>The rest is simple</u>:

$$y(t) = C(t)\left(\alpha_1 \underline{x}_1 + \alpha_2 \underline{x}_2\right) + D(t)\left(\alpha_1 u_1 + \alpha_2 u_2\right)$$

$$= \alpha_1 \left(C(t)\,\underline{x}_1 + D(t)u_1\right) + \alpha_2 \left(C(t)\,\underline{x}_2 + D(t)u_2\right) = \alpha_1 y_1 + \alpha_2 y_2$$

One of the most important implications of the notion of a "system state" is that knowledge of that state at a given time, say t_0, eliminates the need to know anything about the system prior to time t_0.

Therefore, for a system described by the model

$$\dot{\underline{x}} = A(t)\,\underline{x} + \underline{B}(t)\underline{u}$$
$$\underline{y} = C(t)\,\underline{x} + D(t)\,\underline{u}$$

if one is only interested in the output and state for $t > t_0$, one <u>only</u> needs to know $\underline{x}(t_0)$ and $u(t)$ for $t \geq t_0$.

Accordingly the state at time t_0 and the input for $t \geq t_0$ can be uniquely mapped to the state trajectory x for $t \geq t_0$ and output y for $t \geq t_0$, i.e.

$$(\underline{x}_0, \underline{u}) \longmapsto (\underline{x}, \underline{y})$$ "initial-state-and-input to state-and-output map"

where $\underline{x}(t_0) = \underline{x}_0$

and $\underline{u}, \underline{x}, \underline{y}$ are functions of time defined on $[t_0, \infty)$.

<u>Fact:</u> The mapping $(\underline{x}_0, \underline{u}) \mapsto (\underline{x}, \underline{y})$ corresponding to the system

$$\begin{aligned} \dot{x} &= A(t)\, x + B(t)\, u \\ y &= C(t)\, x + D(t)\, u \end{aligned}$$

is a linear mapping.

<u>Proof:</u> Exercise.

__Fact__: If the model is given by $\overset{\circ}{x} = Ax + Bu$, $y = Cx + Du$

where A, B, C, D are constant matrices, then the initial-state-and-input to state-and-output mapping

$$(\underline{x}_0, \underline{u}) \longmapsto (\underline{x}, \underline{y})$$

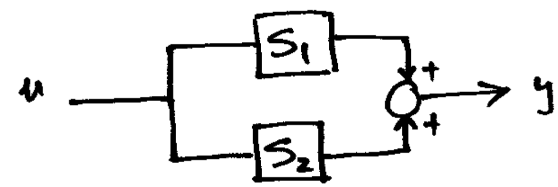
does not depend on the initial time t_0.

__Proof:__ Exercise.

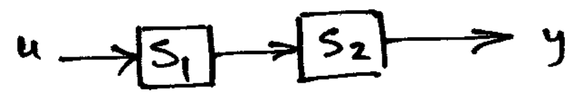
Interconnections of Linear Systems

I. State-Space Descriptions

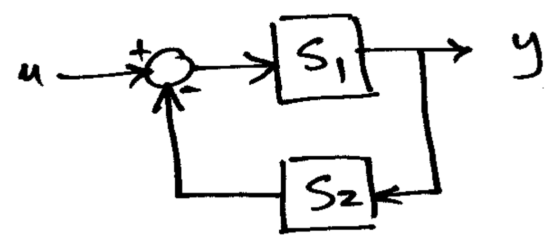
a. Parallel Connection



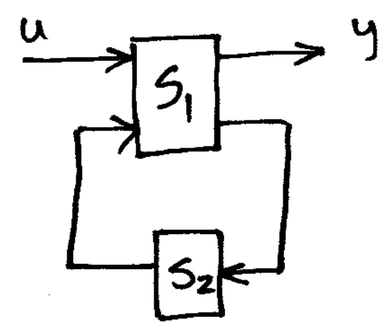
b. Series Connection



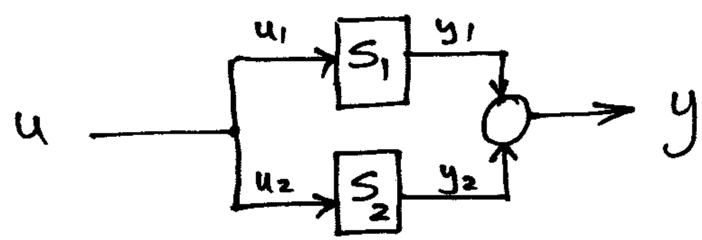
c. Feedback Connection



d. General Feedback Connection



a. Parallel Connection



$$S_1 := \begin{cases} \dot{\underline{x}}_1 = A_1 \underbrace{\underline{x}_1}_{n_1 \times 1} + B_1 \underbrace{\underline{u}_1}_{q \times 1} \\ \underline{y}_1 = C_1 \underline{x}_1 + D_1 \underline{u}_1 \end{cases}$$

$$S_2 := \begin{cases} \dot{\underline{x}}_2 = A_2 \underbrace{\underline{x}_2}_{n_2 \times 1} + B_2 \underline{u}_2 \\ \underline{y}_2 = C_2 \underline{x}_2 + D_2 \underline{u}_2 \end{cases}$$

<u>Note</u>: $\underline{u}_1$ and $\underline{u}_2$ must have same size

$\underline{y}_1$ and $\underline{y}_2$ must have same size

<u>Key</u>: For the interconnected system, the vector $\underline{x} := \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}$ qualifies as a "state".

We therefore want to <u>express the state-equations in terms of $\underline{x}$</u>, i.e.

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}$$
$$\underline{y} = C\underline{x} + D\underline{u}$$

A is $(n_1+n_2) \times (n_1+n_2)$
B is $(n_1+n_2) \times q$
C is $p \times (n_1+n_2)$

To compute $\mathring{\underline{x}} = \begin{bmatrix} \mathring{\underline{x}}_1 \\ \mathring{\underline{x}}_2 \end{bmatrix}$, we can compute $\mathring{\underline{x}}_1$ and $\mathring{\underline{x}}_2$ separately.

$$\mathring{\underline{x}}_1 = A_1 \underline{x}_1 + B_1 u_1 = A_1 \underline{x}_1 + B_1 \underline{u}$$

$$\mathring{\underline{x}}_2 = A_2 \underline{x}_2 + B_2 u_2 = A_2 \underline{x}_2 + B_2 \underline{u}$$

$$\therefore \quad \mathring{\underline{x}} = \underbrace{\begin{bmatrix} \mathring{\underline{x}}_1 \\ \mathring{\underline{x}}_2 \end{bmatrix}}_{\mathring{\underline{x}}} = \underbrace{\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}}_{\underline{x}} + \underbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}_{B} u$$

$$A = \begin{array}{c c} & \begin{array}{cc} \overbrace{\quad}^{n_1} & \overbrace{\quad}^{n_2} \end{array} \\ \begin{array}{c} n_1\{ \\ n_2\{ \end{array} & \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \end{array} , \qquad B = \begin{array}{c c} & \overbrace{\quad}^{q} \\ \begin{array}{c} n_1\{ \\ n_2\{ \end{array} & \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \end{array}$$

Output equations:

$$\underline{y} = \underline{y}_1 + \underline{y}_2 = \left(C_1 \underline{x}_1 + D_1 u_1\right) + \left(C_2 \underline{x}_2 + D_2 \underline{u}_2\right)$$
$$= C_1 \underline{x}_1 + C_2 \underline{x}_2 + (D_1 + D_2) u$$

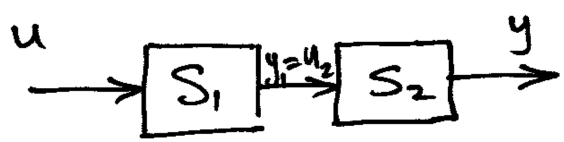
In matrix form:

$$y = p\Big\{ \underbrace{\begin{bmatrix} \overbrace{C_1}^{n_1} & \overbrace{C_2}^{n_2} \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}}_{\underline{x}} + \underbrace{\overbrace{(D_1 + D_2)}^{p \times q}}_{D} u$$

To summarize,

$$\boxed{\begin{array}{l} \text{The state-space description of a parallel interconnection is} \\ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (D_1 + D_2) u \end{array}}$$

b. Series Connection



$$S_i := \begin{cases} \dot{\underline{x}}_i = A_i \underline{x}_i + B_i \underline{u}_i \\ \underline{y}_i = C_i \underline{x}_i + D_i \underline{u}_i \end{cases}$$

The new state-vector is $\underline{x} = \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \end{bmatrix}$

State-Equations:

$$\dot{x}_1 = A_1 x_1 + B_1 u_1 = A_1 x_1 + B_1 u$$

$$\dot{x}_2 = A_2 x_2 + B_2 u_2 = A_2 x_2 + B_2 (C_1 x_1 + D_1 u)$$

$$= B_2 C_1 x_1 + A_2 x_2 + B_2 D_1 u$$

In matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u$$

Output Equations:

$$y = C_2 x_2 + D_2 u_2 = C_2 x_2 + D_2 (C_1 x_1 + D_1 u)$$

$$= D_2 C_1 x_1 + C_2 x_2 + D_2 D_1 u$$

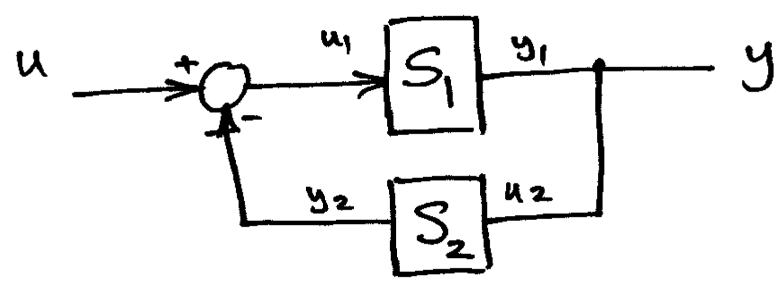
In matrix form:

$$y = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (D_2 D_1) u$$

The state-space description of a series connection is:

$$\boxed{\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u \\ y &= \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (D_2 D_1) u \end{aligned}}$$

c. Feedback Connection



$$S_i := \begin{cases} \mathring{x}_i = A_i x_i + B_i u_i \\ y_i = C_i x_i + D_i u_i \end{cases}$$

$\underline{x} := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the new state vector

$$\begin{aligned} \mathring{x}_1 &= A_1 x_1 + B_1 u_1 = A_1 x_1 + B_1 (u - y_2) \\ &= A_1 x_1 + B_1 u - B_1 (C_2 x_2 + D_2 u_2) \\ &= A_1 x_1 + B_1 u - B_1 (C_2 x_2 + D_2 (C_1 x_1 + D_1 u_1)) \\ &\vdots \end{aligned}$$

We must solve for u_1 first

$$
\begin{aligned}
u_1 &= u - y_2 \\
&= u - (C_2 x_2 + D_2 u_2) \\
&= u - C_2 x_2 - D_2 (C_1 x_1 + D_1 u_1) \\
&= u - C_2 x_2 - D_2 C_1 x_1 - D_2 D_1 u_1
\end{aligned}
$$

or $(I + D_2 D_1) u_1 = u - C_2 x_2 - D_2 C_1 x_1$

There exists a unique u_1 iff $I - D_2 D_1$ is nonsingular

We make this assumption

$\therefore \quad u_1 = Y u - Y D_2 C_1 x_1 - Y C_2 x_2$

where $Y := (I + D_2 D_1)^{-1}$

Now we can substitute for u_1.

$$\mathring{x}_1 = A_1 x_1 + B_1 u_1 = A_1 x_1 + B_1 \left(Yu - YD_2 C_1 x_1 - YC_2 x_2 \right)$$

$$= \left(A - B_1 Y D_2 C_1 \right) x_1 - B_1 Y C_2 x_2 + B_1 Y u$$

$$\mathring{x}_2 = A_2 x_2 + B_2 u_2 = A_2 x_2 + B_2 \left(C_1 x_1 + D_1 u_1 \right)$$

$$= B_2 C_1 x_1 + A_2 x_2 + B_2 D_1 u_1$$

$$= B_2 C_1 x_1 + A_2 x_2 + B_2 D_1 \left(Yu - YD_2 C_1 x_1 - YC_2 x_2 \right)$$

$$= \left(B_2 C_1 - B_2 D_1 Y D_2 C_1 \right) x_1 + \left(A_2 - B_2 D_1 Y C_2 \right) x_2 + B_2 D_1 Y u$$

$$= B_2 (I - D_1 Y D_2) C_1 \, x_1 + \left(A_2 - B_2 D_1 Y C_2 \right) x_2 + B_2 D_1 Y u$$

In matrix form,

$$\begin{bmatrix} \mathring{x}_1 \\ \mathring{x}_2 \end{bmatrix} = \begin{bmatrix} A - B_1 Y D_2 C_1 & -B_1 Y C_2 \\ B_2 (I - D_1 Y D_2) C_1 & A_2 - B_2 D_1 Y C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 Y \\ B_2 D_1 Y \end{bmatrix} u$$

Output equations

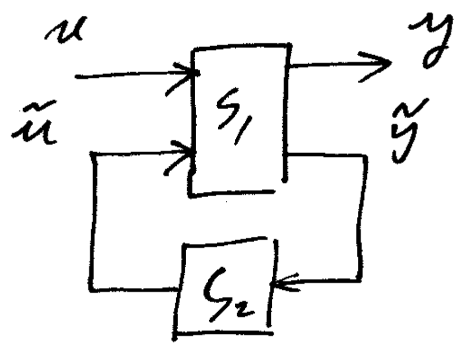
$$y = C_1 x_1 + D_1 u_1$$
$$= C_1 x_1 + D_1 \left(Y u - Y D_2 C_1 x_1 - Y C_2 x_2 \right)$$
$$= \left(C_1 - D_1 Y D_2 C_1 \right) x_1 - D_1 Y C_2 x_2 + D_1 Y u$$

In matrix form

$$y = \begin{bmatrix} (I - D_1 Y D_2) C_1 & -D_1 Y C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_1 Y u$$

$$\boxed{\begin{array}{l} \text{The state-space description of a feedback interconnection is} \\ \begin{bmatrix} \mathring{x}_1 \\ \mathring{x}_2 \end{bmatrix} = \begin{bmatrix} A - B_1 Y D_2 C_1 & -B_1 Y C_2 \\ B_2 (I - D_1 Y D_2) C_1 & A_2 - B_2 D_1 Y C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 Y \\ B_2 D_1 Y \end{bmatrix} u \\ y = \begin{bmatrix} (I - D_1 Y D_2) C_1 & -D_1 Y C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (D_1 Y) u \end{array}}$$

d. General Feedback Connection



S_1: $\qquad \dot{x}_1 = A_1 x_1 + \begin{bmatrix} B_{11} & B_{12} \end{bmatrix} \begin{bmatrix} u \\ \tilde{u} \end{bmatrix}$

$$\begin{bmatrix} y \\ \tilde{y} \end{bmatrix} = \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} x_1 + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} u \\ \tilde{u} \end{bmatrix}$$

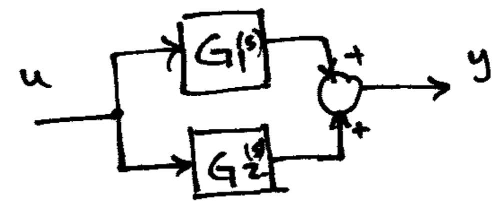
S_2: $\qquad \dot{x}_2 = A_2 x_2 + B_2 u_2$

$$y_2 = C_2 x_2 + D_2 u_2$$

Exercise: Derive state-space description of interconnected system.

Interconnections of LTI Systems : Transfer Function Approach

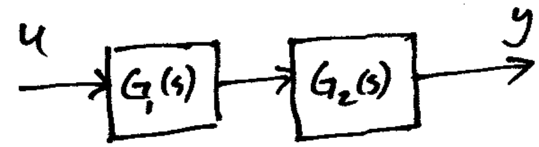
a. Parallel Connection



$$y = \underbrace{\left(G_1(s) + G_2(s)\right)}_{G(s)} u$$

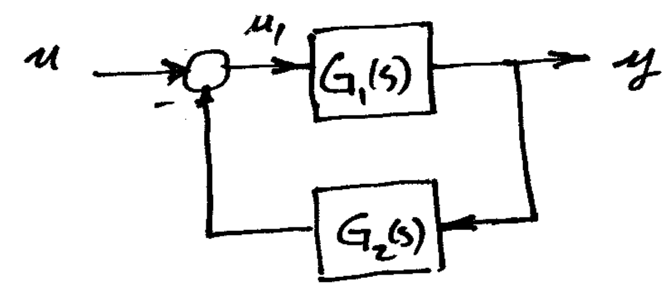
$G_1(s)$ is $q \times p$
$G_2(s)$ is $q \times p$

b. Series Connection



$$\begin{aligned} y &= G_2(s)\, u_2 \\ &= G_2(s)\left(G_1(s)\, u\right) \\ &= \underbrace{G_2(s)\, G_1(s)}_{G(s)}\, u \end{aligned}$$

c. Feedback Connection



$$y(s) = G_1(s)\, u_1(s)$$
$$= G_1(s)\big(u(s) - G_2(s)\, y(s)\big)$$

$$\therefore \big(I + G_1(s)\, G_2(s)\big) y = G_1(s)\, u(s)$$

We assume $\big(I + G_1(s) G_2(s)\big)^{-1}$ exists (i.e. $\det\big(I + G_2(s) G_1(s)\big) \neq 0$)

Therefore, $y(s) = \underbrace{\big(I + G_1(s)\, G_2(s)\big)^{-1} G_1(s)}_{G(s)}\, u(s)$

Note that:

1. $G(s) = \big(I + G_1(s) G_2(s)\big)^{-1} G_1(s) = G_1(s)\big(I + G_2(s) G_1(s)\big)^{-1}$

2. Making the assumption that $\det(I + G_2 G_1) \neq 0$ was essential for the closed-loop mathematical formulation to make sense. To see this ...

Consider the example:

$$G_1(s) = \begin{bmatrix} \frac{1-s}{s} & \frac{1}{s+1} \\ \frac{1}{s} & \frac{-s}{s+1} \end{bmatrix}, \qquad G_2(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I + G_1(s)G_2(s) = \begin{bmatrix} \frac{1}{s} & \frac{1}{s+1} \\ \frac{1}{s} & \frac{1}{s+1} \end{bmatrix} \quad \text{and} \quad \det\left(I + G_1(s)G_2(s)\right) \equiv 0$$

From the block diagram, $\left(I + G_1(s)G_2(s)\right) y(s) = G_1(s)\, u(s)$

so
$$\begin{bmatrix} \frac{1}{s} & \frac{1}{s+1} \\ \frac{1}{s} & \frac{1}{s+1} \end{bmatrix} \underbrace{\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix}}_{y(s)} = \begin{bmatrix} \frac{1-s}{s} & \frac{1}{s+1} \\ \frac{1}{s} & \frac{-s}{s+1} \end{bmatrix} \underbrace{\begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}}_{u(s)}$$

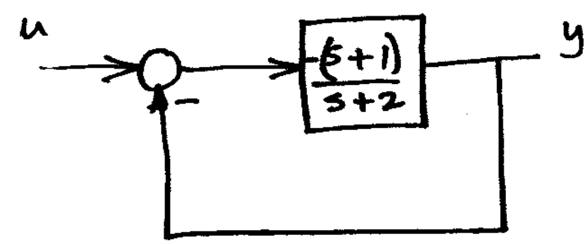
if $\cancel{y}\; u(s) = \begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix}$, then we have $\begin{bmatrix} \frac{1}{s} & \frac{1}{s+1} \\ \frac{1}{s} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1-s}{s^2} \\ \frac{1}{s^2} \end{bmatrix}$

for which there is <u>no solution</u> !

We've seen that $\det(I + G_2(s)G_1(s)) \not\equiv 0$ is absolutely essential.

Even when $(I + G_1(s)G_2(s))^{-1}$ exists the transfer function from $u(s)$ to another point in the loop may not be proper.

Example:



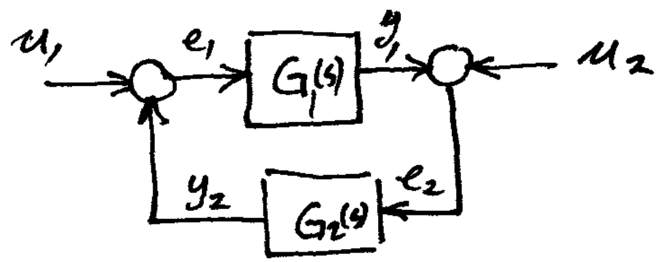
Here $\det(I + G_1(s)G_2(s)) = 1 + G_1(s) = \frac{s+2}{s+2} + \frac{-(s+1)}{s+2} = \frac{1}{s+2} \not\equiv 0$

However $y(s) = (I + G_1(s)G_2(s))^{-1} G_1(s)\, u = \frac{G_1(s)}{1+G_1(s)} = \frac{\frac{-(s+1)}{s+2}}{\frac{1}{s+2}} = -(s+1)$

Improper transfer functions do not correspond to "good" systems.

Problem ?? NOISE !

We want to impose conditions on $G_1(s)$ and $G_2(s)$ so that the transfer function from any point in the loop to any other point is proper. (Well-posedness)



$u_1 \mapsto e_1$: $(I+G_2(s)G_1(s))^{-1}$

$u_1 \mapsto e_2$: $G_1(s)(I+G_2(s)G_1(s))^{-1}$

$u_1 \mapsto y_1$: $G_1(s)(I+G_2(s)G_1(s))^{-1}$

$u_1 \mapsto y_2$: $G_2(s)G_1(s)(I+G_2(s)G_1(s))^{-1}$

$u_2 \mapsto e_1$: $G_2(s)(I+G_1(s)G_2(s))^{-1}$

$u_2 \mapsto e_2$: $(I+G_1(s)G_2(s))^{-1}$

$u_2 \mapsto y_1$: $G_1(s)G_2(s)(I+G_1(s)G_2(s))^{-1}$

$u_2 \mapsto y_2$: $G_2(s)(I+G_1(s)G_2(s))^{-1}$

$\therefore$ Well-posedness iff $\begin{cases} (I+G_2(s)G_1(s))^{-1} \text{ is proper and} \\ (I+G_1(s)G_2(s))^{-1} \text{ is proper} \end{cases}$

The conditions for well-posedness can be simplified further:

<u>Claim:</u> $(I + G_1(s)G_2(s))^{-1}$ is proper iff $(I + G_2(s)G_1(s))^{-1}$ is proper.

<u>Proof:</u> We use the identity

$$(I + G_1(s)G_2(s))^{-1} + G_1(s)G_2(s)(I + G_1(s)G_2(s))^{-1} = I \quad \text{(prove)}$$

$$\therefore \ (I + G_1(s)G_2(s))^{-1} = I - G_1(s)G_2(s)(I + G_1(s)G_2(s))^{-1}$$

$$= I + G_1(s)(I + G_2(s)G_1(s))^{-1}G_2(s)$$

From this, if $(I + G_2(s)G_1(s))^{-1}$ is proper, then $(I + G_1(s)G_2(s))^{-1}$ is proper (because $G_1(s)$ and $G_2(s)$ are proper).

We've shown

$$(I + G_2(s)G_1(s))^{-1} \text{ is proper} \Longrightarrow (I + G_1(s)G_2(s))^{-1} \text{ is proper}$$

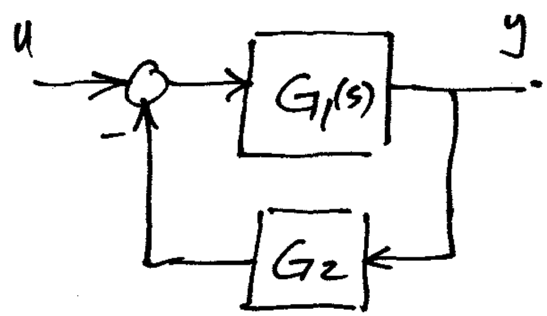
Proving the other direction ($\Leftarrow$) is exactly the same.

Q.E.D.

To summarize, to check well-posedness of closed loop check $(I + G_1(\infty) G_2(\infty))$ is nonsingular. Equivalently

$$\det(I + G_1(\infty) G_2(\infty)) = \det(I + G_2(\infty) G_1(\infty)) \neq 0$$

<u>Example:</u>



$$G_1(s) = \begin{bmatrix} \frac{s+1}{s+2} & \frac{3s^2}{s^2-1} \\ \frac{1}{s} & 3 \end{bmatrix}$$

$$G_2(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{1}{s+2} \\ 1 & \frac{s}{s+1} \end{bmatrix}$$

$$I + G_1(\infty) G_2(\infty) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix}$$ which is nonsingular

$\det\left(\begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix}\right) \neq 0$. Therefore, this interconnection is well-posed.

Given $(I + G_1(s)G_2(s)) =: M(s)$ which is invertible.

How do we check whether $(I + G_1(s)\,G_2(s))^{-1} = M^{-1}(s)$ is proper?

The transfer function matrix $M(s)$ can be written

$$M(s) = \underbrace{M_\infty}_{\text{constant}} + \underbrace{M_{sp}(s)}_{\text{strictly proper}}$$

e.g.

$$\underbrace{\begin{bmatrix} \frac{s+1}{s+2} & \frac{1}{s} \\ 1 & \frac{s}{s+1} \end{bmatrix}}_{M(s)} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{M_\infty} + \underbrace{\begin{bmatrix} \frac{-1}{s+2} & \frac{1}{s} \\ 0 & \frac{-1}{s+1} \end{bmatrix}}_{M_{sp}(s)}$$

Clearly, $M_\infty = M(\infty)$

<u>Theorem:</u> $(I + G_1(s)\, G_2(s))^{-1}$ is proper if and only if $(I + G_1(\infty)\, G_2(\infty))$ is nonsingular.

Note: Checking singularity of a constant matrix is easier that computing the inverse of a rational matrix.

<u>Proof:</u> ($\Rightarrow$)

Suppose $M^{-1}(s)$ is proper. This implies $M^{-1}(\infty) =$ finite const.

But this implies that $(M(\infty))^{-1}$ exists.

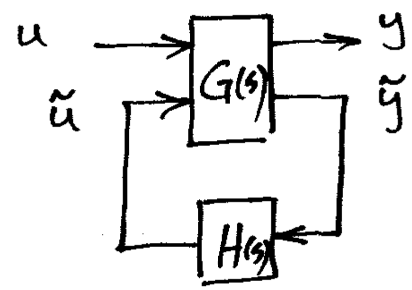
$M(\infty) = M_\infty + M_{sp}(\infty) = M_\infty$. Therefore, M_∞^{-1} exists.

($\Leftarrow$) Suppose M_∞^{-1} exists.

We will show that $M^{-1}(s) = (M_\infty + M_{sp}(s))^{-1}$ is proper. $M^{-1}(\infty) = (M_\infty + M_{sp}(\infty))^{-1} = M_\infty^{-1}$.

Therefore $M^{-1}(\infty)$ is a finite constant. Q.E.D.

d. Generalized Feedback Connection



$$\begin{bmatrix} y(s) \\ \tilde{y}(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} u(s) \\ \tilde{u}(s) \end{bmatrix}$$

$$y(s) = G_{11}(s)\, u(s) + G_{12}(s)\, \tilde{u}(s)$$

$$\tilde{u}(s) = H(s)\left(G_{21}(s)\, u(s) + G_{22}(s)\, \tilde{u}(s)\right)$$

$$\therefore \left(I - H(s)\, G_{22}(s)\right) \tilde{u}(s) = H(s)\, G_{21}(s)\, u(s)$$

For well-posedness $(I + H(s)\, G_{22}(s))^{-1}$ must exist and be proper.

With this assumption, $\tilde{u}(s) = (I - H(s) G_{22}(s))^{-1} H(s) G_{21}(s)\, u$

Substituting,

$$y(s) = \underbrace{\left\{ G_{11}(s) + G_{12}(s)\left(I - H(s) G_{22}(s)\right)^{-1} H(s)\, G_{21}(s) \right\}}_{G(s)} u(s)$$

Solution of Linear State-Space Equations

$$\dot{x} = A(t)\,x + B(t)\,u \qquad x(t_0) = x_0$$
$$y = C(t)\,x + D(t)\,u$$

The solution of $\dot{x} = A(t)\,x + B(t)\,u(t) \quad x(t_0) = x_0$

can be decomposed into two parts:

(a) The zero-state solution, i.e. the solution of

$$\dot{x}_{ZS} = A(t)\,x_{ZS} + B(t)\,u(t) \qquad x_{ZS}(t_0) = 0$$

(b) The zero-input solution, i.e. the solution of

$$\dot{x}_{ZI} = A(t)\,x_{ZI} \qquad x_{ZI}(t_0) = x_0$$

In other words, $x(t)$ which solves $\begin{cases} \dot{x} = A(t)x + B(t)u \\ x(t_0) = x_0 \end{cases}$

is given by $x(t) = x_{ZI}(t) + x_{ZS}(t)$.

To see this,

$$\dot{x}(t) = \dot{x}_{ZI}(t) + \dot{x}_{ZS}(t)$$

$$= A(t)\, x_{ZI}(t) + A(t)\, x_{ZS}(t) + B(t)u$$

$$= A(t)\left(x_{ZI}(t) + x_{ZS}(t)\right) + B(t)u$$

$$= A(t)\, x(t) + B(t)u$$

Also $x(t_0) = x_{ZI}(t_0) + x_{ZS}(t_0)$

$$= x_0 + 0 = x_0$$

This proves the solution $x(t)$ can be decomposed into 2 parts.

We will therefore try to find each solution separately.

We begin with the zero-input solution.

We want to solve the differential equation

$$\dot{x} = A(t)\, x \qquad x(t_0) = x_0$$

Without imposing the condition $x(t_0) = x_0$, the differential equation $\dot{x} = A(t)\, x$ has an infinite number of solutions.

Indeed,

<u>Theorem:</u> The set of solutions of $\dot{x} = A(t)\, x$ is an n-dimensional vector space, where n is the number of state-variables.

What is a vector space?

What is the dimension of a vector space?

⋮

Look at chapter 3.

Review:

Def: A vector space over a field of scalars $\mathcal{F}$ is a <u>set</u>, denoted by $\mathcal{X}$, of elements (called vectors) s.t. 2 properties are well-defined.

1. Vector addition: $x_1 + x_2 \in \mathcal{X} \quad \forall x_1, x_2 \in \mathcal{X}$
2. Scalar multiplication: $\alpha x \in \mathcal{X} \quad \forall \alpha \in \mathcal{F}, \forall x \in \mathcal{X}$

<u>and</u> these two operations satisfy the following:

1. $x_1 + x_2 = x_2 + x_1 \quad \forall\, x_1, x_2 \in \mathcal{X}$
2. $(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3) \quad \forall\, x_1, x_2, x_3 \in \mathcal{X}$
3. There exists an element $\underline{0} \in \mathcal{X}$ satisfying $x + \underline{0} = x, \ \forall x \in \mathcal{X}$
4. $\forall\, x \in \mathcal{X}$, there is $\bar{x} \in \mathcal{X}$ such that $x + \bar{x} = \underline{0}$
5. $\alpha(\beta x) = (\alpha\beta) x \quad \forall\, \alpha, \beta \in \mathcal{F}$ and $\forall\, x \in \mathcal{X}$
6. $\alpha(x_1 + x_2) = \alpha x_1 + \alpha x_2 \quad \forall\, \alpha \in \mathcal{F}$ and $\forall\, x_1, x_2 \in \mathcal{X}$
7. $(\alpha + \beta) x = \alpha x + \beta x \quad \forall\, \alpha, \beta \in \mathcal{F}$ and $\forall\, x \in \mathcal{X}$
8. There is an element $\underline{1} \in \mathcal{F}$ such that $\underline{1} \cdot x = x \ \forall x \in \mathcal{X}$

Examples:

$(\mathbb{R}^n, \mathbb{R})$	n-tuples of real numbers
$(\mathbb{C}^n, \mathbb{C})$	n-tuples of complex numbers
$(\mathbb{R}_n[s], \mathbb{R})$	nth order polynomials with real coef.
$(\mathbb{R}_n(s), \mathbb{R})$	nth order rational functions with real coef.

Def: A set of vectors $\{x_1, \dots, x_n\}$ in a vector space over $\mathcal{F}$, $(\mathcal{X}, \mathcal{F})$ is said to be <u>linearly</u> <u>dependent</u> if there exist scalars $\alpha_1, \dots, \alpha_n \in \mathcal{F}$, not all zero such that
$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = \underline{0}.$$
Otherwise, $\{x_1, \dots, x_n\}$ is linearly independent.

<u>Note</u>: If $\{x_1, \dots, x_n\}$ is linearly dependent, then at least one of the vectors can be expressed as a linear combination of the others.

So if $\alpha_1 \neq 0$, then $x_1 = \frac{1}{\alpha_1}(-\alpha_2 x_2 \cdots - \alpha_n x_n)$

<u>Note</u>: A set $\{x_1, \dots, x_n\}$ is linearly independent iff
$$\alpha_1 x_1 + \cdots \alpha_n x_n = \underline{0} \quad \text{implies} \quad \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

<u>Example</u>: $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is linearly indep. in $(\mathbb{R}^2, \mathbb{R})$

Pf: Suppose $\alpha_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\left.\begin{aligned} 2\alpha_1 + \alpha_2 &= 0 \\ \alpha_1 + \alpha_2 &= 0 \end{aligned}\right\} \Rightarrow \alpha_1 = \alpha_2 = 0$$

<u>Example</u>: The columns of $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ are linearly dependent

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

in matrix form: $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = 0$

There are many nonzero solutions. For example $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Example: $\{ p_1(s) = s^2+3 \,,\, p_2(s) = 4s^2+1 \}$ is linearly independent on $(\mathbb{R}_2[s], \mathbb{R})$.

Pf: Suppose $\alpha_1 p_1(s) + \alpha_2 p_2(s) = 0$

$$\alpha_1 (s^2+3) + \alpha_2 (4s^2+1) = 0$$

$$\Rightarrow (\alpha_1 + 4\alpha_2)s^2 + (3\alpha_1 + \alpha_2) = 0$$

$$\Rightarrow \begin{cases} \alpha_1 + 4\alpha_2 = 0 \\ 3\alpha_1 + \alpha_2 = 0 \end{cases}$$

$$\Rightarrow \alpha_1 = \alpha_2 = 0$$

Example: $\{ 1 \,,\, s \,,\, 3s+1 \}$ is linearly dependent on $(\mathbb{R}[s], \mathbb{R})$

$$\underset{\alpha_1}{1}(1) + \underset{\alpha_2}{3}(s) + \underset{\alpha_3}{(-1)}(3s+1) = \underline{0}$$

<u>Definition</u>: A set of vectors $\{x_1, \ldots, x_n\}$ spans the vector space $(\mathcal{X}, \mathcal{F})$ if every element of $\mathcal{X}$ can be written as: $x = \alpha_1 x_1 + \cdots + \alpha_n x_n \quad \alpha_i \in \mathcal{F}$.

<u>Definition</u>: A set of vectors in a vector space is said to form a basis if

(a) The set spans $\mathcal{X}$.
(b) The set is linearly independent.

<u>Example</u>: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $(\mathbb{R}^2, \mathbb{R})$

<u>Fact</u>: Any two bases for a vector space must have the same number of elements. This is called the "dimension" of the vector space.

Examples: $\{1, s, s^2\}$ is a basis for $(\mathbb{R}_2[s], \mathbb{R})$

$\{1, s, 2s+1\}$ is not a basis for $(\mathbb{R}_2[s], \mathbb{R})$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right\}$ cannot be a basis for $(\mathbb{R}^3, \mathbb{R})$

$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $(\mathbb{R}^2, \mathbb{R})$

$\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is also a basis for $(\mathbb{R}^2, \mathbb{R})$

Solutions of $\dot{x} = A(t)x$

<u>Theorem:</u> All solutions of $\dot{x} = A(t)x$ form an n-dimensional Vector space over the field of real numbers.

<u>Proof:</u> Let $\psi_i(t)$ be a solution to $\dot{x} = A(t)x$ such that

$$\psi_i(t_0) = e_i := \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{ith position}.$$

<u>Claim:</u> The set $\{\psi_i\}_{i=1,\dots,n}$ forms a basis for the set of solutions of $\dot{x} = A(t)x$.

We must show 2 things to prove the claim

1. $\{\psi_i\}_{i=1,\dots,n}$ is linearly independent
2. $\operatorname{span}\{\psi_i\}_{i=1,\dots,n}$ = set of all solutions to $\dot{x} = A(t)x$

To show 1, suppose $\alpha_1 \psi_1 + \cdots + \alpha_n \psi_n \equiv 0$.

Then $\alpha_1 \psi_1(t) + \cdots + \alpha_n \psi_n(t) = 0 \quad \forall t$. In particular,

for $t = t_0$ $\quad \alpha_1 \psi_1(t_0) + \cdots + \alpha_n \psi_n(t_0) = 0$

$$\Rightarrow \alpha_1 e_1 + \cdots + \alpha_n e_n = 0$$

$$\Rightarrow \alpha_1 = \cdots = \alpha_n = 0 \qquad (\text{by linearly indep. of } e_i\text{'s})$$

To show 2, let ψ be any solution to $\dot{x} = A(t)\,x$

$\psi(t_0) \in \mathbb{R}^n$, and $\psi(t_0) = \alpha_1 e_1 + \cdots + \alpha_n e_n$ for some α_i

Now $\alpha_1 \psi_1 + \cdots + \alpha_n \psi_n$ is a solution to $\dot{x} = A(t)\,x$

Furthermore $\left(\alpha_1 \psi_1 + \cdots + \alpha_n \psi_n\right)(t_0) = \alpha_1 e_1 + \cdots + \alpha_n e_n = \psi(t_0)$

$\therefore \psi(t) = \alpha_1 \psi_1(t) + \cdots + \alpha_n \psi_n(t) \quad \forall t$. (Existence & Uniqueness)

Q.E.D.

<u>Example:</u> $\dot{x} = \begin{bmatrix} -1 & 0 \\ t & 0 \end{bmatrix} x$

$\psi_1(t) \equiv \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\psi_2(t) = \begin{bmatrix} -e^{-t} \\ e^{-t} + te^{-t} \end{bmatrix}$ are both solutions

They are linearly indep. Let's check this...

$$\alpha_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -e^{-t} \\ e^{-t} + te^{-t} \end{bmatrix} = 0 \implies \begin{bmatrix} -\alpha_2 e^{-t} \\ \alpha_1 + \alpha_2(e^{-t} + te^{-t}) \end{bmatrix} \equiv 0$$

$-\alpha_2 e^{-t} = 0 \implies \alpha_2 = 0$. This implies $\alpha_1 = 0$.

$\therefore$ All solutions have the form:

$$\psi(t) = \alpha_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -e^{-t} \\ e^{-t} + te^{-t} \end{bmatrix}$$

<u>Definition</u>: An $n \times n$ matrix function Ψ is said to be a <u>fundamental matrix</u> of $\dot{x} = A(t)x$ if the columns of Ψ consist of n linearly independent solutions of $\dot{x} = A(t)x$.

<u>Example</u>: For the system $\dot{x} = \begin{bmatrix} -1 & 0 \\ t & 0 \end{bmatrix} x$ we have seen that $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -e^{-t} \\ e^{-t} + te^{-t} \end{bmatrix}$ are linearly indep. solutions.

Thus $\Psi_1(t) = \begin{bmatrix} 0 & -e^{-t} \\ 1 & e^{-t} + te^{-t} \end{bmatrix}$ is a fundamental matrix.

$\Psi_2(t) = \begin{bmatrix} 0 & e^{-t} \\ 10 & 100 + e^{-t} + te^{-t} \end{bmatrix}$ is also a fundamental matrix.

Theorem: Every fundamental matrix Ψ is nonsingular for all $t \in (-\infty, \infty)$.

Proof: Exercise. The idea behind the proof is that if $\Psi(t_0)$ is singular, its columns are linearly dependent. Since the columns are initial conditions for the solutions which form the columns of Ψ, then these solutions must be linearly dependent. This is a contradiction.

Example:

$\Psi(t) = \begin{bmatrix} 0 & -e^{-t} \\ 1 & e^{-t}+te^{-t} \end{bmatrix}$ is a fundamental matrix.

$$\det(\Psi(t)) = e^{-t} \neq 0 \quad \forall\, t \in \mathbb{R}.$$

Given any fundamental matrix of $\mathring{x} = A(t)x$, say Ψ, then clearly Ψ satisfies the matrix equation

$$\mathring{\Psi}(t) = A(t)\,\Psi(t)$$

<u>Theorem:</u> A matrix function Ψ is a fundamental matrix of $\mathring{x} = A(t)x$ if and only if

(a) $\mathring{\Psi}(t) = A(t)\,\Psi(t)$

(b) $\Psi(t)$ is nonsingular for some $t \in \mathbb{R}$.

<u>Proof:</u> Exercise.

Given a fundamental matrix Ψ of $\mathring{x} = A(t)x$. How do we get the unique solution of : $\mathring{x} = A(t)x$, $x(t_0) = x_0$?

The unique solution must be of the form

$$x = \begin{bmatrix} \Psi_1 & \cdots & \Psi_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = \Psi \underline{\alpha} \quad \text{for some } \alpha_i \in \mathbb{R}$$

How do we compute $\underline{\alpha}$?

$$\mathring{x} = (\Psi \underline{\alpha})^{\circ} = \mathring{\Psi} \underline{\alpha} = A(t) \Psi \underline{\alpha} = A(t) x$$

$x(t_0) = \Psi(t_0) \underline{\alpha} = x_0$. Therefore, $\underline{\alpha} = \Psi^{-1}(t_0) x_0$

Hence, $\boxed{x(t) = \Psi(t) \Psi^{-1}(t_0) x_0}$

Define, $\Phi(t,t_0) := \Psi(t)\Psi^{-1}(t_0)$. $\Phi(t,t_0)$ is said to be the <u>state transition matrix</u> of $\dot{x} = A(t)x$.

Properties of the state-transition matrix:

(a) $\Phi(t,t) = I$

(b) $\Phi^{-1}(t,t_0) = \Phi(t_0,t)$ (Because $\Phi^{-1}(t,t_0) = \left(\Psi(t)\Psi^{-1}(t_0)\right)^{-1}$

$= \Psi(t_0)\Psi^{-1}(t)$

(c) $\Phi(t_2,t_0) = \Phi(t_2,t_1)\Phi(t_1,t_0)$

<u>Remark:</u> $\Phi(t,t_0)$ is uniquely determined by $A(t)$.

<u>Example:</u> For the system $\dot{x} = \begin{bmatrix} -1 & 0 \\ t & 0 \end{bmatrix} x$ we have seen $\Psi(t) = \begin{bmatrix} 0 & -e^{-t} \\ 1 & e^{-t} + te^{-t} \end{bmatrix}$

$$\Phi(t,0) = \Psi(t)\,\Psi^{-1}(0) = \begin{bmatrix} 0 & -e^{-t} \\ 1 & e^{-t} + te^{-t} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} e^{-t} & 0 \\ 1 - e^{-t} - te^{-t} & 1 \end{bmatrix}$$

Suppose $x(0) = x_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Then $x(t) = \Phi(t,0)\,x_0 = \begin{bmatrix} e^{-t} & 0 \\ 1 - e^{-t} - te^{-t} & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2e^{-t} \\ 5 - 2e^{-t} - 2te^{-t} \end{bmatrix}$

Properties of $\Phi(t,t_0)$

a) $\frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\Phi(t,t_0)$

b) $\Phi(t_0,t_0) = I \qquad \forall t_0$

c) $\Phi(t,t_0)$ is unique

Proof:

a) $\frac{\partial}{\partial t}\Psi(t)\Psi^{-1}(t_0) = \left(\frac{d}{dt}\Psi(t)\right)\Psi^{-1}(t_0) = A(t)\Psi(t)\Psi^{-1}(t_0) = A(t)\Phi(t,t_0)$

b) $\Phi(t_0,t_0) = \Psi(t_0)\Psi^{-1}(t_0) = I$

c) Let $\Phi_1(t,t_0) = \Psi_1(t)\Psi_1^{-1}(t_0)$, and $\Phi_2(t,t_0) = \Psi_2(t)\Psi_2^{-1}(t_0)$

where $\Psi_1(t)$ and $\Psi_2(t)$ are two fundamental matrices.

Since columns of $\Psi_1(t)$ form a basis for solution space

$$\Psi_2(t) = \Psi_1(t)\underbrace{[\underline{\alpha}_1 \cdots \underline{\alpha}_n]}_{P} = \Psi_1(t)P.$$

P is invertible. (Prove this!)

$$\therefore\ \Phi_2(t,t_0) = \Psi_1(t)P\left(\Psi_1(t_0)P\right)^{-1} = \Psi_1(t)\Psi_1^{-1}(t_0) = \Phi_1(t,t_0)$$

Summary: The solution of $\dot{x} = A(t)x$, $x(t_0) = x_0$ is given by $x(t) = \Phi(t,t_0)x_0$, where $\Phi(t,t_0)$ is the state-transition matrix.

The state-transition matrix can be characterized in two ways:

1. $\Phi(t,t_0) = \Psi(t)\Psi^{-1}(t_0)$ for any fundamental sol'n Ψ
2. The unique solution of

$$\frac{\partial \Phi}{\partial t}(t,\tau) = A(t)\Phi(t,\tau) \qquad \forall \tau$$
$$\Phi(\tau,\tau) = I \qquad \forall \tau$$

Solution of $\dot{x} = A(t)x + B(t)u \ , \ x(t_0) = x_0$

<u>Def:</u> Let $\phi(t; t_0, x_0, u)$ denote the solution of

$\dot{x} = A(t)x + B(t)u$ when $x(t_0) = x_0$.

<u>Theorem:</u> The solution of $\dot{x} = A(t)x + B(t)u \ , \ x(t_0) = x_0$ is given by

$$\phi(t; t_0, x_0, u) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

where $\Phi(t,\tau)$ is the state transition matrix of $\dot{x} = A(t)x$.

Solution of $\dot{x} = A(t)x + B(t)u$, $x(t_0) = x_0$

<u>Theorem:</u> The solution of $\dot{x} = A(t)x + B(t)u$, $x(t_0) = x_0$ is given by

$$\phi(t; t_0, x_0, \cancel{x_0}^{\,u}) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

where $\Phi(t,\tau)$ is the state-transition matrix associated with $\dot{x} = A(t)x$.

<u>Proof:</u>

$$\frac{d}{dt}\phi(t; t_0, x_0, u) = \frac{d}{dt}\left\{\Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau\right\}$$

$$= \frac{d}{dt}\left\{ \Phi(t,t_0)x_0 + \Phi(t_0,t_0)\int_{t_0}^{t} \Phi(t_0,\tau)B(\tau)u(\tau)\,d\tau \right\}$$

$$= A(t)\Phi(t,t_0)x_0 + A(t)\Phi(t,t_0)\int_{t_0}^{t} \Phi(t_0,\tau)B(\tau)u(\tau)\,d\tau$$

$$+ \Phi(t,t_0)\,\Phi(t_0,t)B(t)u(t)$$

$$= A(t)\left\{ \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau \right\} + B(t)u(t)$$

$$= A(t)\,\phi(t;t_0,x_0,u) + B(t)u$$

$\therefore$ $\phi(t;t_0,x_0,u)$ is indeed a solution. Also $\phi(t_0;t_0,x_0,u) = x_0$.

<u>Note</u>: The solution can be decomposed into 2 parts

$$\phi(t; t_0, x_0, u) = \underbrace{\Phi(t, t_0)\, x_0}_{\text{zero-input solution}} + \underbrace{\int_{t_0}^{t} \Phi(t, \tau) B(\tau) u(\tau)\, d\tau}_{\text{zero-state solution}}$$

The output $\quad y(t) = C(t)\, x + D(t)\, u$

$$= C\, \Phi(t, t_0) x_0 + \int_{t_0}^{t} C(t) \Phi(t, \tau) B(\tau) u(\tau)\, d\tau + D(t)\, u$$

<u>Recall</u>: For a system which is linear and is relaxed at $t = t_0$

$$y(t) = \int_{t_0}^{t} G(t,\tau)\, u(\tau)\, d\tau$$

What is $G(t,\tau)$ for
$$\dot{x} = A(t)x + B(t)u \qquad ?$$
$$y = C(t)x + D(t)u$$

$G(t,\tau)$ is the response at time t to a δ applied at $t = \tau$.

$$G(t,\tau) = \int_{t_0}^{t} C(t)\,\Phi(t,s)\,B(s)\,u(s)\,ds + D(t)\,\delta(t-\tau) \qquad \text{where } u(s) = \delta(s-\tau)$$

$$\boxed{G(t,\tau) = C(t)\,\Phi(t,\tau)\,B(\tau) + D(\tau)\,\delta(t-\tau)}$$

The Time-Invariant Case

$$\dot{x} = Ax + Bu \qquad \text{What is } \Phi(t,\tau)\text{ ?}$$
$$y = Cx + Du$$

Look at the scalar case:

$\dot{x} = ax$, a fundamental matrix is $\Psi(t) = e^{at}$

$$\Phi(t,\tau) = e^{at} \cdot e^{-a\tau} = e^{a(t-\tau)}. \quad \text{(depends on } t-\tau\text{)!}$$

For this case, we can see that e^{at} is a fundamental matrix from the Taylor series expansion

$$e^{at} = 1 + at + \frac{a^2}{2}t^2 + \frac{a^3}{3!}t^3 + \cdots$$

$$\frac{d}{dt}\left(e^{at}\right) = a + a^2 t + \frac{a^3}{2!}t^2 + \cdots$$
$$= a\left(1 + at + \frac{a^2}{2!}t^2 + \cdots\right) = ae^{at}$$

The same procedure would apply in the matrix case if the series can be shown to converge

If we take

$$\Psi(t) := I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \cdots$$

$$\begin{aligned}\frac{d}{dt}\Psi(t) &:= A + A^2 t + \frac{A^3}{2!}t^2 + \cdots \\ &= A\left(I + At + \frac{A^2}{2!}t^2 + \cdots\right) \\ &= A\,\Psi(t)\end{aligned}$$

$\therefore$ If the series expression for $\Psi(t)$ can be shown to converge, then $\Psi(t)$ will be a fundamental matrix.

<u>Fact:</u>

Let $\sum_{i=0}^{\infty} \alpha_i \lambda^i$ have a radius of convergence ρ.

If $\bar{A}$ is $n \times n$ matrix with all of its eigenvalues having absolute values less than ρ, then the series $\sum_{i=0}^{\infty} \alpha_i \bar{A}^i$ converges.

Let's apply this to our problem

$\sum_{i=0}^{\infty} \frac{1}{i!} \lambda^i$ converges for all λ. In fact $\sum_{i=0}^{\infty} \frac{1}{i!} \lambda^i = e^{\lambda}$

Applying the above fact with $\bar{A} = At$ we have that

$\sum_{i=0}^{\infty} \frac{1}{i!} (At)^i$ converges for all t.

Thus the series $\sum_{i=0}^{\infty} \frac{1}{i!}(At)^i = I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \dots$

is always well defined (for all A and for all t).

We will define it to be a "matrix exponential", i.e.

$$\boxed{e^{At} := I + At + \frac{A^2t^2}{2!} + \frac{A^3}{3!}t^3 + \dots}$$

Note that e^{At} is a <u>matrix</u>. It has the same size at A.

<u>Summary</u>: If we take $\Psi(t) = e^{At}$, then $\Psi(t)$ will be a fundamental solution of $\mathring{x} = Ax$.

Properties of matrix exponentials:

Let X be $n \times n$ matrix.

1. If $X = 0$, $e^X = I$
2. If Y is $n \times n$ matrix, then
   $e^{X+Y} = e^X \cdot e^Y$ if and only if $XY = YX$
3. $\frac{d}{dt} e^{Xt} = X e^{Xt}$
4. $X e^{Xt} = e^{Xt} X$
5. $\left(e^X\right)^{-1} = e^{-X}$

<u>Proof</u>

1. $e^X = I + X + \frac{X^2}{2!} + \cdots$. So $X = 0 \Rightarrow e^X = I$

2. Omitted

3. $$\frac{d}{dt} e^{Xt} = \frac{d}{dt}\left(I + Xt + \frac{X^2 t^2}{2!} + \cdots\right)$$
$$= X + X^2 t + \frac{X^3 t^2}{2!} + \cdots$$
$$= X\left(I + Xt + \frac{X^2 t^2}{2!} + \cdots\right)$$
$$= X e^{Xt}$$

4. Same as 3.

5. Let $Y = -X$. Clearly $XY = YX$. Apply 2.

How does one compute e^{At} ?

We have seen one method : Taylor Series

This method does not give closed-form solution.

We will look at other methods which allow us to compute any function of a square matrix.

To do this we must introduce/review some concepts from linear algebra. These are:

a) Eigenvalues and eigenvectors
b) Jordan Form of a matrix
c) Minimal polynomials, characteristic polynomials
d) Cayley-Hamilton Theorem
e) Functions of a square matrix

Eigenvalues & eigenvectors

Let A be $n\times n$ matrix whose elements are in $\mathbb{C}$.

Def: A scalar $\lambda \in \mathbb{C}$ is said to be an eigenvalue of A if there exists $x \neq 0$ in $\mathbb{C}^n$ such that $Ax = \lambda x$. Any nonzero vector satisfying this equation is said to an eigenvector corresponding to the eigenvalue λ.

Remark: Eigenvectors are not unique. If x is an eigenvector of A ~~**~~ corresponding to the eigenvalue λ, then αx is also an eigenvector corresponding to λ.

How do we find the eigenvalues?

Solve for all λ satisfying $Ax = \lambda x \qquad x \neq 0$.

Equivalently, $(\lambda I - A)x = 0$

This equation has a nontrivial solution if and only if $(\lambda I - A)$ is singular (i.e. $\det(\lambda I - A) = 0$).

<u>Conclusion:</u> The eigenvalues of A are all $\lambda \in \mathbb{C}$ satisfying $\det(\lambda I - A) = 0$.

$\lambda I - A$ is $n \times n$ matrix. <u>$\overbrace{\det(\lambda I - A)}^{\Delta(\lambda) :=}$ is a polynomial in λ.</u>

It is called the <u>characteristic polynomial of A.</u>

$\Delta(\lambda)$ has degree n. Therefore, there are n eigenvalues of A.

Example:

$A = [-2]$. Clearly -2 is the only eigenvalue since

$\det(\lambda I - A) = \det(\lambda + 2) = \lambda + 2$. Now $\lambda + 2 = 0 \Rightarrow \lambda = -2$ is an eig.

Any nonzero number is an eigenvector associated with $\lambda = -2$.

Example:

$A = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}$. $\quad \Delta(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -3 \\ 1 & \lambda \end{bmatrix}$

$= \lambda^2 - \lambda + 3$

$\therefore$ eigenvalues are $\frac{1}{2} \pm \frac{\sqrt{11}}{2} j = \lambda_{1,2}$

$\lambda_1 = \frac{1}{2} + j\frac{\sqrt{11}}{2} \Rightarrow v_1 = \begin{bmatrix} \sqrt{11} - j \\ 2j \end{bmatrix}$

$\lambda_2 = \frac{1}{2} - j\frac{\sqrt{11}}{2} \Rightarrow v_2 = \begin{bmatrix} \sqrt{11} + j \\ -2j \end{bmatrix}$

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$A x = \lambda x$. Any nonzero vector satisfying this equation is said to an eigenvector corresponding to the eigenvalue λ.

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$$= \lambda^2 - \lambda + 3$$

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$$\lambda_1 = \frac{1}{2} + j\frac{\sqrt{11}}{2} \Rightarrow v_1 = \begin{bmatrix} \sqrt{11} - j \\ 2j \end{bmatrix}$$

$$\lambda_2 = \frac{1}{2} - j\frac{\sqrt{11}}{2} \Rightarrow v_2 = \begin{bmatrix} \sqrt{11} + j \\ -2j \end{bmatrix}$$

The Jordan Form:

Let A be $n \times n$ (possibly complex) matrix.

There are 2 cases:

Case 1 A has distinct eigenvalues.

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of. Let $v_1, \ldots, v_n$ be the corresponding eigenvectors.

Theorem: The set $\{v_1, \ldots, v_n\}$ is linearly independent.

Pf: See text book.

Now $Av_i = \lambda_i v_i \quad i = 1, \ldots, n.$

This can be written in matrix form:

$$A[v_1\ v_2 \dots v_n] = [v_1\ v_2\ \cdots\ v_n]\begin{bmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{bmatrix}$$

Define $Q := [v_1\ \cdots\ v_n]$. Q is $n\times n$ nonsingular.

Define $\hat{A} := \begin{bmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{bmatrix}$

We have $AQ = Q\hat{A}$. Since Q is invertible

$\boxed{A = Q\hat{A}Q^{-1}.}$ This is called the diagonalization of A.

Note that $\hat{A} = Q^{-1}AQ$.

$\hat{A}$ has another interpretation:

It is the matrix representation of the linear operator A with respect the basis $\{v_1, \dots, v_n\}$.

<u>Case 2</u>: The eigenvalues of A are <u>not</u> distinct.

In this case there are 2 possibilities

(a) If λ_i is an eigenvalue repeated m_i times, there exists m_i linearly independent eigenvector associated with λ_i

(b) For some eigenvalue λ repeated m times, there does <u>not</u> exist m linearly independent eigenvectors associated with λ. In other words

$[\lambda I - A]$ has a null space of dimension <u>less than m</u>.

- For possibility (a) we can proceed exactly as in the case of distinct eigenvalues. $A = Q \hat{A} Q^{-1}$ where $\hat{A}$ is diagonal.
- For possibility (b) <u>there does not exist Q</u> such that $A = Q \hat{A} Q^{-1}$ with $\hat{A}$ diagonal, i.e. A cannot be diagonalized!

Let A be $n \times n$ complex matrix. Let $\lambda_1, \ldots, \lambda_m$ be eigenvalues of A with multiplicities $n_1, \ldots, n_m$ respectively.

<u>Theorem</u>: (Jordan Canonical Form)

There exists a nonsingular $n \times n$ matrix Q such that

$$A = Q \hat{A} Q^{-1}$$

where

$$\hat{A} = \begin{bmatrix} \hat{A}_1 & & \\ & \ddots & \\ & & \hat{A}_m \end{bmatrix}, \quad \text{and} \quad \hat{A}_i = \begin{bmatrix} \hat{A}_{i1} & & \\ & \ddots & \\ & & \hat{A}_{ir_i} \end{bmatrix} \quad \text{where}$$

$$\hat{A}_{ij} = \begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}.$$

Also $\bar{n}_i := \dim \hat{A}_{i1} \geq \dim \hat{A}_{i2} \geq \cdots \geq \dim \hat{A}_{ir_i}$.

$\hat{A}_{i1}, \ldots, \hat{A}_{ir_i}$ are called the Jordan blocks associated with λ_i.

Examples of basic Jordan ~~Block and~~ Matrices

$$\hat{A}_i = \left[\begin{array}{ccc|c} \lambda_i & 1 & & \\ 0 & \lambda_i & 1 & 0 \\ & & \lambda_i & \\ \hline & 0 & & \lambda_i \end{array}\right] ; \qquad \hat{A}_i = \left[\begin{array}{cc|cc} \lambda_i & 1 & & 0 \\ & \lambda_i & & \\ \hline & 0 & \lambda_i & 1 \\ & & & \lambda_i \end{array}\right]$$

$$\hat{A}_i = \left[\begin{array}{cc|c|c} \lambda_i & 1 & & 0 \\ 0 & \lambda_i & & \\ \hline & & \lambda_i & \\ \hline & 0 & & \lambda_i \end{array}\right] , \qquad \hat{A}_i = \left[\begin{array}{c|c|c|c} \lambda_i & & 0 & \\ \hline & \lambda_i & & \\ \hline & & \lambda_i & \\ \hline 0 & & & \lambda_i \end{array}\right]$$

$$\hat{A}_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \lambda_i & 1 \\ & & & \lambda_i \end{bmatrix}$$

<u>Def</u>: We define the largest dimension of the Jordan Blocks associated with λ_i to be the <u>index of λ_i</u>. It is denoted by $\bar{n}_i$.

<u>Example</u>: Suppose the characteristic poly. of A is $(\lambda-2)^4(\lambda-3)^2$

All the possible Jordan forms $\hat{A}$ are given by the following 10 matrices:

	$\bar{n}_1=4$	$\bar{n}_1=3$	$\bar{n}_1=2$	$\bar{n}_1=1$
$\bar{n}_2=2$	$\left[\begin{array}{cccc:cc} 2&1&&&&\\ &2&1&&&\\ &&2&1&&\\ &&&2&&\\ \hdashline &&&&3&1\\ &&&&&3 \end{array}\right]$	$\left[\begin{array}{ccc:c:cc} 2&1&&&&\\ &2&1&&&\\ &&2&&&\\ \hdashline &&&2&&\\ \hdashline &&&&3&1\\ &&&&&3 \end{array}\right]$	$\left[\begin{array}{cc:cc:cc} 2&1&&&&\\ &2&&&&\\ \hdashline &&2&1&&\\ &&&2&&\\ \hdashline &&&&3&1\\ &&&&&3 \end{array}\right]$	$\left[\begin{array}{c:c:c:c:cc} 2&&&&&\\ \hdashline &2&&&&\\ \hdashline &&2&&&\\ \hdashline &&&2&&\\ \hdashline &&&&3&1\\ &&&&&3 \end{array}\right]$
$\bar{n}_2=1$	$\left[\begin{array}{cccc:c:c} 2&1&&&&\\ &2&1&&&\\ &&2&1&&\\ &&&2&&\\ \hdashline &&&&3&\\ \hdashline &&&&&3 \end{array}\right]$	$\left[\begin{array}{ccc:c:c:c} 2&1&&&&\\ &2&1&&&\\ &&2&&&\\ \hdashline &&&2&&\\ \hdashline &&&&3&\\ \hdashline &&&&&3 \end{array}\right]$	$\left[\begin{array}{cc:cc:c:c} 2&1&&&&\\ &2&&&&\\ \hdashline &&2&1&&\\ &&&2&&\\ \hdashline &&&&3&\\ \hdashline &&&&&3 \end{array}\right]$	$\left[\begin{array}{c:c:c:c:c:c} 2&&&&&\\ \hdashline &2&&&&\\ \hdashline &&2&&&\\ \hdashline &&&2&&\\ \hdashline &&&&3&\\ \hdashline &&&&&3 \end{array}\right]$

and

$$\bar{n}_1 = 2$$

$$\bar{n}_2 = 2 \quad \begin{bmatrix} 2 & 1 & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & 2 & & \\ & & & & 3 & 1 \\ & & & & & 3 \end{bmatrix}$$

$$\bar{n}_2 = 1 \quad \begin{bmatrix} 2 & 1 & & & & \\ & 2 & & & & \\ & & 2 & & & \\ & & & 2 & & \\ & & & & 3 & \\ & & & & & 3 \end{bmatrix}$$

Functions of a Square Matrix

Let $f(\lambda)$ be polynomial in λ. Let A be $n \times n$ matrix.

Then $f(A)$ is well defined.

For example if $f(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_0$

then $f(A) = A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_0 I$

Let $A = Q\hat{A}Q^{-1}$ where $\hat{A}$ is the Jordan form

$$f(A) = (Q\hat{A}Q^{-1})^n + \alpha_{n-1}(Q\hat{A}Q^{-1})^{n-1} + \cdots + \alpha_0 I$$

$$= Q(\hat{A}^n + \alpha_{n-1}\hat{A}^{n-1} + \cdots + \alpha_0 I)Q^{-1}$$

$$= Q(f(\hat{A}))Q^{-1}$$

$f(\hat{A})$ is easier to compute than $f(A)$ since $f(\hat{A}) = \begin{bmatrix} f(\hat{A}_1) & & \\ & \ddots & \\ & & f(\hat{A}_m) \end{bmatrix}$

<u>Example:</u> Let $A = \begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix}$. Compute $f(A)$ if $f(\lambda) = \lambda^2 + 2$

$$A = \begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} -2 & \\ & -3 \end{bmatrix}}_{\hat{A}} \underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_{Q^{-1}}$$

$$f(A) = Q\, f(\hat{A})\, Q^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f(-2) & 0 \\ 0 & f(-3) \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 0 & 11 \end{bmatrix}$$

By direct calculation: $f(A) = A^2 + 2I = \begin{bmatrix} 4 & 5 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 0 & 11 \end{bmatrix}$

<u>Def</u>: The <u>minimal polynomial</u> of a matrix A is the monic polynomial $\psi(\lambda)$ of the smallest degree such that $\psi(A) = 0$.

<u>Remark</u>: "Monic" means the coefficient of the highest power is 1. Examples of monic polynomials: $\lambda^3 - 4$, $\lambda^2 - 3\lambda + 1$, ...

- The above definition implies that the minimal polynomial is unique. This is in fact true! (We will see why)

- The definition also seems to imply that given any matrix A there exists a polynomial such that $p(A) = 0$. This is indeed the case. (We shall prove this as well).

We begin by finding the minimal polynomial of a Jordan block.

<u>Lemma:</u> Let $J = \begin{bmatrix} \bar{\lambda} & 1 & & \\ & \bar{\lambda} & \ddots & \\ & & \ddots & 1 \\ & & & \bar{\lambda} \end{bmatrix}$ be $\bar{n} \times \bar{n}$. Then

a) $\psi(\lambda) := (\lambda - \bar{\lambda})^{\bar{n}}$ satisfies $\psi(J) = 0$

b) The is no polynomial $f(\lambda)$ with degree less than $\bar{n}$ such that $f(J) = 0$

c) If $f(\lambda)$ is a polynomial such that $f(J) = 0$, then

$$f(\lambda) = \psi(\lambda) \cdot h(\lambda) \quad \text{where } h(\lambda) \text{ is a polynomial.}$$

(In other words, $f(J) = 0 \Rightarrow \psi \mid f$)

<u>Comments:</u> This lemma says that $\psi(\lambda)$ is a minimal polynomial of J. Part(c) implies it is unique. Why? If $f(\lambda)$ is also a minimal polynomial, then $f(\lambda) = \psi(\lambda)\, h(\lambda)$. Minimality implies $h(\lambda) = 1$.

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<u>Proof:</u> (a) $\Psi(J) = (J - \bar{\lambda} I)^{\bar{n}} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}^{\bar{n}}$

Note that $\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}^{2} = \begin{bmatrix} 0 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & \ddots & 0 \\ & & & & 0 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}^{3} = \begin{bmatrix} 0 & 0 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & 1 \\ & & & \ddots & \ddots & 0 \\ & & & & \ddots & 0 \\ & & & & & 0 \end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}^{\bar{n}-1} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ & \ddots & & 0 \\ & & \ddots & \vdots \\ & & & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}^{\bar{n}} = O$$

(b)

$$J = \begin{bmatrix} * & * & & & \\ & * & * & & \\ & & \ddots & \ddots & \\ & & & & * \\ & & & & * \end{bmatrix}, \quad J^2 = \begin{bmatrix} * & * & * & & & \\ & * & * & * & & \\ & & * & * & \ddots & \\ & & & \ddots & \ddots & * \\ & & & & & * \\ & & & & & * \end{bmatrix}, \quad J^3 = \begin{bmatrix} * & * & * & * & & \\ & * & * & * & * & \\ & & * & * & \ddots & * \\ & & & \ddots & \ddots & * \\ & & & & & * \\ & & & & & * \end{bmatrix}$$

$$J^{\bar{n}-1} = \begin{bmatrix} * & * & * & \cdots & * \\ & * & * & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & & * \\ & & & & * \end{bmatrix}$$

Clearly $\alpha_{\bar{n}-1} J^{\bar{n}-1} + \alpha_{\bar{n}-2} J^{\bar{n}-2} + \cdots + \alpha_0 I = \mathbf{0}$ <u>cannot</u> be satisfied for any $\alpha_0, \ldots, \alpha_{\bar{n}-1}$. Hence, there does not exist a poly. $f(\lambda)$ with degree less that $\bar{n}$ such that $f(J) = 0$. This proves part (b).

(c) Let $f(\lambda)$ be a poly. such that $f(J) = 0$. There exists poly. $h(\lambda)$ and $g(\lambda)$ such that $f(\lambda) = \Psi(\lambda) h(\lambda) + g(\lambda)$ where $\deg g(\lambda) < \deg \Psi(\lambda)$. Now $g(J) = f(J) - \Psi(J) h(J) = 0$. This implies that $g(\lambda) = 0$.

<u>Theorem</u>: Let A be $n \times n$ matrix with eigenvalues $\lambda_1, \dots, \lambda_m$ having indices $\bar{n}_1, \dots, \bar{n}_m$ respectively. The minimal polynomial of A is $\Psi(\lambda) = (\lambda - \lambda_1)^{\bar{n}_1} \cdot (\lambda - \lambda_2)^{\bar{n}_2} \cdot \dots \cdot (\lambda - \lambda_m)^{\bar{n}_m}$

<u>Proof</u>: Let $A = Q\hat{A}Q^{-1}$ where $\hat{A}$ is the Jordan form of A.

$\Psi(A) = Q\Psi(\hat{A})Q^{-1}$. *Hence $\Psi(\lambda)$ is a minimal poly. of A iff it is the minimal poly. of $\hat{A}$.*

We now show that $\Psi(\lambda)$ is the minimal poly. of $\hat{A}$.

$$\Psi(\hat{A}) = \begin{bmatrix} \Psi(\hat{A}_1) & & \\ & \ddots & \\ & & \Psi(\hat{A}_m) \end{bmatrix}, \text{ and } \Psi(\hat{A}_i) = \begin{bmatrix} \Psi(\hat{A}_{i1}) & & \\ & \ddots & \\ & & \Psi(\hat{A}_{ir_i}) \end{bmatrix}$$

By the previous lemma $\Psi(\hat{A}_{i1}) = 0$. Similarly, $\Psi(\hat{A}_{ij}) = 0$.

$\therefore \Psi(\hat{A}) = 0$. *How do we know $\Psi(\lambda)$ is the minimal poly. ?*

Let $f(\lambda)$ be the minimal poly. of $\hat{A}$. Clearly $f(\hat{A}_{i1})=0$

But $(\lambda-\lambda_i)^{\bar{n}_i}$ is the minimal poly. of $\hat{A}_{i1}$ (by lemma).

Hence $(\lambda-\lambda_i)^{\bar{n}_i}$ must be a factor of $f(\lambda)$.

This means that $\psi(\lambda) = (\lambda-\lambda_1)^{\bar{n}_1} \cdots\cdots (\lambda-\lambda_m)^{\bar{n}_m}$

is a factor of the minimal poly. of $\hat{A}$.

Since $\psi(\hat{A})=0$ then $\psi(\lambda)$ $\underline{\underline{\text{is}}}$ the minimal poly. of $\hat{A}$.

Q.E.D.

<u>Examples:</u>

a) $\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ has the minimal poly. $(\lambda+2)^2(\lambda+1)^2$

b) $\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ has the minimal poly. $(\lambda+2)^3$

Examples (cont.)

c) $\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & & 0 \end{bmatrix}$ has minimal poly. $(\lambda+1)(\lambda-1)^3(\lambda)$

d) $$\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{\hat{A}} \underbrace{\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}}_{Q^{-1}}$$

$\therefore$ A has minimal poly. $(\lambda-1)^2(\lambda-2)$

e) $\begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$ has minimal polynomial $(\lambda-3)^2$

<u>Theorem</u>: (Cayley - Hamilton Theorem)

Every $n \times n$ matrix A satisfies its own characteristic equation, i.e. if $\Delta(\lambda) := \det(\lambda I - A)$, then $\Delta(A) = 0$.

<u>Pf</u>: Let $\Psi(\lambda) = (\lambda-\lambda_1)^{\bar{n}_1}(\lambda-\lambda_2)^{\bar{n}_2}\cdots(\lambda-\lambda_m)^{\bar{n}_m}$ be the minimal polynomial of A. Clearly, $\Psi(\lambda)$ is a factor of $\Delta(\lambda)$

since $\Delta(\lambda) = c(\lambda-\lambda_1)^{n_1}(\lambda-\lambda_2)^{n_2}\cdots(\lambda-\lambda_m)^{n_m}$, and $\bar{n}_i \le n_i$.

$\therefore \Delta(\lambda) = \Psi(\lambda)\, h(\lambda)$ for some $h(\lambda)$.

It follows that $\Delta(A) = \Psi(A)\, h(A) = 0$

Q.E.D.

<u>Example</u>: $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$. $\Delta(\lambda) = \det(\lambda I - A) = \lambda^3 - 2\lambda^2 + 3$

$$\Delta(A) = A^3 - 2A^2 + 3I = \begin{bmatrix} 1 & 8 & -2 \\ -2 & -1 & -2 \\ 2 & 4 & -1 \end{bmatrix} - 2 \cdot \begin{bmatrix} 2 & 4 & -1 \\ -1 & 1 & -1 \\ 1 & 2 & 1 \end{bmatrix} + 3 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

<u>Theorem</u>: Let A be $n \times n$ matrix having eigenvalues $\lambda_1, \dots, \lambda_m$ with indices $\bar{n}_1, \dots, \bar{n}_m$ respectively. Let $\Delta(\lambda) = \det(\lambda I - A) = \prod_{i=1}^{m} (\lambda - \lambda_i)^{n_i}$ so that $n_1, \dots, n_m$ are the algebraic multiplicities of $\lambda_1, \dots, \lambda_m$.

Then if $f(\lambda)$ and $g(\lambda)$ are two polynomials, the following holds

1. $f(A) = g(A) \iff f^{(l)}(\lambda_i) = g^{(l)}(\lambda_i) \quad \begin{array}{l} l = 0, \dots, \bar{n}_i - 1 \\ i = 1, \dots, m \end{array}$

2. $f(A) = g(A) \Leftarrow f^{(l)}(\lambda_i) = g^{(l)}(\lambda_i) \quad \begin{array}{l} l = 0, \dots, n_i - 1 \\ i = 1, \dots, m \end{array}$

<u>Proof</u>: (1) $f(A) = \cancel{f}g(A) \iff (f-g)(A) = 0 \iff (f-g)(\lambda) = \underbrace{\prod_{i=1}^{m} (\lambda - \lambda_i)^{\bar{n}_i}}_{\Psi(\lambda)} h(\lambda)$

$\iff (f-g)^{(l)}(\lambda_i) = 0 \quad \begin{array}{l} l = 0, \dots, \bar{n}_i - 1 \\ i = 1, \dots, m \end{array}$

(2) The proof of (2) follows from the fact that $n_i \geq \bar{n}_i$ and from part (1).

Q.E.D.

<u>Terminology:</u> Two functions f and g are said to agree on the spectrum of A if

$$f^{(\ell)}(\lambda_i) = g^{(\ell)}(\lambda_i) \qquad \begin{array}{l} \ell = 0, \dots, n_i - 1 \\ i = 1, \dots, m \end{array}$$

<u>Implications of theorem:</u>

Given a polynomial $f(\lambda)$ of a degree higher than $n-1$. Then we can always find a polynomial $g(\lambda)$ of order $n-1$ such that $g(A) = f(A)$.

How do we find $g(\lambda)$?

Pick any $n-1$ order polynomial which agrees with $f(\lambda)$ on the spectrum of A. This polynomial is the desired $g(\lambda)$.

Examples:

Let $A = \begin{bmatrix} 2 & 6 \\ 0 & -3 \end{bmatrix}$. Find $A^4 + 3A^2 - 2A + 3I$.

Let $f(\lambda) = \lambda^4 + 3\lambda^2 - 2\lambda + 3$, we want to find $f(A)$.

$n = 2$, therefore we pick $g(\lambda) = \alpha_1 \lambda + \alpha_0$

Spectrum of A: $\lambda_1 = 2$, $\lambda_2 = -3$

We want to pick α_1, α_0 so that $f(\lambda)$ and $g(\lambda)$ would agree on the spectrum of A.

$$\left.\begin{aligned} g(2) &= 2\alpha_1 + \alpha_0 = f(2) = 27 \\ g(-3) &= -3\alpha_1 + \alpha_0 = f(-3) = 117 \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha_1 &= -18 \\ \alpha_0 &= 63 \end{aligned}$$

$$\therefore f(A) = g(A) = -18A + 63 \cdot I = \begin{bmatrix} 27 & -108 \\ 0 & 117 \end{bmatrix}.$$

By direct calculation: $\begin{bmatrix} 2 & 6 \\ 0 & -3 \end{bmatrix}^4 + 3\begin{bmatrix} 2 & 6 \\ 0 & -3 \end{bmatrix}^2 - 2\begin{bmatrix} 2 & 6 \\ 0 & -3 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 27 & -108 \\ 0 & 117 \end{bmatrix}$

We've seen that $A^4 + 3A^2 - 2A + 3I = \underbrace{-18A + 63I}_{n-1}$.

$\boxed{\text{We can express any polynomial of } n \times n \text{ matrix as an } n-1 \text{ order poly. of that matrix.}}$

There's another way to see this fact using Cayley-Hamilton theorem directly:

The characteristic poly of A is $\Delta(\lambda) = (\lambda - 2)(\lambda + 3) = \lambda^2 + \lambda - 6$

By Cayley-Hamilton, $A^2 + A - 6I = 0$, which implies $\underline{\underline{A^2 = -A + 6I}}$

Now $A^4 = A^2 \cdot A^2 = A^2(-A + 6I) = A(-A^2 + 6A) = A(-(-A + 6I) + 6A)$

$= A(7A - 6I) = 7A^2 - 6A = 7(-A + 6I) - 6A = -13A + 42I$

$A^2 = -A + 6I$

$\therefore A^4 + 3A^2 - 2A + 3I = (-13A + 42I) + 3(-A + 6I) - 2A + 3I$

$= -18A + 63I$

Functions of a Square Matrix

Let A be $n \times n$ matrix. Let $f : X \to \mathbb{C}$ where $X \subset \mathbb{C}$ contains the spectrum of A. Let $g(\lambda)$ be any polynomial which agrees with $f(\lambda)$ on the spectrum of A. Then we define $f(A) := g(A)$.

Remark: An $n-1$ order poly. $g(\lambda)$ which agrees with $f(\lambda)$ on the spectrum of A can always be found. Therefore for every function $f(\lambda)$ which is defined on spectrum of A we have $f(A) = \alpha_{n-1} A^{n-1} + \cdots + \alpha_0 I$ for some $\alpha_0, \ldots, \alpha_{n-1}$

Example:

Let $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. Find e^A.

$f(\lambda) = e^{\lambda}$ and we want to find $f(A)$.

The spectrum of A is $\lambda_1 = -1$, $\lambda_2 = 2$

We can find a 1st order polynomial which agrees with $f(\lambda)$ on the spectrum.

Let $g(\lambda) = \alpha_0 + \alpha_1 \lambda$

To find α_0 and α_1:

$$\left.\begin{aligned} g(\lambda_1) &= \alpha_0 + \alpha_1(-1) = f(\lambda_1) = e^{-1} \\ g(\lambda_2) &= \alpha_0 + \alpha_1(2) = f(\lambda_2) = e^{2} \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha_0 &= \frac{2e^{-1}+e^{2}}{3} \\ \alpha_1 &= \frac{e^{2}-e^{-1}}{3} \end{aligned}$$

$$\therefore f(A) = g(A) = \frac{2e^{-1}+e^{2}}{3} I + \frac{e^{2}-e^{-1}}{3} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} e^{-1} & 0 \\ 0 & e^{2} \end{bmatrix}$$

This is the same answer we get as when we use the series definition.

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Let A be $n \times n$ matrix. Let $f : X \to \mathbb{C}$ where $X \subset \mathbb{C}$ contains the spectrum of A. Let $g(\lambda)$ be any polynomial which agrees with $f(\lambda)$ on the spectrum of A. Then we define $f(A) := g(A)$.

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Example :

Let $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$. Find e^A.

$f(\lambda) = e^{\lambda}$ and we want to find $f(A)$.

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Let $g(\lambda) = \alpha_0 + \alpha_1 \lambda$

To find α_0 and α_1 :

$$\left.\begin{aligned} g(\lambda_1) &= \alpha_0 + \alpha_1(-1) = f(\lambda_1) = e^{-1} \\ g(\lambda_2) &= \alpha_0 + \alpha_1(2) = f(\lambda_2) = e^{2} \end{aligned}\right\} \Rightarrow \quad \begin{aligned} \alpha_0 &= \frac{2e^{-1} + e^2}{3} \\ \alpha_1 &= \frac{e^2 - e^{-1}}{3} \end{aligned}$$

$$\therefore f(A) = g(A) = \frac{2e^{-1} + e^2}{3} I + \frac{e^2 - e^{-1}}{3} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} e^{-1} & 0 \\ 0 & e^2 \end{bmatrix}$$

This is the same answer we get as when we use the series definition.

<u>Example</u>: Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Find e^{At}.

Let $f(\lambda) = e^{\lambda t}$. We need to find $f(A)$.

$\Delta(\lambda) = \lambda^2(\lambda - 1)$. We need to find a <u>2nd order</u> polynomial which agrees with $f(\lambda)$ on the spectrum of A. Let $g(\lambda) = \alpha_0 + \alpha_1 \lambda + \alpha_2 \lambda^2$

$$\left.\begin{aligned} g(1) &= \alpha_0 + \alpha_1 + \alpha_2 = f(1) = e^t \\ g(0) &= \alpha_0 = f(0) = 1 \\ \frac{d}{d\lambda} g(0) &= \alpha_1 + 2\alpha_2(0) = \frac{df}{d\lambda}(0) = t e^{\lambda t}\Big|_{\lambda=0} = t \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha_0 &= 1 \\ \alpha_1 &= t \\ \alpha_2 &= e^t - t - 1 \end{aligned}$$

$$f(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + t \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (e^t - t - 1) \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} e^t & 1 - e^t & 2(e^t - 1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Example:</u> Let $f(\lambda) = \frac{1}{\lambda}$. Compute $f(A)$ if A is a Jordan Block.

Let $A = \begin{bmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_1 \end{bmatrix}$. The characteristic poly. is $(\lambda - \lambda_1)^n$

Let $g(\lambda)$ be $n-1$ th order polynomial. Instead of expressing $g(\lambda)$ as $\alpha_0 + \cdots + \alpha_{n-1}\lambda^{n-1}$ it is more convenient to express it as

$$g(\lambda) = \alpha_0 + \alpha_1(\lambda - \lambda_1) + \alpha_2(\lambda - \lambda_1)^2 + \cdots + \alpha_{n-1}(\lambda - \lambda_1)^{n-1}$$

<u>Note that:</u> $g(\lambda_1) = \alpha_0$

$$\frac{dg}{d\lambda}(\lambda_1) = \alpha_1 \cdot (1!)$$

$$\vdots$$

$$\frac{d^{n-1}g}{d\lambda}(\lambda_1) = \alpha_{n-1} \cdot (n-1)!$$

$$g(\lambda_1) = (\alpha_0)\cdot(0!) = f(\lambda_1) = \frac{1}{\lambda_1}$$

$$\frac{dg}{d\lambda}(\lambda_1) = (\alpha_1)(1!) = \frac{df}{d\lambda}(\lambda_1) = \frac{-1}{\lambda_1^2}$$

$$\frac{d^2 g}{d\lambda^2}(\lambda_1) = (\alpha_2)(2!) = \frac{d^2 f}{d\lambda^2}(\lambda_1) = \frac{2!}{\lambda_1^3}$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{d^{n-1}}{d\lambda^{n-1}}(\lambda_1) = (\alpha_{n-1})[(n-1)!] = \frac{d^{n-1} f}{d\lambda^{n-1}}(\lambda_1) = \frac{(-1)^{n-1}\cdot(n-1)!}{\lambda^n}$$

$$\therefore \quad \alpha_0 = \frac{1}{\lambda_1}, \quad \alpha_1 = \frac{-1}{\lambda_1^2}, \quad \dots, \quad \alpha_{n-1} = \frac{(-1)^{n-1}}{\lambda^n}$$

$$f(A) = g(A) = \frac{1}{\lambda_1} I + \frac{-1}{\lambda_1^2}(A - \lambda_1 I) + \frac{1}{\lambda_1^3}(A - \lambda_1 I)^2 + \dots + \frac{(-1)^{n-1}}{\lambda^n}(A - \lambda_1 I)^{n-1}$$

$$= \begin{bmatrix} \frac{1}{\lambda_1} & \frac{-1}{\lambda_1^2} & \frac{1}{\lambda_1^3} & \cdots & \frac{(-1)^{n-1}}{\lambda_1^n} \\ & \frac{1}{\lambda_1} & \frac{-1}{\lambda_1^2} & \frac{1}{\lambda_1^3} \cdots & \frac{(-1)^{n-2}}{\lambda_1^{n-1}} \\ & & \frac{1}{\lambda_1} & \ddots & \vdots \\ & & & & \frac{1}{\lambda} \end{bmatrix}.$$

This is exactly A^{-1}!

$$\therefore f(A) = A^{-1} \text{ when } f(\lambda) = \frac{1}{\lambda}.$$

Important Observation:

From the previous example we see that if $\hat{A}$ is a Jordan block of the form: $\begin{bmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & 1 & \\ & & \ddots & 1 \\ & & & \lambda_1 \end{bmatrix}$, then $f(\hat{A})$ is given by:

$$f(\hat{A}) = \begin{bmatrix} f(\lambda_1) & \frac{f'(\lambda_1)}{1!} & \cdots & \frac{f^{(n-1)}(\lambda_1)}{(n-1)!} \\ 0 & f(\lambda_1) & \cdots & \frac{f^{(n-2)}(\lambda_1)}{(n-2)!} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & f(\lambda_1) \end{bmatrix}$$

We will use this observation to compute $e^{\hat{A}t}$ when $\hat{A}$ is a Jordan block. In this case, $f(\lambda) = e^{\lambda t}$

$$f(\lambda_1) = e^{\lambda_1 t}, \quad f'(\lambda_1) = te^{\lambda_1 t}, \dots, f^{(n-1)}(\lambda_1) = t^{n-1} e^{\lambda_1 t}$$

Therefore,
$$f(\hat{A}) = e^{\hat{A}t} = \begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & \cdots & t^{n-1}\frac{e^{\lambda_1 t}}{(n-1)!} \\ & e^{\lambda_1 t} & \cdots & t^{n-2}\frac{e^{\lambda_1 t}}{(n-2)!} \\ & & \ddots & \vdots \\ & & & e^{\lambda_1 t} \end{bmatrix}$$

This implies that for any matrix $A = Q\hat{A}Q^{-1}$, entries of A ~~is~~ are linear combinations of terms of the form:

$$e^{\lambda_1 t}, e^{\lambda_2 t}, \dots, e^{\lambda_n t}, te^{\lambda_1 t}, \dots, te^{\lambda_n t}, \dots, t^{n-1}e^{\lambda_1 t}, \dots, t^{n-1}e^{\lambda_n t}$$

Back to $\begin{cases} \dot{x} = Ax + Bu \\ x(t_0) = x_0 \end{cases}$

Let's look at the zero input response:

$$x(t) = \phi(t; 0, x_0, 0) = e^{At} x_0$$

We would expect $x(t)$ to be a linear combination of terms of the form:

$$e^{\lambda_1 t}, e^{\lambda_2 t}, \ldots, e^{\lambda_n t}, t e^{\lambda_1 t}, \ldots, t^{n-1} e^{\lambda_n t} \quad \text{(Modes)}$$

What is the relative weight of each of these terms?

This brings us to the topic of modal decomposition ....

Modal Decomposition:

$x(t) = e^{At} x_0$. We explore the case when A is diagonalizable.

Let Q be the $n \times n$ matrix consisting of the eigenvectors corresponding with $\lambda_1, \dots, \lambda_n$. i.e.

$$Q = [q_1 \cdots q_n] \quad \text{where} \quad A q_i = \lambda_i q_i$$

Q is invertible. Now $A = Q \hat{A} Q^{-1}$ where

$$\hat{A} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

$$e^{At} x_0 = Q e^{\hat{A}t} Q^{-1} x_0 = Q \begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{bmatrix} Q^{-1} x_0$$

If we define $Q^{-1} x_0 =: x_0'$ then

$$e^{At}x_0 = [q_1 \cdots q_n]\begin{bmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{bmatrix} x_0' \quad , \quad \text{equivalently}$$

$$\boxed{e^{At}x_0 = x_{01}'\left(q_1 e^{\lambda_1 t}\right) + x_{02}'\left(q_2 e^{\lambda_2 t}\right) + \cdots + x_{0n}'\left(q_n e^{\lambda_n t}\right)}$$

This is the modal decomposition of the state-response.

<u>Remarks</u>: 1. $x_0' = Q^{-1}x_0$ $\left(x_0 = Q x_0'\right)$. This means that x_0' is the representation of x_0 with respect to the basis $\{q_1, \cdots, q_n\}$.

2. If A has repeated eigenvalules, the modes will have terms like $t^k e^{\lambda_i t}$.

<u>Example:</u> Given $\begin{bmatrix} \overset{\circ}{x}_1 \\ \overset{\circ}{x}_2 \\ \overset{\circ}{x}_3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 4 & 0 & 1 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $X(0) = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Express the solution $x(t)$ in terms of its 3 modes.

$\lambda_1 = 3.732$, $\lambda_2 = 0.2679$, $\lambda_3 = 1$

$q_1 = \begin{bmatrix} 0.6321 \\ 0.5247 \\ -0.5701 \end{bmatrix}$, $q_2 = \begin{bmatrix} 0.1003 \\ -0.785 \\ -0.6113 \end{bmatrix}$, $q_3 = \begin{bmatrix} 0 \\ 0.707 \\ 0.707 \end{bmatrix}$

$x_0 = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} x_0'$, $x_0' = \begin{bmatrix} -0.579 \\ 13.6259 \\ 14.1421 \end{bmatrix}$

$$x(t) = -0.579 \begin{bmatrix} 0.6321 \\ 0.5247 \\ -0.5701 \end{bmatrix} e^{3.732t} + 13.6259 \begin{bmatrix} 0.1003 \\ -0.785 \\ -0.6113 \end{bmatrix} e^{0.2679t} + 14.1421 \begin{bmatrix} 0 \\ 0.707 \\ 0.707 \end{bmatrix} e^{t}$$

For LTI systems of the form $\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$ $\quad x(0) = x_0$

we could have taken an alternative approach .... "Frequency Domain"

Let's take the Laplace transform of the state equation and try to solve for x and y.

$$sX(s) - x(0) = A\,X(s) + B\,U(s)$$

$$\Rightarrow (sI - A)\,X(s) = x(0) + B\,U(s)$$

$$X(s) = \underbrace{(sI-A)^{-1}x(0)}_{\substack{\text{zero-input} \\ \text{solution}}} + \underbrace{(sI-A)^{-1}B\,U(s)}_{\substack{\text{zero-state} \\ \text{solution}}}$$

$$\mathcal{L}\{e^{At}x_0\} \qquad\qquad \mathcal{L}\left\{\int_0^t e^{A(t-\tau)}B\,u(\tau)\,d\tau\right\}$$

This analysis implies that

$$\boxed{\mathcal{L}\{e^{At}\} = (sI-A)^{-1}}$$

Yet another way to compute e^{At} !

Remark: $(sI-A)^{-1}$ is always strictly proper.

The output equations:

$$y(t) = Cx(t) + Du(t)$$

$$Y(s) = CX(s) + DU(s)$$

$$= C\left((sI-A)^{-1}x_0 + (sI-A)^{-1}BU(s)\right) + DU(s)$$

$$\boxed{Y(s) = \underbrace{C(sI-A)^{-1}x_0}_{\text{zero input solution}} + \underbrace{\left[C(sI-A)^{-1}B + D\right]U(s)}_{\text{zero-state solution}}}$$

Note: $C(sI-A)^{-1}B + D$ is the system transfer function

Equivalent Dynamical Systems

<u>Given:</u>
$$\begin{aligned} \mathring{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad x(0) = x_0 \qquad (\Sigma)$$

Define $\bar{x} := Q^{-1}x$ where Q is any $n \times n$ matrix (invertible)

Then
$$\begin{aligned} Q\mathring{\bar{x}} &= AQ\bar{x} + Bu \\ y &= CQ\bar{x} + Du \end{aligned}$$

$$\Rightarrow \quad \begin{aligned} \mathring{\bar{x}} &= Q^{-1}AQ\,\bar{x} + Q^{-1}Bu \\ y &= CQ\,\bar{x} + Du \end{aligned}$$

Defining:
$$\begin{aligned} \bar{A} &= Q^{-1}AQ, & \bar{B} &= Q^{-1}B \\ \bar{C} &= CQ, & \bar{D} &= D \end{aligned}$$

we have
$$\begin{aligned} \mathring{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} + \bar{D}u \end{aligned} \qquad \bar{x}_0 := Q^{-1}x_0 = \bar{x}(0) \qquad (\bar{\Sigma})$$

System (Σ) and system $(\bar{\Sigma})$ are equivalent.

Q is called an equivalence transformation.

<u>From input-output view point</u>: Two equivalent dynamical equations give the same zero-state response. i.e.

The impulse response and therefore the transfer function remain unchanged by an equivalence transformation.

To prove this: The impulse response is given by:

$$G(t) = Ce^{At}B + D\delta(t)$$

$$\begin{aligned} \bar{G}(t) = \bar{C}e^{\bar{A}t}\bar{B} + \bar{D}\delta(t) &= (CQ)e^{Q^{-1}AQt}(Q^{-1}B) + D\delta(t) \\ &= CQQ^{-1}e^{At}QQ^{-1}B + D\delta(t) \\ &= Ce^{At}B + D\delta(t) \\ &= G(t) \end{aligned}$$

We can also arrive at the same conclusion from the transfer function.

$$
\begin{aligned}
\bar{G}(s) = \bar{C}(sI-\bar{A})^{-1}\bar{B} + \bar{D} &= CQ(sQ^{-1}Q - Q^{-1}AQ)^{-1}Q^{-1}B + D \\
&= CQ\left[Q^{-1}(sI-A)Q\right]^{-1}Q^{-1}B + D \\
&= CQQ^{-1}(sI-A)^{-1}QQ^{-1}B + D \\
&= C(sI-A)^{-1}B + D \\
&= G(s)
\end{aligned}
$$

<u>From an internal view point:</u>

For any initial condition for a system (Σ), there exists an initial condition for the equivalent system $(\bar{\Sigma})$ such that the outputs due to the initial conditions are identical.

For example: let (Σ): $\dot{x} = Ax + Bu$ $\quad x(0) = x_0$
$y = Cx + Du$

Let $(\bar{\Sigma})$: $\quad \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} + \bar{D}u \end{aligned} \qquad \bar{x}(0) = \bar{x}_0$

If we take $\bar{x}_0$ to be $Q^{-1}x_0$, then the zero-input response for $\bar{\Sigma}$ is equal to the zero-input response of Σ.

Indeed,

For $\bar{\Sigma}$: $\quad y(t) = \bar{C}e^{\bar{A}t}\bar{x}_0 = CQ\,e^{Q^{-1}\bar{A}Qt}\,Q^{-1}x_0$

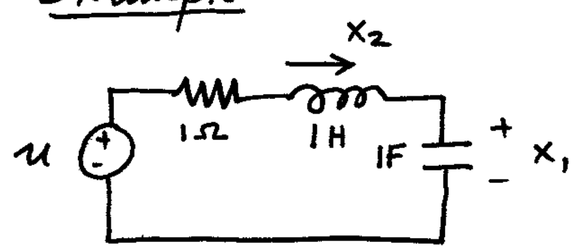
$$= CQ\,Q^{-1}e^{At}Q\,Q^{-1}x_0$$

$$= Ce^{At}x_0 = \text{zero-input response of } \Sigma.$$

Why do we care about equivalent dynamical equations?

1. Equivalent dynamical equations arise when the <u>same</u> <u>system</u> is described using different state variables.

<u>Example</u>:



$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_2 + u \\ y &= x_1 \end{aligned}\right\} \Rightarrow \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0], D = 0$$

Suppose for the same system we use the following state variables:

$\bar{x}_1 :=$ capacitor voltage + resistor voltage

$\bar{x}_2 :=$ resistor voltage

$$\left.\begin{aligned} \dot{\bar{x}}_1 &= \bar{x}_2 + (u - \bar{x}_1) \\ \dot{\bar{x}}_2 &= u - \bar{x}_1 \end{aligned}\right\} \Rightarrow \quad \bar{A} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \bar{C} = [1 \quad -1], \bar{D} = 0$$

$$y = \bar{x}_1 - \bar{x}_2$$

$$\Sigma: \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad \text{and} \quad \bar{\Sigma}: \begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x} + \bar{D}u \end{cases}$$

are two descriptions of the <u>same</u> <u>system</u>.

They are equivalent description, Indeed if we take

$$x = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \bar{x}$$ and substitute for x in equations (Σ)

we get the equations for $\bar{x}$ given by $(\bar{\Sigma})$.

$$\dot{\bar{x}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \bar{x} + 0u$$

$$\therefore \left.\begin{aligned} \dot{\bar{x}} &= \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & -1 \end{bmatrix} \bar{x} + 0u \end{aligned}\right\}$$ These equations are identical to $(\bar{\Sigma})$.

2. An equivalent system description may give more insight about those properties which are preserved by the equivalence transformations.

e.g. given
$$\mathring{x} = Ax + Bu$$
$$y = Cx$$

Let Q be chosen so that $A = Q\hat{A}Q^{-1}$ and $\hat{A}$ is diagonal.

Defining $\bar{x} = Q^{-1}x$ we have

$$\mathring{\bar{x}} = \underbrace{Q^{-1}AQ}_{\hat{A}}\,\bar{x} + Q^{-1}Bu \quad \Longrightarrow \quad \mathring{\bar{x}} = \hat{A}\bar{x} + \bar{B}u$$

(This is easier to solve)

Once we solve for $\bar{x}(t)$, $x(t) = Q\,\bar{x}(t)$.

We've seen that if two dynamical equations are equivalent they will be input-output equivalent (i.e. zero-state equivalent)

Does this also work the other way around? No!

Consider $\Sigma := \begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$ $\qquad \bar{\Sigma} := \begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ y &= \bar{C}\bar{x} \end{aligned}$

where $\bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ B' \end{bmatrix}$

$\bar{C} = [C \quad 0']$

Clearly, $C(sI-A)^{-1}B = [C \quad 0'] \begin{bmatrix} sI-A & 0 \\ 0 & sI \end{bmatrix}^{-1} \begin{bmatrix} B \\ B' \end{bmatrix}$

So Σ & $\bar{\Sigma}$ are zero-state equivalent (have the same transfer function)

yet Σ and $\bar{\Sigma}$ are <u>not</u> equivalent state equations. Why?

We have seen that:

1. Equivalent state-space systems have the same transfer function
2. State space systems which are not equivalent (and may not even have the same dimension) could have the same transfer function.

<u>Question 1:</u> Given two state-space systems Σ and $\bar{\Sigma}$ with realizations $\begin{matrix} \dot{x} = Ax + Bu \\ y = Cx + Du \end{matrix}$ and $\begin{matrix} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x} + \bar{D}u \end{matrix}$ respectively.

When will these two systems have the same transfer function?

<u>Question 2:</u> Given a transfer function description, how can we get at least one state-space description which gives that transfer function?

We start by answering the first question:

<u>Theorem:</u> Using the previous notation suppose Σ has dimension n and $\bar{\Sigma}$ has dimension $\bar{n}$. Then Σ and $\bar{\Sigma}$ have the same input-output transfer function if and only if

(a) $D = \bar{D}$

(b) $CA^iB = \bar{C}\bar{A}^i\bar{B} \qquad i = 0, \ldots$

<u>Proof:</u>

Transfer function of $\Sigma = C(sI-A)^{-1}B + D$

$$= C\left(\frac{1}{s}I + \frac{A}{s^2} + \frac{A^2}{s^3} + \cdots\right)B + D$$

$$= \frac{1}{s}CB + \frac{1}{s^2}CAB + \frac{1}{s^3}CA^2B + \cdots + D$$

Transfer function of $\bar{\Sigma} = \bar{C}(sI-\bar{A})^{-1}\bar{B} + \bar{D}$

$$= \frac{1}{s}\bar{C}\bar{B} + \frac{1}{s^2}\bar{C}\bar{A}\bar{B} + \frac{1}{s^3}\bar{C}\bar{A}^2\bar{B} + \cdots + \bar{D}$$

$\therefore$ The two transfer functions are equal if and only if

$$D = \bar{D} \quad \text{and} \quad C A^i B = \bar{C} \bar{A}^i \bar{B} \qquad i = 0, 1, 2, \ldots$$

To answer the 2nd question, ~~we~~ recall that we already know how to find a state-space representation of a SISO transfer function. In fact if $G(s) = d + \dfrac{b_0 + b_1 s + \cdots + b_{n-1} s^{n-1}}{a_0 + a_1 s + \cdots + a_{n-1} s^{n-1} + s^n}$

then a state-space realization is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ & & & & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + [d] u$$

Suppose we have a MIMO transfer function, i.e. $G(s)$ is a transfer matrix. How do we get a state-space realization which gives $G(s)$ as its transfer function?

In more precise terms, we want to answer the question:

<u>Given a real rational transfer matrix $G(s)$, find matrices A, B, C, D such that $G(s) = C(sI-A)^{-1}B + D$.</u>

For simplicity, let $G(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix}$ where

$g_{ij}(s)$ has the realization $\left[\begin{array}{c|c} A_{ij} & b_{ij} \\ \hline c_{ij} & d_{ij} \end{array}\right] = g(s)$.

We can combine the realizations for g_{ij}'s to get one realization for $G(s)$.

Consider the realization

$$\left[\begin{array}{cccc|cc} A_{11} & & & 0 & b_{11} & 0 \\ & A_{12} & & & 0 & b_{12} \\ & & A_{21} & & b_{21} & 0 \\ 0 & & & A_{22} & 0 & b_{22} \\ \hline c_{11} & c_{12} & 0 & 0 & d_{11} & d_{12} \\ 0 & 0 & c_{21} & c_{22} & d_{21} & d_{22} \end{array}\right] =: \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

$$C(s\underline{I}-A)^{-1}B + D = \begin{bmatrix} c_{11} & c_{12} & 0 & 0 \\ 0 & 0 & c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} (s\underline{I}-A_{11})^{-1} & 0 & 0 & 0 \\ 0 & (s\underline{I}-A_{12})^{-1} & 0 & 0 \\ 0 & 0 & (s\underline{I}-A_{21})^{-1} & 0 \\ 0 & 0 & 0 & (s\underline{I}-A_{22})^{-1} \end{bmatrix} \begin{bmatrix} b_{11} & 0 \\ 0 & b_{12} \\ b_{21} & 0 \\ 0 & b_{22} \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & c_{12} & 0 & 0 \\ 0 & 0 & c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} (s\underline{I}-A_{11})^{-1} b_{11} & 0 \\ 0 & (s\underline{I}-A_{12})^{-1} b_{12} \\ (s\underline{I}-A_{21})^{-1} b_{21} & 0 \\ 0 & (s\underline{I}-A_{22})^{-1} b_{22} \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11}(s\underline{I}-A_{11})^{-1} b_{11} + d_{11} & c_{12}(s\underline{I}-A_{12})^{-1} b_{12} + d_{12} \\ c_{21}(s\underline{I}-A_{21})^{-1} b_{21} + d_{21} & c_{22}(s\underline{I}-A_{22})^{-1} b_{22} + d_{22} \end{bmatrix} = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix}$$

<u>Example:</u> Find a state space realization for

$$G(s) = \begin{bmatrix} \frac{3s^2+2}{s^3+4s^2+1} & \frac{s+1}{s-1} \\ \frac{s+4}{s+1} & \frac{(s+2)}{s^2+3s-1} \\ \frac{4s}{s^2+2s} & \frac{1}{s} \end{bmatrix}$$

Note that $\frac{s+1}{s-1} = 1 + \frac{+2}{s-1}$

$\frac{s+4}{s+1} = 1 + \frac{3}{s+1}$

$$G(s) = \left[\begin{array}{ccc|c|c|cc|cc|c|cc} 0 & 1 & 0 & & & & & & & & 0 & \\ 0 & 0 & 1 & & & & & & & & 0 & \\ -1 & 0 & -4 & & & & & & & & 1 & \\ \hline & & & 1 & & & & & & & & 1 \\ \hline & & & & -1 & & & & & & 1 & \\ \hline & & & & & 0 & 1 & & & & & 0 \\ & & & & & 1 & -3 & & & & & 1 \\ \hline & & & & & & & 0 & 1 & & 0 & \\ & & & & & & & 0 & -2 & & 1 & \\ \hline & & & & & & & & & 0 & & 1 \\ \hline +2 & 0 & +3 & +2 & & & & & & & 0 & 1 \\ & & & & +3 & +2 & +1 & & & & 1 & 0 \\ & & & & & & & 0 & +4 & +1 & 0 & 0 \end{array}\right]$$

Controllability & Observability of State-Space Systems

Given: $\dot{x} = A(t)\,x + B(t)\,u \qquad (\Sigma)$

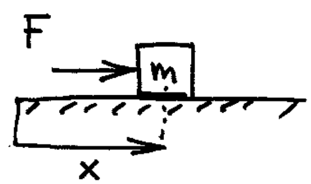
<u>Definition</u> (Controllability)

The state-space description (Σ) is said to be <u>controllable</u> <u>at time t_0</u>, if there exists a finite time $t_1 > t_0$ such that for any $x_0 \in \mathbb{R}^n$ and $x_1 \in \mathbb{R}^n$ there exists an input such that $\phi(t_1; t_0, x_0, u) = x_1$.

According to the definition, for a controllable system the input must be able to steer the state of the system from x_0 at time t_0 to x_1 at time t_1. The exact trajectory followed is not important.

If Σ is time-invariant, controllability at time t_0 implies controllability at any time t_0'. Therefore, in this case we shall say that Σ is controllable.

Example

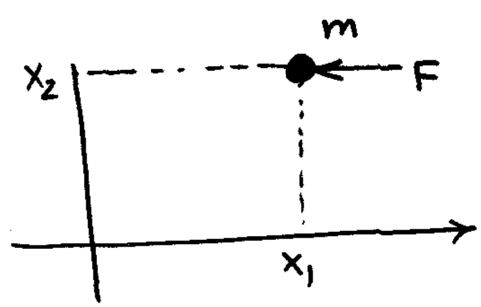
1. 

Frictionless surface

If the position x and the velocity ~~$\dot{x}$~~ u are the state-variables this system is controllable. Any position and velocity $\begin{bmatrix} x_1 \\ u_1 \end{bmatrix}$ are attainable. How do we choose F?

First ~~chose~~ choose the direction of F so that it points towards x_1. Then apply sufficient force F so that desired velocity (u_1) is reached, then set F to zero. Eventually, the position x_1 will be reached.

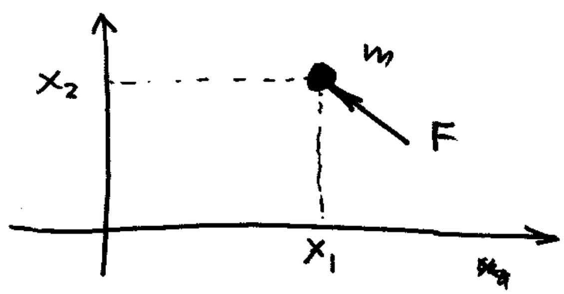
<u>Example:</u>



mass in $\mathbb{R}^2$

Eventhough any x_1 and $v_1 = \mathring{x}_1$ can be attained, most positions in the x_2 direction cannot be reached. This system is therefore uncontrollable.

<u>Example</u>



Every position x_1 and velocity $v_1 := \mathring{x}_1$ can be attained. Also

" " x_2 and velocity $v_2 := \mathring{x}_2$ " " .

But ... not all states can be attained.

Controllability can be determined from A, B matrices. To develop this theory we need to understand the linear independence of functions of time.

Let $f_i : [t_1, t_2] \longrightarrow \mathbb{C}^p$, $i = 1, \dots, n$

Then $\{f_i\}_{i=1,\dots,n}$ is linearly independent on $[t_1, t_2]$ over the field of complex numbers if

$$\alpha_1 f_1 + \dots + \alpha_n f_n = 0_{[t_1,t_2]} \Longrightarrow \alpha_1 = \dots = \alpha_n = 0$$
$$\alpha_i \in \mathbb{C}$$

Note that $\alpha_1 f_1 + \dots + \alpha_n f_n = 0$ here means that

$\alpha_1 f_1(t) + \dots + \alpha_n f_n(t) = 0$ for all $t \in [t_1, t_2]$

$f_i : [t_1, t_2] \longrightarrow \mathbb{C}^p$ can be represented by $1 \times p$ row vector for each t

$$f_i(t) = [f_{i1}(t) \quad \cdots \quad f_{ip}(t)]$$

With this notation, let $F(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$ $n \times p$ matrix

$$F : [t_1, t_2] \longrightarrow \mathbb{C}^{n \times p}$$

<u>Theorem</u>: Let f_i, for $i = 1, \dots, n$, be $1 \times p$ complex-valued continuous functions on $[t_1, t_2]$. Let F be a matrix function on $[t_1, t_2]$ having f_i as its ith row as shown above. Then $f_1, \dots, f_n$ are linearly independent on $[t_1, t_2]$ if and only if $\int_{t_1}^{t_2} F(t) F^*(t)\, dt$ is nonsingular.

<u>Proof:</u> ($\Leftarrow$) Suppose $\int_{t_1}^{t_2} F(t)F^*(t)\,dt$ is nonsingular. We want to show that $f_1, \dots, f_n$ are linearly independent on $[t_1, t_2]$.

i.e. that: $\alpha F(t) = 0 \Rightarrow \alpha = 0$ (where $\alpha = [\alpha_1 \cdots \alpha_n]$)

So suppose $\alpha F(t) = 0$. Then $\int_{t_1}^{t_2} \alpha F(t)\, F^*(t)\, dt = 0$

But $\int_{t_1}^{t_2} \alpha F(t) F^*(t)\, dt = \alpha \int_{t_1}^{t_2} F(t) F^*(t)\, dt$.

Therefore by the nonsingularity of $\int_{t_1}^{t_2} F(t) F^*(t)\, dt$ we have $\alpha = 0$.

($\Rightarrow$) Suppose $f_1, \dots, f_n$ are linearly independant on $[t_1, t_2]$.

Then $\alpha F(t) = 0$ on $[t_1, t_2] \Rightarrow \alpha = 0$.

Now suppose, $\alpha \int_{t_1}^{t_2} F(t) F^*(t)\, dt = 0$. Then $\int_{t_1}^{t_2} \alpha F(t) F^*(t) \alpha^*\, dt = 0$

$\Rightarrow \int_{t_1}^{t_2} (\alpha F(t))(\alpha F(t))^*\, dt = 0 \Rightarrow \alpha F(t) = 0$ on $[t_1, t_2]$. $\therefore \alpha = 0$.

We shall now look at conditions for controllability of the time-invariant system: $\mathring{x} = Ax + Bu$

Let's start by assuming $x_0 = 0$ at $t=0$. We want to be able to devise an input u from $t=0$ to $t=t_f$ so that $x(t_f) = x_1$.

Clearly u will depend on x_1.

Now $x(t_f) = \int_0^{t_f} e^{A(t_f-\tau)} B\, u(\tau)\, d\tau$

If we define $e^{A(t_f-\tau)}B =: F(\tau)$ we have

$$x(t_f) = \int_0^{t_f} F(\tau)\, u(\tau)\, d\tau$$

If $F(\tau)$ has linearly independent rows on $[0, t_f]$, then we can choose $u(\tau) = F^*(\tau)\left[\int_0^{t_f} F(\tau)F^*(\tau)\, d\tau\right]^{-1} x_1$

For this input, $x(t_f) = \int_0^{t_f} F(\tau)\, u(\tau)\, d\tau$

$$= \int_0^{t_f} F(\tau)\, F^*(\tau) \left[\int_0^{t_f} F(\tau) F^*(\tau)\, d\tau \right]^{-1} x_1\, d\tau$$

$$= \int_0^{t_f} F(\tau) F^*(\tau)\, d\tau \left[\int_0^{t_f} F(\tau) F^*(\tau)\, d\tau \right]^{-1} x_1 = x_1$$

Therefore, the linear independence of the rows of $e^{A(t_f-\tau)}B$ on $[0, t_f]$ is sufficient for controllability (from $x_0 = 0$).

If $x_0 \neq 0$, then $x(t_f) = e^{At_f} x_0 + \int_0^{t_f} F(\tau)\, u(\tau)\, d\tau$

In this case, to make $x(t_f) = x_1$ we adjust $u(\tau)$ ~~to~~:

$$u(\tau) = F^*(\tau) \left[\int_0^{t_f} F^*(\tau) F(\tau)\, d\tau \right]^{-1} \left(x_1 - e^{At_f} x_0 \right)$$

It turns out the linear independence of the rows of $F(\tau) = e^{A(t_f - \tau)}B$ on $[t_1, t_2]$ is also a necessary condition for stability.

To see this, we will show that if the system is controllable and $\alpha F(\tau) = 0$ on $[t_1, t_2]$ then $\alpha = 0$.

So let $\alpha F(\tau) = 0$. By controllability there exists $u(\tau)$ such that

$$x_1 = \alpha^* = \int_0^{t_f} F(\tau)\, u(\tau)\, d\tau.$$ Multiplying from the left by α we get

$$\alpha \alpha^* = \int_0^{t_f} \alpha F(\tau)\, u(\tau)\, d\tau = 0$$

This implies ~~$\alpha\alpha'$~~ $\alpha = 0$.

Example: Let us consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} u$$

Suppose $x_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and we want to reach $x_1 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ in 1 time unit $(t_f = 1)$.

Use matlab:

$$\int_0^{t_f} F(\tau) F^*(\tau)\, d\tau = \int_0^1 e^{A(1-\tau)} B B^T e^{A^T(1-\tau)}\, d\tau$$

$$u(t) = F^*(t) \left[\int_0^1 e^{A(1-\tau)} B B^T e^{A^T(1-\tau)}\, d\tau \right]^{-1} \left(\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} - e^{A} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

<u>Fact 1:</u> $e^{A(t_f-\tau)}B$ has linearly independent rows on $[0,t_f]$ iff $e^{-A\tau}$ has linearly independent rows on $[0,t_f]$.

This is because $e^{A(t_f-\tau)}B = e^{At_f}\cdot e^{-A\tau}B$, and e^{At_f} is invertible for all t_f and A.

<u>Fact 2:</u> $e^{-A\tau}B$ has linearly independent rows on $[0,t_f]$ iff $e^{A\tau}B$ has linearly independent rows on $[0,t_f]$.

<u>Proof:</u> $\alpha e^{-A\tau}B = 0$ on $[0,t_f]$

$$\Leftrightarrow \quad \alpha\left[I - A\tau + \frac{A^2\tau^2}{2!} - \frac{A^3\tau^3}{3!} + \cdots\right]B = 0 \quad \text{on } [0,t_f]$$

$$\Leftrightarrow \quad \alpha\left[B \;\; -AB \;\; A^2B \;\; -A^3B \;\; \cdots\right] = 0$$

$$\Leftrightarrow \quad \alpha\left[B \;\; AB \;\; A^2B \;\; A^3B \;\; \cdots\right] = 0$$

$$\Leftrightarrow \quad \alpha\left[I + A\tau + \frac{A^2\tau^2}{2!} + \frac{A^3\tau^3}{3!} + \cdots\right]B = 0 \quad \text{on } [0,t_f]$$

$$\Leftrightarrow \quad \alpha e^{A\tau}B = 0 \quad \text{on } [0,t_f].$$

<u>Fact 3</u>: $e^{A\tau}B$ has linearly independent rows on $[0, t_f]$ iff it has linearly independent rows on any interval $[t_1, t_2]$.

<u>Proof</u>: $\alpha\,[I + A\tau + \frac{A^2\tau^2}{2!} + \dots]B = 0$ on $[0, t_f]$

$\Longleftrightarrow$ $\alpha\,[B \;\; AB \;\; A^2B \;\; \dots] = 0$

$\Longleftrightarrow$ $\alpha\,[I + A\tau + \frac{A^2\tau^2}{2!} + \dots]B = 0$ on $[t_1, t_2]$

<u>Implications</u>: If $\dot{x} = Ax + Bu$ is controllable; then it is controllable from any initial state x_0 to any final state x_1 in <u>any</u> time $t_f > 0$. (t_f can be arbitrarily small).

In proving Fact 2 and Fact 3 we used the following result,

$$\alpha e^{At} B = 0 \text{ on } [t_1, t_2] \quad \text{iff} \quad \alpha [B \;\; AB \;\; A^2B \;\; \cdots] = 0$$

This statement is generalizable to the following statement: (Thm 5.3 in text)

Let f_i $i = 1, \ldots, n$ be analytic on $[t_1, t_2]$. Let $F = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$.

Then $\{f_i\}$ are linearly independent on $[t_1, t_2]$ if and only if

$$\rho [F(t_0) \;\; F^{(1)}(t_0) \;\; F^{(2)}(t_0) \;\; \cdots] = n \quad \text{for some } t_0 \in [t_1, t_2].$$

The proof follows immediately from the fact that f_i's can be expressed in a Taylor series expansion at $t = t_0$.

We have seen that $\dot{x} = Ax + Bu$ is controllable iff the rows of $e^{At}B$ are linearly independent on any interval $[t_1, t_2]$, $t_2 > t_1$. We now look at other equivalent conditions:

Theorem (Controllability):

$\dot{x} = Ax + Bu$ is controllable if and only if any of the following holds:

1. The rows of $e^{At}B$ are linearly independent on any interval $[t_1, t_2]$.
2. The rows of $(sI-A)^{-1}B$ are linearly independent.
3. $\int_0^{t_1} e^{A\tau} B B^* e^{A^*\tau} d\tau =: W_{ct}$ is invertible. ~~for any t > 0~~ for any $t_1 > 0$.
4. The controllability matrix

$$[B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B] =: \mathcal{U}$$

has full rank.

<u>Proof:</u> We've seen that controllability of $\dot{x} = Ax + Bu$ is equivalent to 1.

$(1 \Leftrightarrow 2)$: $\alpha e^{At}B = 0$ for $t \in [t_1, t_2]$ $\Leftrightarrow$ $\alpha e^{At}B = 0$ for $t \in [0, \infty)$

$\Leftrightarrow \mathcal{L}\{\alpha e^{At}B\} = 0 \Leftrightarrow \alpha(sI - A)^{-1}B = 0$

$(1 \Leftrightarrow 3)$: The rows of $e^{At}B$ are linearly independent on $[t_1, t_2]$

$\Leftrightarrow$ the rows of $e^{At}B$ are linearly independent on $[0, t_1]$.

$\Leftrightarrow \int_0^{t_1} (e^{At}B)(e^{At}B)^* \, dt$ is nonsingular

$(1 \Leftrightarrow 4)$: $\alpha e^{At}B = 0$ on $[t_1, t_2]$ $\Leftrightarrow$ $\alpha e^{At}B = 0$ on $(-\infty, \infty)$

$\Leftrightarrow \alpha\left[I + At + \frac{A^2t^2}{2!} + \cdots\right]B = 0$ on $(-\infty, \infty)$

$\Leftrightarrow \alpha[B \;\; AB \;\; A^2B \;\; A^3B \;\; \cdots] = 0$

(Cayley-Hamilton)

$\Leftrightarrow \alpha[B \;\; AB \;\; A^2B \cdots A^{n-1}B] = 0$

Reachability:

Given the time-invariant system $\dot{x} = Ax + Bu$ where the state-space has dimension n.

Def: A vector $x \in \mathbb{C}^n$ is said to be reachable if there exists a finite time $t_f > 0$ and an input $\{u(t) : 0 \le t \le t_f\}$ such that

$$\phi(t_f ; 0, 0, u) = x .$$

Note that x is reachable if for some $t_f > 0$

$$x = \int_0^{t_f} e^{A(t_f - \tau)} B \, u(\tau) \, d\tau \qquad \text{for some } u .$$

<u>Fact</u>: The set of all states in $\mathbb{C}^n$ which are reachable in time t_f forms a subspace.

<u>Proof</u>: Let x_1 and x_2 in $\mathbb{C}^n$ be 2 reachable states (in time t_f). Then there exist inputs u_1 and u_2 such that

$$x_1 = \int_0^{t_f} e^{A(t_f-\tau)} B\, u_1(\tau)\, d\tau \quad \text{and}$$

$$x_2 = \int_0^{t_f} e^{A(t_f-\tau)} B\, u_2(\tau)\, d\tau$$

$\therefore$ For any $\alpha_1, \alpha_2 \in \mathbb{C}$ we have

$$(\alpha_1 x_1 + \alpha_2 x_2) = \int_0^{t_f} e^{A(t_f-\tau)} B \underbrace{(\alpha_1 u_1(\tau) + \alpha_2 u_2(\tau))}_{u} d\tau$$

Hence $\alpha_1 x_1 + \alpha_2 x_2$ is reachable in time t_f.

<u>Fact:</u> The system $\dot{x} = Ax + Bu$ is controllable if and only if the whole state-space is reachable.

<u>Pf:</u> ($\Rightarrow$) Obvious

($\Leftarrow$) Given x_0 and x_1 in $\mathbb{C}^n$. By reachability there exists a time t_f and an input $\tilde{u}$ so that the state $\left(x_1 - e^{At_f}x_0\right)$ is reached in time t_f. In other words,

$$x_1 - e^{At_f}x_0 = \int_0^{t_f} e^{A(t_f-\tau)}B\,\tilde{u}(\tau)\,d\tau$$

This implies that

$$x_1 = e^{At_f}x_0 + \int_0^{t_f} e^{A(t_f-\tau)}B\,\tilde{u}(\tau)\,d\tau$$

which means that $\tilde{u}$ drives the state from x_0 to x_1 in time t_f.

Q.E.D.

Characterization of the Reachable Subspace

Define: $W_{t_f} := \int_0^{t_f} e^{A\tau} B B^* e^{A^*\tau}\, d\tau$

$$\mathcal{U} := [B \;\; AB \;\; \cdots \;\; A^{n-1}B]$$

Note that $W_{t_f} = \int_0^{t_f} F(\tau) F^*(\tau)\, d\tau$ where $F(\tau) = e^{A(t_f-\tau)} B$

Indeed, $\int_0^{t_f} e^{A(t_f-\tau)} B B^* e^{A^*(t_f-\tau)}\, d\tau = \int_{\sigma=t_f}^{\sigma=0} e^{A\sigma} B B^* e^{A^*\sigma} (-d\sigma)$

$$= W_{t_f}$$

<u>Notation</u>: Given a matrix $\overset{m\times n}{A}$, the <u>range of A</u> is defined by

$$\mathcal{R}(A) = \{ y : y = Ax \text{ for some } x \}$$

<u>Notation:</u> The null space of A is defined by
<u>Cont.</u>

$$\mathcal{N}(A) = \{x : Ax = 0\}$$

Both $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are subspaces of $\mathbb{C}^m$ and $\mathbb{C}^n$ respectively.

Let S be a subset of $\mathbb{C}^n$. A vector $x \in \mathbb{C}^n$ is <u>perpendicular</u> to S if $x^*s = 0$ for all $s \in S$. This is expressed in the notation: $x \perp S$.

We denote $\{x \in \mathbb{C}^n : x \perp S\}$ by $S^\perp$.

Clearly, $S^\perp$ is a subspace of $\mathbb{C}^n$.

Given $\dot{x} = Ax + Bu$

Goal: Show that

$$\text{Reachable subspace} = \mathcal{R}(\mathcal{U}) = \mathcal{R}(W_{t_f}) \quad \text{for any } t_f > 0$$

Lemma: $\mathcal{R}(W_{t_f}) = \{\alpha^* : \alpha F(\tau) = 0 \text{ on } [0, t_f]\}^\perp$

Proof: $x \in \mathcal{R}(W_{t_f})$ iff $x \in \mathcal{N}(W_{t_f}^*)^\perp$

iff ($\alpha x = 0$ whenever $W_{t_f}^* \alpha^* = 0$)

iff ($\alpha x = 0$ whenever $\alpha W_{t_f} = 0$)

Claim: $\alpha W_{t_f} = 0$ iff $\alpha F(\tau) = 0$ on $[0, t_f]$

$$\alpha W_{t_f} = 0 \Rightarrow \alpha \int_0^{t_f} e^{A(t_f - \tau)} B B^* e^{A^*(t_f - \tau)} d\tau = 0$$

$$\Rightarrow \alpha \int_0^{t_f} F(\tau) F^*(\tau)\, d\tau\, \alpha^* = 0$$

$$\Rightarrow \int_0^{t_f} [\alpha F(\tau)][F^*(\tau)\alpha^*]\, d\tau = 0$$

$$\Rightarrow \alpha F(\tau) = 0 \text{ on } [0, t_f]$$

Also $\alpha F(\tau) = 0$ on $[0, t_f] \Rightarrow \alpha \int_0^{t_f} F(\tau) F^*(\tau)\, d\tau = 0$

$$\Rightarrow \alpha W_{t_f} = 0$$

Proof (cont.) It follows that $x \in \mathcal{R}(W_{t_f})$ iff

$$\underbrace{(\alpha x = 0 \text{ whenever } \alpha F(\tau) = 0 \text{ on } [0, t_f])}$$

This condition is equivalent to $x \in \left\{ \alpha^* : \alpha F(\tau) = 0 \text{ on } [0, t_f] \right\}^{\perp}$.

Q.E.D.

Theorem: $\mathcal{R}(W_{t_f})$ = Reachable subspace (in t_f).

Pf: $x(t_f) = \int_0^{t_f} F(\tau) u(\tau)\, d\tau$. If $x_1 \in \mathcal{R}(W_{t_f})$, $x_1 = W_{t_f} z$

Let $\tilde{u}(\tau) = F^*(\tau) z$. For this input,

$$x(t_f) = \int_0^{t_f} F(\tau) F^*(\tau) z \, d\tau = W_{t_f} z = x_1.$$

Hence, $\mathcal{R}(W_{t_f}) \subset$ Reachable subspace (in t_f)

Let $x \in$ subspace reachable in t_f

Then $x = \int_0^{t_f} F(\tau) u(\tau) \, d\tau$ for some input u

Suppose $\alpha^* \in \{ \alpha^* : \alpha F(\tau) = 0 \text{ on } [0, t_f] \}$

then $\alpha x = \int_0^{t_f} \alpha F(\tau) u(\tau) d\tau = 0$

$\therefore x \in \{ \alpha^* : \alpha F(\tau) \text{ on } [0, t_f] \}^{\perp} = \mathcal{R}(W_{t_f})$

Q.E.D.

Controllability Canonical Decomposition

Given $\dot{x} = Ax + Bu$

Suppose $\rho(\mathcal{U}) = n_1 < n$. (There are unreachable states).

Using a linear transformation $(x = Qz)$ we can shed more light on the reachable subspace of the state-space.

Starting with $\mathcal{U} = [B \;\; AB \;\; A^2B \cdots A^{n-1}B]$

let $\{q_1, \ldots, q_{n_1}\}$ be any vectors which span the range of $\mathcal{U}$.

For example, these can be any n_1 linearly independent columns of $\mathcal{U}$.

Let $\{q_{n_1+1}, \ldots, q_n\}$ be <u>any</u> vectors such that $\{q_1, \ldots, q_{n_1}, q_{n_1+1}, \ldots, q_n\}$ is linearly independent.

The set $\{q_1, \ldots, q_n\}$ forms a basis for the state-space.

We want to express the state-variables in terms of this basis.

i.e. Let $x = [q_1 \cdots q_n] z$. z is the representation of x in terms of the new basis.

Defining $Q := [q_1 \cdots q_n]$ we can write the state-equations in terms of z:

$$\frac{d}{dt}(Qz) = A(Qz) + Bu \quad \text{or}$$

$$Q\mathring{z} = AQz + Bu$$

$$\mathring{z} = \underbrace{Q^{-1}AQ}_{\bar{A}}\, z + \underbrace{Q^{-1}B}_{\bar{B}}\, u$$

A and $\bar{A}$ are related by:

$$Q\bar{A} = AQ \quad , \quad \text{or} \quad [q_1 \cdots q_n]\bar{A} = A[q_1 \cdots q_n]$$

Let's partion $\bar{A} = [\bar{a}_1 \ \bar{a}_2 \ \cdots \ \bar{a}_n]$

The above relation between A & $\bar{A}$ says that

$$[q_1 \cdots q_n]\,\bar{a}_i = A q_i$$

$\therefore$ <u>$\bar{a}_i$ is the representation of Aq_i with respect to $\{q_1, \cdots, q_n\}$</u> !

There are two cases:

a. <u>$1 \le i \le n_1$</u> In this case $q_i \in \mathcal{R}(\mathcal{U})$

So $q_i = \mathcal{U}\alpha$ for some α

$\quad = B\alpha_1 + AB\alpha_2 + \cdots + A^{n-1}B\alpha_n$

$\therefore Aq_i = AB\alpha_1 + A^2B\alpha_2 + \cdots + A^nB\alpha_n \in \mathcal{R}(\mathcal{U})$ (By Cayley-Hamilton)

Since $Aq_i \in \mathcal{R}(\mathcal{U})$ and since $\{q_1, \dots, q_{n_1}\}$ span $\mathcal{R}(\mathcal{U})$

$$[q_1 \cdots q_{n_1} \; q_{n_1+1} \cdots q_n] \underbrace{\begin{bmatrix} \bar{a}_{i1} \\ \vdots \\ \bar{a}_{in_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{\bar{a}_i} = Aq_i$$

$\therefore$ for $i \le n_1$, $\bar{a}_i$ always has the structure $\begin{bmatrix} * \\ \vdots \\ * \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \}\, n_1 \\ \\ \}\, n-n_1 \end{matrix}$

b. <u>$i > n_1$</u>

In this case, $\bar{a}_i$ the representation of Aq_i with respect to $\{q_1, \dots, q_n\}$ has <u>no</u> <u>special</u> <u>structure</u>.

<u>Conclusion:</u>

$$\bar{A} = \begin{array}{c} \left[\begin{array}{ccc|ccc} * & \cdots & * & * & \cdots & * \\ \vdots & \cdots & * & * & \cdots & * \\ * & & & * & \cdots & * \\ 0 & \cdots & 0 & * & & \vdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & * & \cdots & * \end{array}\right] \begin{array}{l} \}\, n_1 \\ \\ \}\, n-n_1 \\ \\ \\ \\ \end{array} \\ \underbrace{\qquad\quad}_{n_1} \;\; \underbrace{\qquad\quad}_{n-n_1} \end{array}$$

$$= \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix}$$

$\bar{A}_c$ is $n_1 \times n_1$

$\bar{A}_{\bar{c}}$ is $(n-n_1) \times (n-n_1)$

$\bar{A}_{12}$ is $n_1 \times (n-n_1)$

We can repeat the above analysis to see the special structure of $\bar{B}$.

Let $\bar{B} = [\bar{b}_1 \cdots \bar{b}_p]$

We can write the relation between B & $\bar{B}$ as follows:

$$[q_1 \cdots q_{n_1}\, q_{n_1+1} \cdots q_n]\, \bar{b}_i = A\, b_i$$

Since $Ab_i \in \mathcal{R}(\mathcal{U})$, Ab_i can be expressed in terms of $\{q_1, \ldots, q_{n_1}\}$ which span $\mathcal{R}(\mathcal{U})$.

$$\therefore \quad [q_1 \cdots q_{n_1} \; q_{n_1+1} \cdots q_n] \begin{bmatrix} \bar{b}_{i1} \\ \vdots \\ \bar{b}_{in_1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = Ab_i$$

It follows that $\bar{B}$ has the structure:

$$\bar{B} = \begin{bmatrix} * & \cdots & * \\ \vdots & & \\ * & \cdots & * \\ 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} * \\ \vdots \\ * \end{matrix}}\right\} n_1 \\ \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} n-n_1 \end{matrix} = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} \qquad \bar{B}_c \text{ is } n_1 \times p$$

Therefore, in terms of the new basis the state-equations can be written as:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix}}_{\bar{A}} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \underbrace{\begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}}_{\bar{B}} u$$

* Note that the dynamics of z_2 <u>cannot</u> <u>be</u> <u>affected</u> by u.

If $z_2(0) = 0$ then $z_2(t) = 0$ for all $t \geq 0$.

* Clearly u affects z_1. In fact, $(\bar{A}_c, \bar{B}_c)$ are a controllable pair.

To see this, $\bar{\mathcal{U}} = [\bar{B}\ \bar{A}\bar{B} \cdots \bar{A}^{n-1}\bar{B}] = \begin{bmatrix} \bar{B}_c & \bar{A}_c\bar{B} & \bar{A}_c^2\bar{B}_c & \cdots & \bar{A}_{n_1}^{n-1}B_c \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$

$$n_1 = \rho[\mathcal{U}) = \rho(\bar{\mathcal{U}}) = \rho([\bar{B}_c\ \bar{A}_c\bar{B}_c \cdots \bar{A}_c^{n-1}\bar{B}_c]) = \rho([\bar{B}_c\ \bar{A}_c\bar{B}_c \cdots \bar{A}_c^{n_1-1}\bar{B}_c])$$

$$= \rho(\bar{\mathcal{U}}_c)$$

where $\bar{\mathcal{U}}_c$ is controllability matrix of $(\bar{A}_c, \bar{B}_c)$

Given the controllability cannonical decomposition:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u \qquad x = Qz$$

If we have an output equation (in the x variables):

$$y = Cx + Du$$

In the (z-variables) this equation becomes

$$y = \underbrace{CQ}_{\bar{C}} z + Du$$

We can partition $\bar{C} = [\underbrace{\bar{C}_c}_{n_1} \quad \underbrace{\bar{C}_{\bar{c}}}_{n-n_1}]$

The state-space description becomes:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \bar{D} u$$

The transfer function between u and y is <u>unaffected</u> by state transformation.

So $\quad G(s) = C(sI-A)^{-1}B + D \quad = \quad \bar{C}(sI-\bar{A})^{-1}\bar{B} + \bar{D}$

$$= \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} sI-\bar{A}_c & -\bar{A}_{12} \\ 0 & sI-\bar{A}_{\bar{c}} \end{bmatrix}^{-1} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} + \bar{D}$$

$$= \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} (sI-\bar{A}_c)^{-1} & (sI-\bar{A}_c)^{-1}\bar{A}_{12}(sI-\bar{A}_{\bar{c}})^{-1} \\ 0 & (sI-\bar{A}_{\bar{c}})^{-1} \end{bmatrix} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} + \bar{D}$$

$$= \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} (sI-\bar{A}_c)^{-1}\bar{B}_c \\ 0 \end{bmatrix} + \bar{D}$$

$$= \bar{C}_c (sI-\bar{A}_c)^{-1}\bar{B}_c + \bar{D}$$

There are two state-space realizations here:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} \bar{A}_c & \bar{B}_c \\ \hline \bar{C}_c & \bar{D}_c \end{array}\right]$$

Dim of state-space n Dim. of state-space n_1

Both realizations give the <u>same</u> transfer function

The realization $\left[\begin{array}{c|c} \bar{A}_c & \bar{B}_c \\ \hline \bar{C}_c & \bar{D}_c \end{array}\right]$ is <u>controllable</u>, whereas $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is <u>not controllable</u>.

Computation of Cont. Decomp. Matrix

How do we compute Q such that $Q^{-1}AQ = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix}$

$$Q^{-1}B = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}$$

<u>Recall</u>: $Q = [q_1 \cdots q_{n_1}\ q_{n_1+1} \cdots q_n]$

where $\{q_1, \ldots, q_{n_1}\}$ span the range of $\mathcal{U}$.

$\{q_{n_1+1}, \ldots, q_n\}$ make Q invertible.

Q can be obtained from the Singular Value Decomposition of $\mathcal{U}$.

$$\mathcal{U} = U \Sigma V^T \quad \text{where} \quad \Sigma = \left[\begin{array}{c:c} \underbrace{\begin{matrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_{n_1} & & \\ & & & 0 & \\ & & & & \ddots \ 0 \end{matrix}}_{\Sigma_1} & 0 \end{array} \right]$$

Σ_1 is $n_1 \times n_1$

Clearly, the first n_1 columns of U span range$(\mathcal{U})$. U is always invertible. <u>Conclusion</u>: Use $Q = U$

<u>Example;</u> Given $G(s) = \begin{bmatrix} \frac{1}{s^2} \\ \frac{1}{s(s+2)} \end{bmatrix}$

A state-space realization is given by

$$A = \left[\begin{array}{cc|cc} 0 & 1 & & \\ 0 & 0 & & \\ \hline & & 0 & 1 \\ & & 0 & -2 \end{array}\right], \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$C = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right], \quad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\operatorname{rank}(\mathcal{U}) = 3 < 4$. $\therefore$ We can get a "smaller" realization.

Another test for controllability

Let $\dot{x} = Ax + Bu$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of A.

<u>Theorem:</u> The given state-space system is controllable if and only if $\rho\,[\lambda_i I - A \quad B] = n$ for $i = 1, \ldots, m$

<u>Proof:</u> ($\Leftarrow$) Suppose the system is <u>not</u> controllable.

Then there exists a transformation $x = Qz$ such that ~~the~~ ~~system~~

$$\bar{A} = Q^{-1} A Q = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix}$$

$$\bar{B} = Q^{-1} B = \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix}$$

Now $\rho\,[\lambda_i I - A \quad B] = \rho\left\{ Q^{-1}[\lambda_i I - A \quad B]\begin{bmatrix} Q & 0 \\ 0 & I \end{bmatrix}\right\}$

$$= \rho\,[\lambda_i I - \bar{A} \quad \bar{B}]$$

$$= \rho\begin{bmatrix} \lambda_i I - \bar{A}_c & -A_{12} & \bar{B}_c \\ 0 & \lambda_i I - \bar{A}_{\bar{c}} & 0 \end{bmatrix}$$

$< n$ $\quad \left\{ \begin{array}{l} \text{if } \lambda_i \text{ is an eigenvalue} \\ \text{of } \bar{A}_{\bar{c}} \end{array} \right.$

($\Rightarrow$) Suppose $\rho[\lambda_i I - A \quad B] < n$ for some λ_i

Then there exists $\alpha \neq 0$ such that

$$\alpha\,[\lambda_i I - A \quad B] = 0$$

This implies $\alpha B = 0$, $\alpha A B = \lambda_i(\alpha B) = 0$, $\alpha A^2 B = \lambda_i^2(\alpha B) = 0$,

$\cdots \quad \alpha A^{n-1} B = \lambda_i^{n-1}(\alpha B) = 0$

$$\therefore \quad \alpha [B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{n-1}B] = 0$$

which implies that the rank of the controllability matrix is less than n, and therefore $\dot{x} = Ax + Bu$ is not controllable.

The above controllability test is sometimes stated in a different form:

<u>Popov-Belevitch-Hautus (PBH) cont. test:</u>

The system $\dot{x} = Ax + Bu$ is <u>not</u> <u>controllable</u> iff there exists a left eigenvector of A, which is orthogonal to all the columns of B.

$$\text{i.e.} \begin{cases} \alpha A = \lambda_i \alpha \\ \alpha \neq 0 \end{cases} \quad \text{and} \quad \alpha B = 0$$

Observability of State-Space Systems

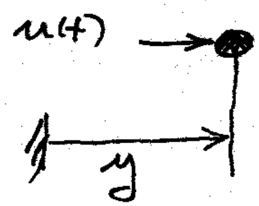
Given
$$\dot{x} = A(t)\,x + B(t)\,u \qquad (\Sigma)$$
$$y = C(t)\,x + D(t)\,u$$

Definition

The system Σ is said to be completely observable at t_0 if there exists a finite time $t_1 > t_0$ such that for any initial state x_0 at time t_0, knolwedge of $A(t), B(t), C(t), D(t)$ and $\{u(t),\ t_0 \le t \le t_1\}$ and $\{y(t),\ t_0 \le t \le t_1\}$ gives sufficient information to determine x_0.

Remark: For time-invariant systems, we can take $t_0 = 0$.

__Example__ (point mass)



$u(t)$ – applied force

$y(t)$ – position

The state equations are :

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = m^{-1} u(t)$$
$$y(t) = x_1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ m^{-1} \end{bmatrix} u$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If we know $u(t)$ we know acceleration $\dot{x}_2$

$$x_2(t) = \int_0^t \dot{x}_2 \, d\tau + x_2(0)$$

Knowledge of $x_1(t)$ allows us to determine $x_1(0)$ and $x_2(0)$.

Indeed, $x_2(0) = \dot{x}_1(t) - \int_0^t \dot{x}_2 \, d\tau$

If in the example, the velocity, x_2, was taken as the output, the state equations become:

$$\begin{bmatrix} \mathring{x}_1 \\ \mathring{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ m^{-1} \end{bmatrix} u$$

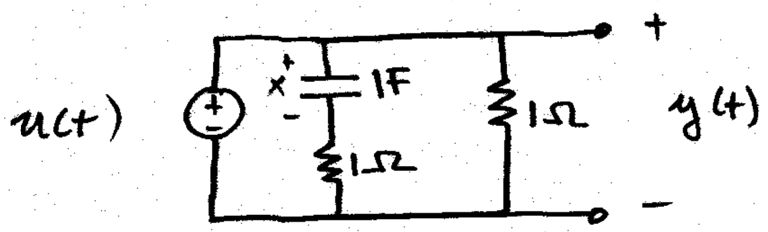
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Knowledge of the velocity of the particle and the acceleration (equivalently $u(t)$) does not give enough information to determine $x_1(0)$, the initial position.

For example, for $u(t) \equiv 0$, $y(t) \equiv 0$

$x_1(0) \equiv x_1(t)$ can be <u>any</u> <u>number</u>.

Example:



For this circuit, the state equations are:

$$\dot{x} = -x + u \qquad A = [-1] \quad B = [1]$$
$$y = u \qquad C = [0] \quad D = [1]$$

No matter what the initial state $x(0)$ is, the output $y(t) = u(t)$. Knowledge of $y(t)$ and $u(t)$ does not lead to the determination of $x(0)$.

For the system
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

If we know $y(t)$ and $u(t)$, then we know

$$y(t) - C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau = \underline{\underline{Ce^{At}x_0}}$$

∴ Observability question reduces to the following:

Does there exist a finite time t_f such that knowledge of $Ce^{At}x_0$ $\quad 0 \le t \le t_f$ gives enough information to determine x_0? Can this be done for all x_0?

<u>Question:</u> Given $Ce^{At}x_0$ for $0 \le t \le t_f$, when can we recover x_0 (uniquely)?

Suppose the <u>columns</u> of Ce^{At} are linearly independent over $[0, t_f]$. Then the rows of $e^{A^*t}C^*$ are linearly independent over $[0, t_f]$. This is equivalent to the invertibility of

$$\int_0^{t_f} e^{A^*\tau} C^* C e^{A\tau}\, d\tau =: W_{0t_f}$$

In this case, given $y(t) = Ce^{At}x_0 \quad 0 \le t \le t_f$

$$W_{0t_f}^{-1} \int_0^{t_f} e^{A^*\tau} C^* y(\tau)\, d\tau = W_{0t_f}^{-1} \int_0^{t_f} e^{A^*\tau} C^* \underbrace{C e^{A\tau} x_0}_{y(\tau)}\, d\tau$$

$$= x_0$$

We have shown that the linear independence of the columns of Ce^{At} over $[0, t_f]$ is sufficient for observability.

This condition is also necessary!

Suppose the system $\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$ is observable.

This implies that for some t_f, there is a <u>unique</u> solution x_0 to the equation $Ce^{At}x_0 = 0, \quad 0 \le t \le t_f$,

(otherwise we would not be able to determine x_0 when $u(t) = 0$ and $y(t) = 0 \quad 0 \le t \le t_f$)

Since $x_0 = 0$ solves $Ce^{At}x_0 = 0$ for $0 \le t \le t_f$

it must be the unique solution.

We have shown that if the system is observable (in t_f), the columns of Ce^{At} will be linearly independent on $[0, t_f]$.

<u>Remark:</u> The exact value of $t_f > 0$ is not important. This is because of the following fact:

The columns of Ce^{At} are linearly independent over $[0, t_f]$, $t_f > 0$ if and only if they are linearly independent over any interval $[t_1, t_2]$ where $t_1 < t_2$.

<u>Pf:</u> $Ce^{At}\beta = 0$ on $[0, t_f]$

$\Leftrightarrow$ $C\left[I + At + \frac{A^2t^2}{2!} + \cdots\right]\beta = 0$ on $[0, t_f]$

$\Leftrightarrow$ $C\beta = 0$, $CA\beta = 0$, $CA^2\beta = 0 \cdots\cdots$.

$\Leftrightarrow$ $C\left[I + At + \frac{A^2t^2}{2!} + \cdots\right]\beta = 0$ on any interval $[t_1, t_2]$

$\Leftrightarrow$ $Ce^{At}\beta = 0$ on $[t_1, t_2]$.

Q.E.D.

From the previous discussion it follows that if the system

$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$ is observable in t_f, then it will be observable in any time $\tilde{t}_f > 0$. In other words, if knowledge of $y(t)$ and $u(t)$ for $0 \le t \le t_f$ allows one to determine x_0 for all x_0 then knowledge of $y(t)$ and $u(t)$ for $0 \le t \le \tilde{t}_f$ for any $\tilde{t}_f > 0$, no matter how small, will also allow the determination of x_0.

There are other tests for the observability of a linear system. We will state those next.

Note that $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$ is observable $\Leftrightarrow$ $\begin{cases} \dot{x} = Ax \\ y = Cx \end{cases}$ is observable

In this case, we will also say that the pair $\{A, C\}$ is observable.

e

<u>Theorem</u> (Observability)

The system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$ is observable if and only if

any one of the following conditions holds :

1. The columns of Ce^{At} are linearly independent on any interval $[t_1, t_2]$
2. The columns of $C(sI-A)^{-1}$ are linearly independent.
3. $\int_0^{t_1} e^{A^*\tau} C^* C e^{A\tau} \, d\tau =: W_{ot_1}$ is invertible for any $t_1 > 0$.
4. The observability matrix

$$V = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has full rank.

<u>Proof:</u> We've shown that observability is equivalent to (1).

(1) $\Leftrightarrow$ (2) (similar to analogous result for controllability)

(1) $\Leftrightarrow$ (3) We have shown this

(1) $\Leftrightarrow$ (4)

The columns of Ce^{At} are linearly independent of $[t_1, t_2]$

$$\Leftrightarrow \left(Ce^{At}\beta = 0 \text{ on } [t_1, t_2] \Rightarrow \beta = 0 \right)$$

$$\Leftrightarrow \left(C\left(I + At + \frac{A^2t^2}{2!} + \dots\right)\beta = 0 \text{ on } [t_1, t_2] \Rightarrow \beta = 0 \right)$$

$$\Leftrightarrow \left(\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} \beta = 0 \Rightarrow \beta = 0 \right)$$

$$\Leftrightarrow \left(\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \beta = 0 \Rightarrow \beta = 0 \right)$$

$\Leftrightarrow$ Y has full rank.

Q.E.D.

Duality of Controllability & Observability

Given the two systems:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad (\Sigma)$$

and

$$\begin{aligned} \dot{z} &= A^* z + C^* v \\ \gamma &= B^* z + D^* v \end{aligned} \qquad (\Sigma^*)$$

Theorem: Σ is controllable if and only if Σ^* is observable.

Proof: Σ is controllable iff $e^{At}B$ has linearly indep. rows on $[0, t_f]$

iff $B^* e^{A^* t}$ has linearly independent columns on $[0, t_f]$

iff Σ^* is observable.

The unobservable subspace

Suppose that $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$ is not observable.

which initial states result in a measurable output ?

How do different initial states reveal themselves ~~to~~ at the output ?

Consider the situation when $u \equiv 0$.

$$\dot{x} = Ax \ , \quad y = Cx$$

<u>Definition</u> A nonzero vector $x_0 \in \mathbb{C}^n$ is said to be <u>unobservable</u>

if $\begin{matrix} \dot{x} = Ax \\ y = Cx \end{matrix}$ and $x(0) = x_0$ implies that

$$y(t) = 0 \qquad t \geq 0.$$

<u>Fact:</u> The set of all unobservable vectors in $\mathbb{C}^n$ together with the zero vector forms a subspace of $\mathbb{C}^n$.

<u>Pf:</u> Let x_1 and x_2 be unobservable.

this implies $Ce^{At}x_i = 0 \qquad i=1,2.$

For any $\alpha_1, \alpha_2 \in \mathbb{C}^n$, $\quad Ce^{At}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 Ce^{At}x_1 + \alpha_2 Ce^{At}x_2 = 0$ on $[t_1, t_2]$.

Q.E.D.

<u>Theorem:</u> The unobservable subspace $= \mathcal{N}(V)$

$$= \mathcal{N}(W_{ot_f})$$

<u>Proof:</u> We first show that the unobservable subspace is equal to $\mathcal{N}(W_{ot_f})$.

β belongs to the unobservable subspace

$\Longleftrightarrow$ for some $t_f > 0$, $\quad Ce^{At}\beta = 0 \quad$ on $[0, t_f]$

$$\Longleftrightarrow \left[\int_0^{t_f} e^{A^*\tau} C^* C e^{A\tau}\, d\tau\right]\beta = 0 \qquad (\text{for some } t_f > 0)$$

$\Longleftrightarrow$ for some $t_f > 0$, $\beta \in \mathcal{N}(W_{ot_f})$

Next we show that the unobservable subspace $= \mathcal{N}(V)$

$$Ce^{At}\beta = 0 \quad \text{on } [0, t_f]$$

$$\Longleftrightarrow C\left[I + At + \frac{A^2t^2}{2!} + \dots\right]\beta = 0 \quad \text{on } [0, t_f]$$

$$\Longleftrightarrow \begin{bmatrix} C \\ CA \\ \vdots \end{bmatrix}\beta = 0$$

$$\Longleftrightarrow \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}\beta = 0$$

Q.E.D.

<u>Theorem:</u> Given $\dot{x} = Ax$, $y = Cx$. Then

The unobservable subspace $= \mathcal{N}(W_{ot_f})$ for all $t_f > 0$

$= \mathcal{N}(V)$

<u>Proof:</u> x_0 is unobservable $\iff \forall\, t_f > 0, \quad Ce^{At}x_0 = 0$ on $[0, t_f]$

$$\iff \int_0^{t_f} e^{A^*\tau} C^* C e^{A\tau} x_0 \, d\tau = 0$$

$$\iff x_0 \in \mathcal{N}(W_{ot_f})$$

$\therefore$ The unobservable subspace $= \mathcal{N}(W_{ot_f})$.

x_0 is unobservable $\iff \forall\, t_f > 0, \quad Ce^{At}x_0 = 0$ on $[0, t_f]$

$$\iff \forall\, t_f > 0, \quad C\left[I + At + \frac{A^2t^2}{2!} + \cdots\right]x_0 \text{ on } [0, t_f]$$

$$\iff Cx_0 = 0, \; CAx_0 = 0, \; \cdots\cdots$$

$$\iff Cx_0 = 0, \; CAx_0 = 0, \; \cdots, \; CA^{n-1}x_0 = 0$$

$$\iff x_0 \in \mathcal{N}(V)$$

Observability Canonical Decomposition

Given
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

If $x = Qz$, then

$$\dot{z} = Q^{-1}AQz + Q^{-1}Bu$$
$$y = CQz + Du$$

Defining $\bar{A} := Q^{-1}AQ$, $\bar{B} = Q^{-1}B$, $\bar{C} := CQ$, $\bar{D} := D$

$$\dot{z} = \bar{A}z + \bar{B}u$$
$$y = \bar{C}z + \bar{D}u$$

Suppose $\rho(V) = n_2 < n$ (System is not observable)

It follows that $\underline{\underline{\dim \mathcal{N}(V) = n - n_2}}$

Choose $Q = [q_1 \cdots q_n]$ as follows:

1. Let $q_{n_2+1}, \dots, q_n$ form a basis for $\mathcal{N}(V)$
2. Choose $q_1, \dots, q_{n_2}$ be any vectors so that $[q_1 \cdots q_n] = Q$ is nonsingular.

With this Q taken as the similarity transformation, what is the representation of A in terms of the new basis? i.e. what is $\bar{A}$?

$Q\bar{A} = AQ$ gives the relation between A and $\bar{A}$

this can be written as

$$[q_1 \cdots q_n][\bar{a}_1 \cdots \bar{a}_n] = \underbrace{[a_1 \cdots a_n]}_{A}[q_1 \cdots q_n]$$

Hence, $Aq_i = [q_1 \cdots q_n]\,\bar{a}_i$

For $i = 1, \ldots, n_2$

$$\bar{a}_i = \begin{bmatrix} \bar{a}_{i1} \\ \vdots \\ \bar{a}_{in} \end{bmatrix} \qquad \text{(no special structure)}$$

For $i = n_2+1, \ldots, n$

$$\bar{a}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \bar{a}_{i,\,n_2+1} \\ \vdots \\ \bar{a}_{in} \end{bmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} n_2 \\ \left.\vphantom{\begin{matrix} \bar{a} \\ \vdots \\ \bar{a} \end{matrix}}\right\} n - n_2 \end{array}$$

This is because $Aq_i \in \mathcal{N}(Y)$ for $i = n_2+1, \ldots, n$ and therefore the representation of Aq_i with respect to $[q_1 \cdots q_n]$ will have zeros in the first n_2 entries.

From this discussion it follows that

$$\bar{A} = \begin{array}{cc} \left[\begin{array}{cc} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{array}\right] & \begin{array}{l} \}\ n_2 \\ \}\ n - n_2 \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{n_2} & \underbrace{\quad}_{n-n_2} \end{array} & \end{array}$$

What about $\bar{C}$?

$\bar{C} = CQ$. But $Cq_{n_2+1} = 0, \ldots, Cq_n = 0$.

$$\therefore \bar{C} = [\underbrace{\bar{C}_o}_{n_2} \quad \underbrace{0}_{n-n_2}]$$

$\bar{B} = Q^{-1}B$ has no special structure. We can, however, partition $\bar{B}$ as follows:

$$\bar{B} := \left[\begin{array}{c} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{array}\right] \begin{array}{l} \}\ n_2 \\ \}\ n - n_2 \end{array}$$

Putting everything together:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \bar{D} u$$

<u>There are 3 important observations:</u>

1. The unobservable subspace (in terms of the new coordinates) is given by $\left\{ \begin{bmatrix} 0 \\ z \end{bmatrix} : z \in \mathbb{C}^{n-n_2} \right\}$
2. The pair $(\bar{A}_o, \bar{C}_o)$ is observable.
3. $\bar{C}(sI-\bar{A})^{-1}\bar{B} + \bar{D} = \bar{C}_o(sI-\bar{A}_o)^{-1}\bar{B}_o + \bar{D}$
   (This allows the reduction of the dimension of the state-space)

How do we compute Q using matlab?

The Singular Value Decomposition of V is

$$V = U \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix} W^T \qquad \text{where} \quad \Sigma_1 = \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_{n_2} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$$

If we write $W := [w_1 \dots w_n]$, then for all i

$$W^T w_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i\text{th entry} \qquad \text{(because } W \text{ is orthonormal)}$$

If $i = n_2+1, \dots, n$ then $V w_i = U \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} W^T w_i = 0$

Since $\{w_{n_2+1}, \dots, w_n\}$ are linearly independent, then the set $\{w_{n_2+1}, \dots, w_n\}$ is a basis for the null space of V.

W is invertible, so we can simply <u>take</u> $\underline{Q = W}$.

Example:

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s^2-1} & \frac{1}{s(s+1)} \end{bmatrix}$$

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccccc|ccc} -1 & & & & & 1 & & \\ & 0 & 1 & & & & 0 & \\ & 1 & 0 & & & & 1 & \\ & & & 0 & 1 & & & 0 \\ & & & 0 & -1 & & & 1 \\ \hline 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\text{rank}(V) = 3$$

PBH Test for Observability

Given: $\dot{x} = Ax + Bu$
$y = Cx + Du$ $\quad (\Sigma)$

Σ is observable iff for every eigenvalue λ of A, $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$ has rank n.

<u>Proof:</u> ($\Rightarrow$) Suppose Σ is observable.

Let $\alpha \in \mathbb{C}^n$ such that $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} \alpha = 0$

Then $\alpha\lambda = A\alpha$ <u>and</u> $C\alpha = 0$.

Then $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \alpha = \begin{bmatrix} C\alpha \\ \lambda C\alpha \\ \vdots \\ \lambda^{n-1} C\alpha \end{bmatrix} = 0$

By observability, α must be zero.

$\therefore$ Observability $\Rightarrow$ $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$ has rank n for all eigenvalues of A

($\Leftarrow$) Suppose is <u>not</u> observable.

$\{A, B, C\}$ are equivalent to $\{\bar{A}, \bar{B}, \bar{C}\}$

where $\qquad \bar{A} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix}$

$$\bar{C} = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix}$$

$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix}$ has rank n iff $\begin{bmatrix} \lambda I - \bar{A} \\ \bar{C} \end{bmatrix}$ has rank n

$$\rho \begin{bmatrix} \lambda I - \bar{A} \\ \bar{C} \end{bmatrix} = \rho \begin{bmatrix} \lambda I - \bar{A}_o & 0 \\ -\bar{A}_{21} & \lambda I - \bar{A}_{\bar{o}} \\ \bar{C}_o & 0 \end{bmatrix} < n$$

for any eigenvalue of $\bar{A}_{\bar{o}}$.

Kalman's Canonical Decomposition

Given $\dot{x} = Ax + Bu$
$\quad\quad y = Cx + Du$

There exists a similarity transformation Q_1 such that

$$\left[\begin{array}{c:c} Q_1^{-1} A Q_1 & Q_1^{-1} B \\ \hdashline C Q_1 & D \end{array}\right] = \left[\begin{array}{cc:c} \bar{A}_c & \bar{A}_{12} & \bar{B}_c \\ 0 & \bar{A}_{\bar{c}} & 0 \\ \hdashline \bar{C}_c & \bar{C}_{\bar{c}} & D \end{array}\right] =: \left[\begin{array}{c:c} \bar{A} & \bar{B} \\ \hdashline \bar{C} & D \end{array}\right]$$

- (A_c, B_c) is controllable
- $\bar{C}_c (sI - \bar{A}_c)^{-1} \bar{B}_c + D = C(sI - A)^{-1} B + D$

There exists $\tilde{Q}_1$ (similarity transformation) so that

$$\left[\begin{array}{c|c} \tilde{Q}_1^{-1}\bar{A}_c\tilde{Q}_1 & \tilde{Q}_1^{-1}\bar{B}_c \\ \hline \bar{C}_c\tilde{Q}_1 & D \end{array}\right] = \left[\begin{array}{cc|c} \bar{A}_{co} & 0 & \bar{B}_{co} \\ \tilde{A} & A_{c\bar{o}} & \bar{B}_{c\bar{o}} \\ \hline \bar{C}_{co} & 0 & D \end{array}\right]$$

There exists $\tilde{Q}_2$ (similarity transf.) so that

$$\left[\begin{array}{c|c} \tilde{Q}_2^{-1}\bar{A}_{\bar{c}}\tilde{Q}_2 & 0 \\ \hline \bar{C}_{\bar{c}}\tilde{Q}_2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_{\bar{c}o} & 0 & 0 \\ \tilde{\tilde{A}} & A_{\bar{c}\bar{o}} & 0 \\ \hline \bar{\bar{C}}_{\bar{c}o} & 0 & 0 \end{array}\right]$$

<u>Comments</u>: 1. $(\bar{A}_{co}, \bar{C}_{co})$ is observable.

2. $(\bar{A}_{co}, \bar{B}_{co})$ is controllable! This follows from the fact that $(\bar{A}_c, \bar{B}_c)$ is controllable

3. $\bar{C}_{co}(sI-\bar{A}_{co})^{-1}\bar{B}_{co} + D = C(sI-A)^{-1}B + D$

Indeed, comment 2 can be justified as follows:

The controllability matrix of $(\bar{A}_c, \bar{B}_c)$ has full rank, which implies that the controllability matrix of $(\tilde{Q}_1^{-1}\bar{A}_c\tilde{Q}_1, \tilde{Q}_1^{-1}\bar{B}_c)$ has full rank, i.e.

$$\begin{bmatrix} \bar{B}_{co} & \bar{A}_{co}\bar{B}_{co} & \cdots & \bar{A}_{co}^{n_1-1}\bar{B}_{co} \\ B_{c\bar{o}} & * & & * \end{bmatrix}$$ has full rank

This implies that

$[\bar{B}_{co} \;\; \bar{A}_{co}\bar{B}_{co} \;\; \cdots \;\; \bar{A}_{co}^{n_1-1}\bar{B}_{co}]$ has full row rank

which means that $(\bar{A}_{co}, \bar{B}_{co})$ is controllable.

Suppose we define

$$Q_2 = \begin{bmatrix} \tilde{Q}_1 & 0 \\ 0 & \tilde{Q}_2 \end{bmatrix}$$

Then

$$\left[\begin{array}{c|c} Q_2^{-1} \bar{A} Q_2 & Q_2^{-1} \bar{B} \\ \hline \bar{C} Q_2 & D \end{array}\right] = \left[\begin{array}{cc|c} \tilde{Q}_1^{-1} \bar{A}_c \tilde{Q}_1 & \tilde{Q}_1^{-1} \bar{A}_{12} \tilde{Q}_2 & \tilde{Q}_1^{-1} \bar{B}_c \\ 0 & \tilde{Q}_2^{-1} \bar{A}_{\bar{c}} \tilde{Q}_2 & 0 \\ \hline \bar{C}_c \tilde{Q}_1 & \bar{C}_{\bar{c}} \tilde{Q}_2 & D \end{array}\right]$$

$$= \left[\begin{array}{cccc|c} \bar{A}_{co} & 0 & \bar{A}_{13} & \bar{A}_{14} & \bar{B}_{co} \\ \tilde{A} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} & \bar{B}_{c\bar{o}} \\ 0 & 0 & A_{\bar{c}o} & 0 & 0 \\ 0 & 0 & \tilde{\tilde{A}} & A_{\bar{c}\bar{o}} & 0 \\ \hline \bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} & 0 & D \end{array}\right]$$

This is called the Kalman Canonical Decomposition

To find a transformation which gives the Kalman decomposition directly from $\left[\begin{array}{c:c} A & B \\ \hdashline C & D \end{array}\right]$ is easy:

Simply take $Q = Q_2 \cdot Q_1$.

<u>Note</u>: If one is only interested in a controllable & observable realization of $\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array}$, there is no need to compute the Kalman canonical decomposition. Instead

(a) Get a controllable subsystem $(\bar{A}_c, \bar{B}_c, \bar{C}_c, \bar{D})$

(b) For $(\bar{A}_c, \bar{B}_c, \bar{C}_c, \bar{D})$ get an observable subsystem $(\bar{A}_{co}, \bar{B}_{co}, \bar{C}_{co}, \bar{D})$, which will be both controllable <u>and</u> observable. It also has the same transfer function as the original system.

<u>Definition:</u> A state-space description $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$ is said

to be <u>reducible</u> if there exists a state space description

$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}u \\ y &= \tilde{C}\tilde{x} + Du \end{aligned}$ of lesser dimension which has the same

transfer function, i.e. $C(sI-A)^{-1}B = \tilde{C}(sI-\tilde{A})^{-1}\tilde{B}$.

Otherwise, the state-space description is said to be <u>irreducible</u> (or <u>minimal</u>).

We have seen that <u>if a state-space realization is minimal then it is both controllable and observable</u>.

The reverse direction is also true. In other words,

<u>If a state-space realization</u> $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$ <u>is both controllable and observable, then it must be minimal.</u>

<u>Proof</u>: Let $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ be such that

$$\tilde{C}(sI-\tilde{A})^{-1}\tilde{B} + \tilde{D} = C(sI-A)^{-1}B + D$$

This implies that $\tilde{D} = D$. Also since $(sI-A)^{-1} = s^{-1}I + s^{-2}A + s^{-3}A^2 + \dots$

It follows that:

$$CBs^{-1} + CABs^{-2} + CA^2Bs^{-3} + \dots = \tilde{C}\tilde{B}s^{-1} + \tilde{C}\tilde{A}\tilde{B}s^{-2} + \tilde{C}\tilde{A}^2\tilde{B}s^{-3} + \dots$$

$$\therefore \; CA^kB = \tilde{C}\tilde{A}^k\tilde{B} \qquad k = 0, 1, \dots$$

$$\therefore \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix} \begin{bmatrix} \tilde{B} & \tilde{A}\tilde{B} & \cdots & \tilde{A}^{n-1}\tilde{B} \end{bmatrix}$$

or $\quad \mathcal{V}\mathcal{U} = \tilde{\mathcal{V}}\tilde{\mathcal{U}}$

Now we can use the following fact:

$$\rho(A)+\rho(B)-n \le \rho(AB) \le \min\{\rho(A), \rho(B)\}$$

to conclude that $\rho(\mathcal{V}\mathcal{U}) = n$.

Therefore $\rho(\tilde{\mathcal{V}}\tilde{\mathcal{U}}) = n$ which in turn implies that $\rho(\tilde{\mathcal{V}}) \ge n$ and $\rho(\tilde{\mathcal{U}}) \ge n$. Therefore ~~there does not~~ the ~~exist~~ dimension of the state-space of $\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$ is greater than or equal to n.

Q.E.D.

<u>Theorem:</u> Let $\{A, B, C, D\}$ and $\{\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}\}$ be two minimal realizations of a transfer function $G(s)$. Then the two realizations are equivalent, i.e. there exists an invertible Q such that $\tilde{A} = Q^{-1} A Q$, $\tilde{B} = Q^{-1} B$ and $\tilde{C} = CQ$.

<u>Proof:</u> Since the two realizations have the same transfer function it follows that $CA^k B = \tilde{C} \tilde{A}^k \tilde{B}$ $\quad k = 0, 1, \cdots$

This implies:

$$\mathcal{V}\mathcal{U} = \tilde{\mathcal{V}}\tilde{\mathcal{U}} \quad \cdots (\mathrm{I})$$

and

$$\mathcal{V}A\mathcal{U} = \tilde{\mathcal{V}}\tilde{A}\tilde{\mathcal{U}} \quad \cdots (\mathrm{II})$$

By minimality, both realizations are controllable and observable. Furthermore, $\rho(\mathcal{U}) = \rho(\mathcal{V}) = \rho(\tilde{\mathcal{U}}) = \rho(\tilde{\mathcal{V}}) = n$

We can solve for $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{V}}$ from (I).

Because $\tilde{\mathcal{V}}$ has column rank (by virtue of observ.), it has a left inverse, namely $(\tilde{\mathcal{V}}^*\tilde{\mathcal{V}})^{-1}\tilde{\mathcal{V}}^*$. Applying this inverse to both sides of (I) we get

$$\tilde{\mathcal{U}} = (\tilde{\mathcal{V}}^*\tilde{\mathcal{V}})^{-1}\tilde{\mathcal{V}}^* \mathcal{V}\mathcal{U} \quad \text{----} \quad (\mathrm{III})$$

Similarly, $\tilde{\mathcal{U}}$ has a right inverse given by $\tilde{\mathcal{U}}^*(\tilde{\mathcal{U}}\tilde{\mathcal{U}}^*)^{-1}$.

Multiplying (I) by this inverse from the right we get

$$\tilde{\mathcal{V}} = \mathcal{V}\underbrace{\mathcal{U}\tilde{\mathcal{U}}^*(\tilde{\mathcal{U}}\tilde{\mathcal{U}}^*)^{-1}}_{=: Q} \quad \text{----} \quad (\mathrm{IV})$$

Q is invertible since $n = \rho(\tilde{\mathcal{V}}) \leq \min\{\rho(\mathcal{V}), \rho(Q)\}$

Multiplying (III) by the right inverse of $\tilde{\mathcal{U}}$ (from the right) we get $I = (\tilde{V}^* \tilde{V})^{-1} \tilde{V}^* V \mathcal{U} \underbrace{\tilde{\mathcal{U}}^* (\tilde{\mathcal{U}} \tilde{\mathcal{U}}^*)^{-1}}_{\text{right inverse of } \tilde{\mathcal{U}}}$

$$= (\tilde{V}^* \tilde{V})^{-1} \tilde{V}^* V Q$$

$\therefore (\tilde{V}^* \tilde{V})^{-1} \tilde{V}^* V = Q^{-1}$. (III) can therefore be written

as $\tilde{\mathcal{U}} = Q^{-1} \mathcal{U}$ .... (III′)

From this equation we get that $\tilde{B} = Q^{-1} B$

From (IV), we get that $\tilde{C} = CQ$

Equation (II) states that: $VA\mathcal{U} = \tilde{V} \tilde{A} \tilde{\mathcal{U}}$

Multiplying from the right by the right inverse of $\tilde{\mathcal{U}}$ and from the left by the left inverse of $\tilde{V}$ we have:

$$\underbrace{(\tilde{V}^* \tilde{V})^{-1} \tilde{V}^* V}_{Q^{-1}} A \underbrace{\mathcal{U} \tilde{\mathcal{U}}^* (\tilde{\mathcal{U}} \tilde{\mathcal{U}}^*)^{-1}}_{Q} = \tilde{A}$$

Q.E.D.