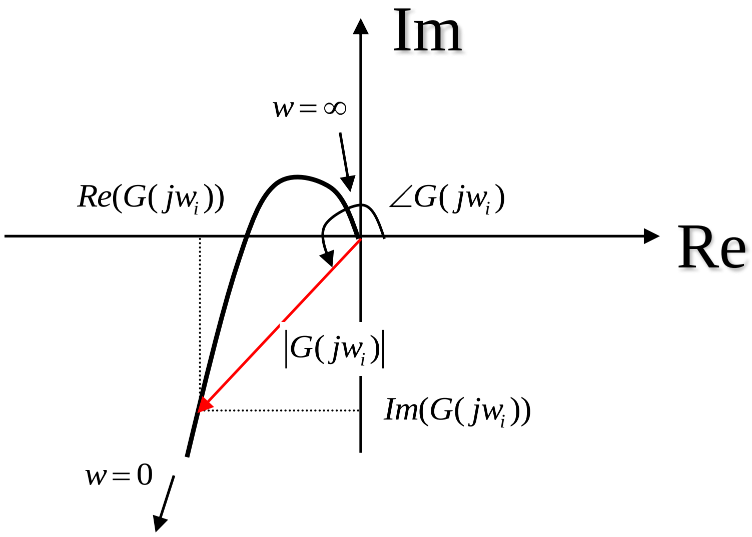
Nyquist Plots

- Plot $|G(j\omega)|\angle(G(j\omega))$ as ω goes from 0 to infinity, the Nyquist plot is the locus of vectors represented
- Convention: positive phase angles are measured counter-clockwise from the real axis



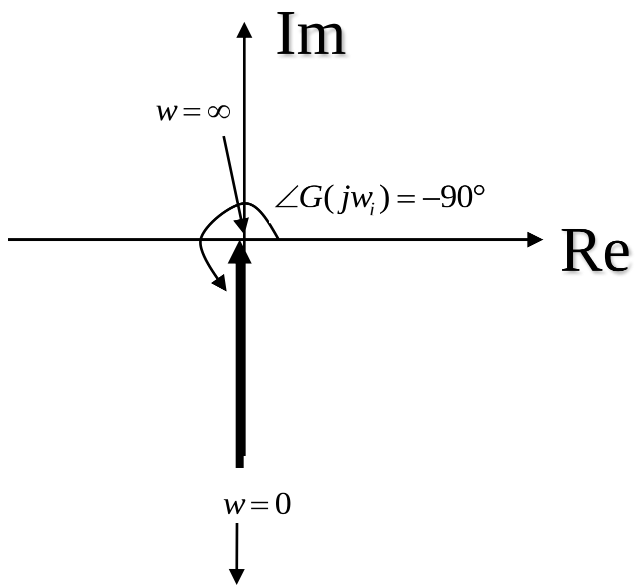
Examples

- Integrator: $G(j\omega) = \frac{1}{j\omega}$

$$|G(j\omega)| = \frac{1}{\omega}$$

$$\angle(G(j\omega)) = \tan^{-1}\left(\frac{-1/\omega}{0}\right) = -90^o$$

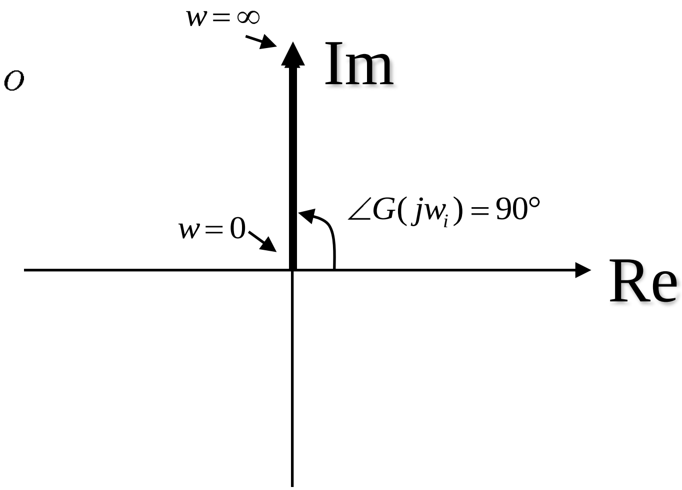
- Nyquist plot is the negative imag. axis



- Derivative: $G(j\omega) = j\omega$

$$|G(j\omega)| = \omega,\ \angle(G(j\omega)) = \tan^{-1}\left(\frac{\omega}{0}\right) = 90^o$$

- Nyquist plot is the positive imag. axis

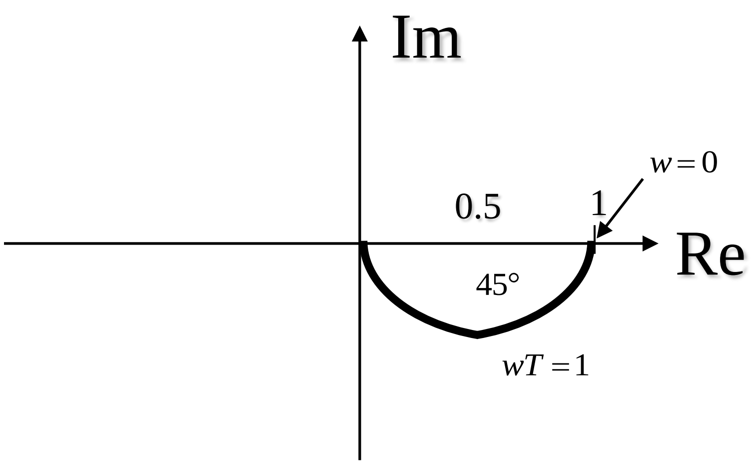


Examples

- First order pole

$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2\tau^2}}$$

$$\angle(G(j\omega)) = -\tan^{-1}(\omega\tau)$$



- Asymptotes (from Bode plot):

$$\omega << \frac{1}{T},\ \ |G(j\omega)| = 1,\ \angle(G(j\omega)) = 0$$

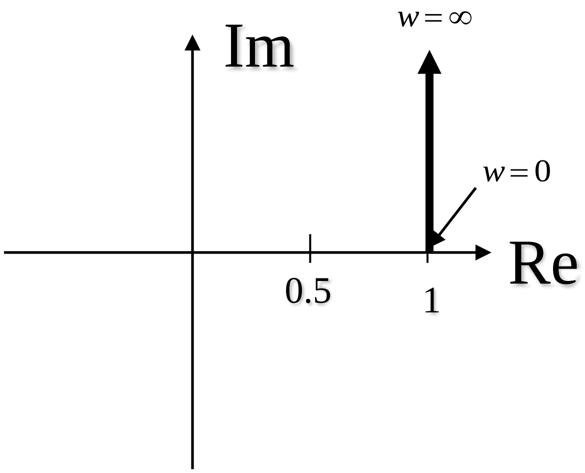
$$\omega \approx \frac{1}{T},\ \ |G(j\omega)| = \frac{1}{\sqrt{2}},\ \angle(G(j\omega)) = -45^O$$

$$\omega >> \frac{1}{T},\ \ |G(j\omega)| = 0,\ \angle(G(j\omega)) = -90^O$$

Example: First order zero

- First order zero: $G(j\omega) = j\omega T + 1$

$$|G(j\omega)| = \sqrt{1 + \omega^2 \tau^2}$$
$$\angle(G(j\omega)) = \tan^{-1}(\omega\tau)$$



- Asymptotes (from Bode plot):

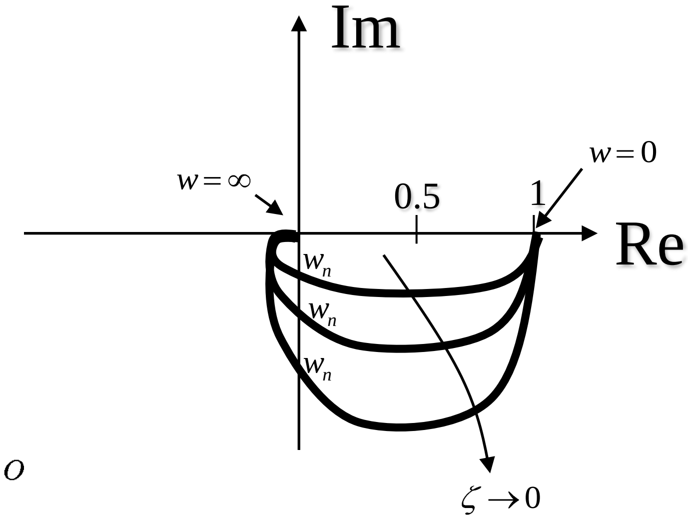
$$\omega << \frac{1}{T},\ |G(j\omega)| = 1,\ \angle(G(j\omega)) = 0$$
$$\omega \approx \frac{1}{T},\ |G(j\omega)| = \sqrt{2},\ \angle(G(j\omega)) = 45^o$$
$$\omega >> \frac{1}{T},\ |G(j\omega)| \approx \infty,\ \angle(G(j\omega)) = 90^o$$

Example: second order poles

- Second-order poles:

$$G(j\omega) = \frac{1}{\left(j\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + 1}$$

$$\angle(G(j\omega)) = -\tan^{-1}\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1-\left(\frac{\omega}{\omega_n}\right)^2}$$



- Asymptotes (from Bode plot):

$$\omega << \omega_n,\ \ |G(j\omega)| = 1,\ \angle(G(j\omega)) = 0$$

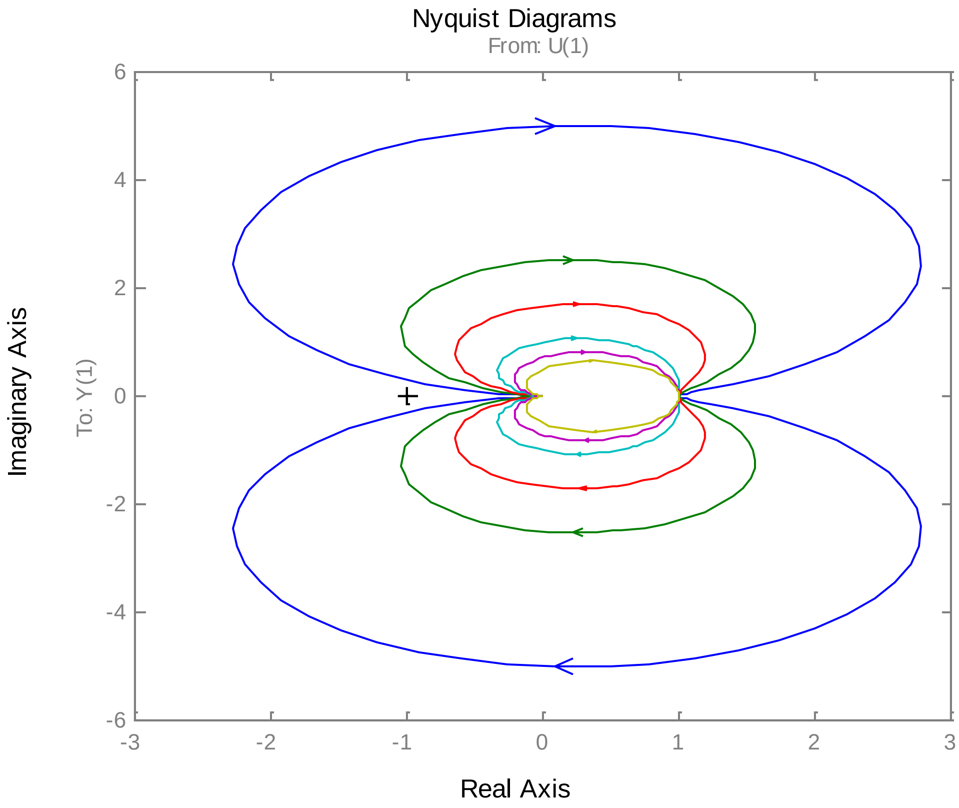
$$\omega \approx \omega_n,\ \ |G(j\omega)| = \frac{}{2\zeta},\ \angle(G(j\omega)) = -90^o$$

$$\omega >> \omega_n,\ \ |G(j\omega)| \approx 0,\ \angle(G(j\omega)) = -180^o$$

- Frequency point whose distance from the origin is a maximum corresponds to the resonant frequency

Example: second order poles

- MATLAB Example
  - ⋆ Note that MATLAB plots both positive and negative frequency plots
  - ⋆ The negative-frequency plot is ALWAYS a mirror of positive Nyquist plot about real axis



MATLAB commands:

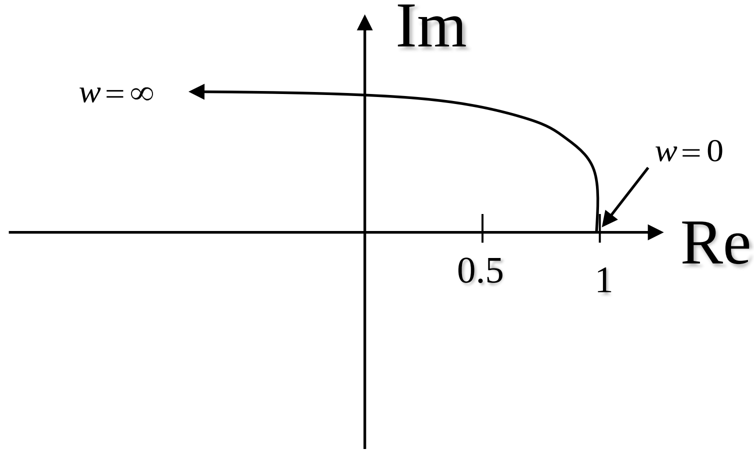
```
for zeta=[0.1 0.2 0.3 0.5 0.7 1]
  nyquist(1,[1 2*zeta*1 1]);
  hold on;
end
```

Examples: second order zeros/ Time delay

- Second-order zeros:

$$G(j\omega) = \left(j\frac{\omega}{\omega_n}\right)^2 + 2\zeta\left(j\frac{\omega}{\omega_n}\right) + 1$$

  ⋆ very different from second orser poles!

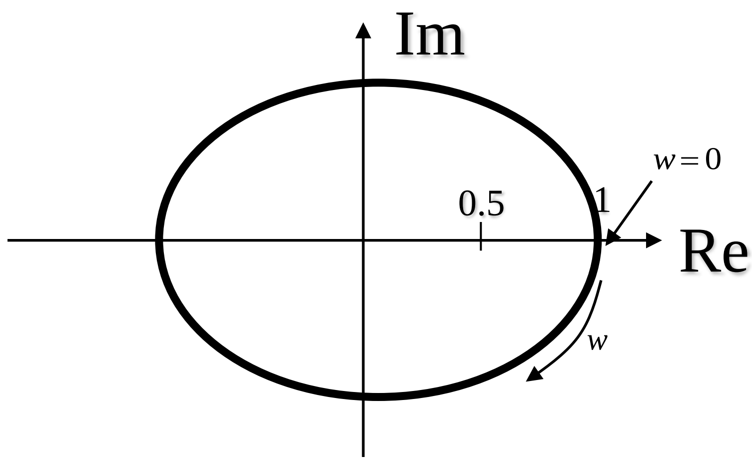


- Time delay

$$G(j\omega) = e^{-j\omega T}$$

  ⋆ From Euler Theorem
  $|G(j\omega)| = 1,\ \angle G(j\omega) = -\omega T$

  ⋆ Forms a circle that spirals forever on top of itself.



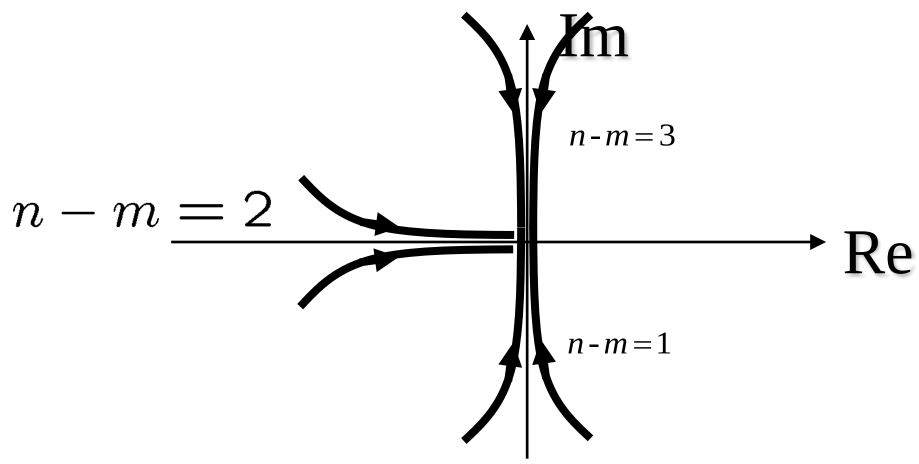
Relative Degree

- Write system as:

$$\frac{b_0 j\omega^m + b_1 j\omega^{m-1} + \cdots}{a_0 j\omega^n + a_1 j\omega^{n-1} + \cdots}$$

then the number (n-m) is the relative degree of the system.

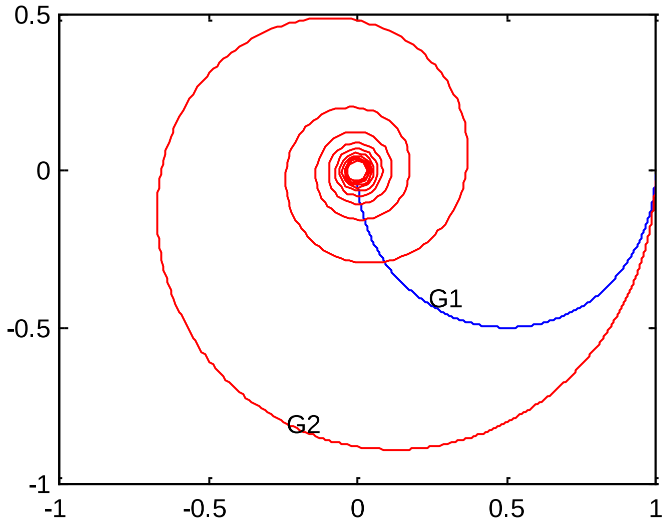
- Relative degree determines the high-frequency asymptote on the Bode plots and the axis the Nyquist plot will converge to:



Cascaded time delay

- Note: MATLAB cannot plot time delays (yet)
- Must write code to create Nyquist plot

$$G_1 = \frac{1}{s+1} \qquad G_2 = \frac{e^{-2s}}{s+1}$$

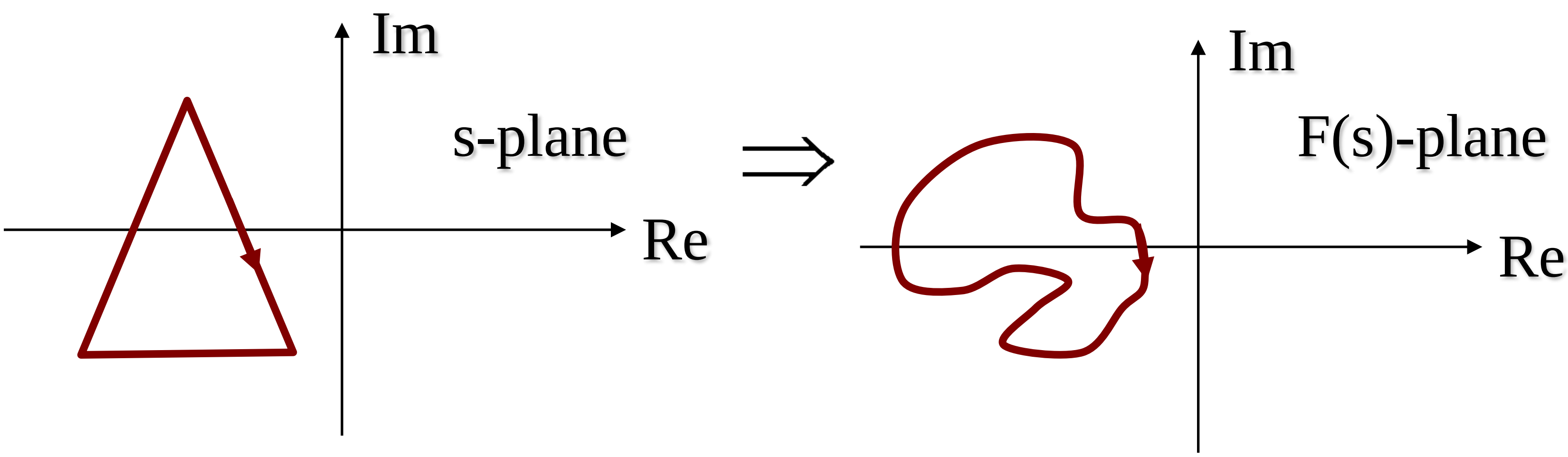


MATLAB commands:

```
delay = 2;
freqs = logspace(-2,1.5,1000);
[mag,pha] = bode(1,[1 1],freqs);
data_no_delay = [];
data_with_delay = [];
for i = 1:length(freqs)
   w = freqs(i);
   G       = mag(i)*exp(pha(i)*pi/180*j);
   G_delay  = mag(i)*exp((pha(i)*pi/180-w*delay)*j);
   data_no_delay = [data_no_delay;[real(G) imag(G)]];
   data_with_delay  = [data_with_delay; [real(G_delay) imag(G_delay)]];
end
plot(data_no_delay(:,1),data_no_delay(:,2),'b'); hold on;
plot(data_with_delay(:,1),data_with_delay(:,2),'r'); hold on;
```

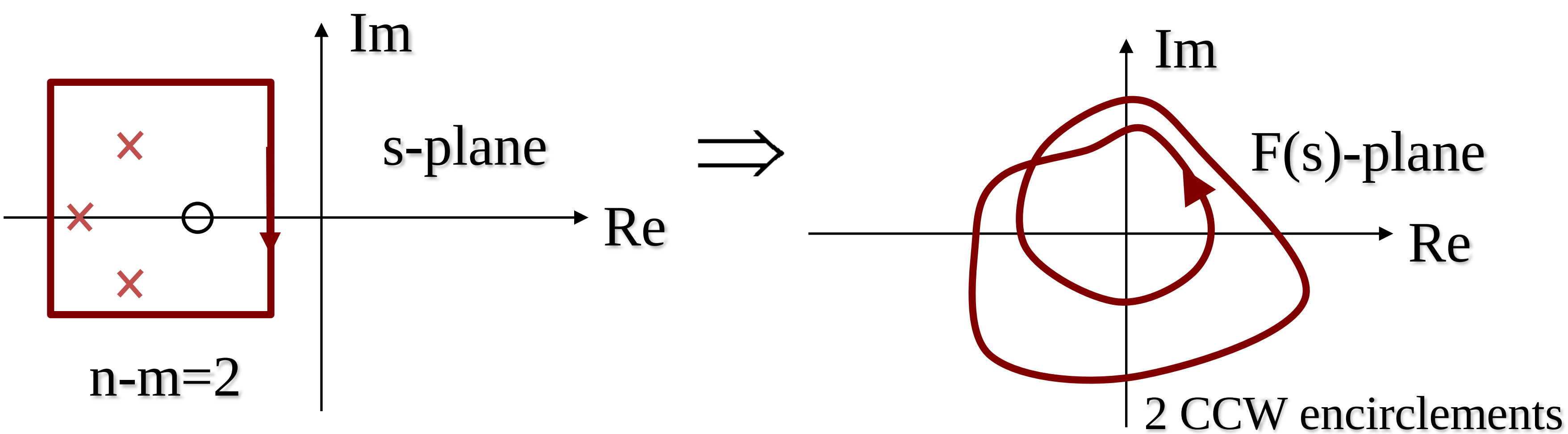
Conformal Mapping

- A contour drawn in the s-plane will correspond to a contour drawn in the F(s) plane:
- The area enclosed by a contour, by definition, is the area to the right as the contour is traversed in the clockwise direction



Conformal Mapping

- The excess of poles of F(s) over zeros (n-m) enclosed by the s-plane contour traversed CW corresponds to the number of times F(s) contour encircles the origin CCW.
  - ⋆ assume we dont draw contour through pole or zero
- A consequence of the Argument Principle (Math 346)



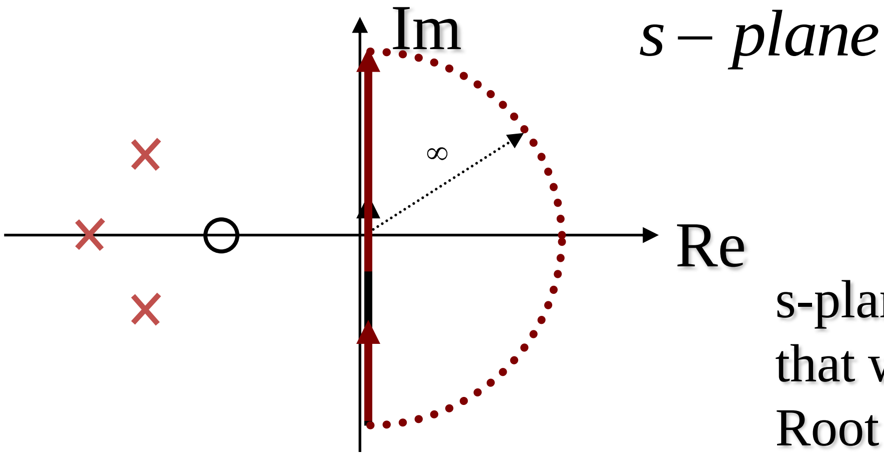
The Mapping Theorem

- Let F(s) be a ratio of polynomials in s, and P and Z are the # of poles and zeros respectively of F(s) that lie inside some closed contour in the s-plane
  - ⋆ The contour doesn't pass through poles or zeros of $F(s)$
  - ⋆ The contour gets traced out in the clockwise direction by a representative point s
  - ⋆ The s-plane contour gets mapped to another contour in $F(s)$ plane
- The Mapping Theorem: The total number, N, of clockwise encirclements of the origin in the F(s) plane is equal to $Z-P$; i.e.,

$$N = Z - P$$

The Mapping Theorem

- If choose contour correctly, can use Mapping Theorem to determine stability
  - ⋆ Choose contour that encloses entire RHP of the s-plane
    - – Contour goes along jw-axis, then circles back with infinite-radius half-circle in a clockwise direction.
    - – If any poles are in the RHP, they show up as CW encirclements of F(s) at origin



assume no jω axis poles for now

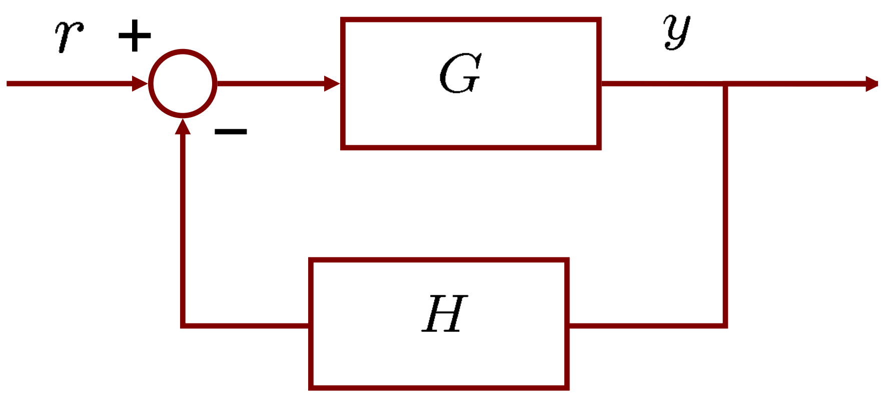
s-plane is the plane that we drew our Root Locus in

Use of the Mapping Theorem

characteristic polynomial

- If $F(s)$ is defined as $F(s) = 1 + G(s)H(s)$, then examining stability of $F(s)$ is the same as examining the number of encirclements of $-1 + j0$ by $G(s)H(s)$ contour.
  - ⋆ We are just shifting the axis.
- If we assume that the relative degree > 0, then we only have to evaluate the contour along $s = j\omega$, because the semicircle part at infinity drops out:
  - ⋆ since $\lim_{s\to\infty} G(s)H(s) = 0$ if $n > m$ and a constant if $n = m$.
  - ⋆ The plot of $1 + G(s)H(s)$ will stay at the same point so we do not have to consider it as a variable as ω goes to ∞

Nyquist Stability Criterion



- If the O.L. T.F. $G(s)H(s)$ has k poles in the RHP and $n \geq m$ then, as ω goes from $-\infty$ to $+\infty$, $G(j\omega)H(j\omega)$ must encircle the -1 point k times in the CCW direction for stability Mathematically

$$N = Z - P$$

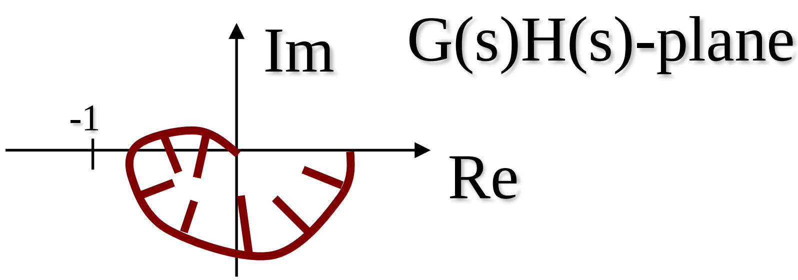
  ⋆ $Z = \#$ of zeros of $1 + G(s)H(s)$ in RHP (i.e Closed Loop poles)
  ⋆ $P = \#$ of poles of $G(s)H(s)$ in RHP (Open Loop poles)
  ⋆ $N = \#$ of CW encirclements of the -1 point by G(s)H(s)

Use of Nyquist Stability Criterion

- Q: How is Nyquist Stability Criterion Used?
- A: To find CL poles

$$\underbrace{Z}_{\substack{\text{\# of unstable}\\ \text{zeros of } 1+GH\\ \text{(closed loop poles)}}} = \underbrace{N}_{\substack{\text{\# of clockwise}\\ \text{encirclements of}\\ \text{the -1 point}}} + \underbrace{P}_{\substack{\text{\# of unstable}\\ \text{open loop}\\ \text{poles}}}$$

- If the O.L. system is stable, then there must be no encirclements of -1 for the C.L. system to be stable

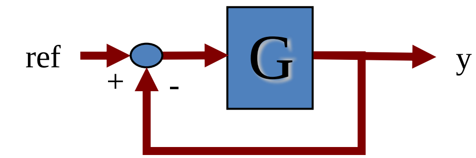


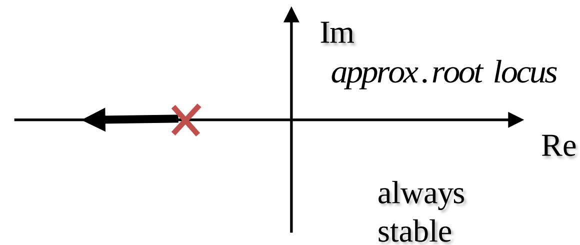
Example: First-order system

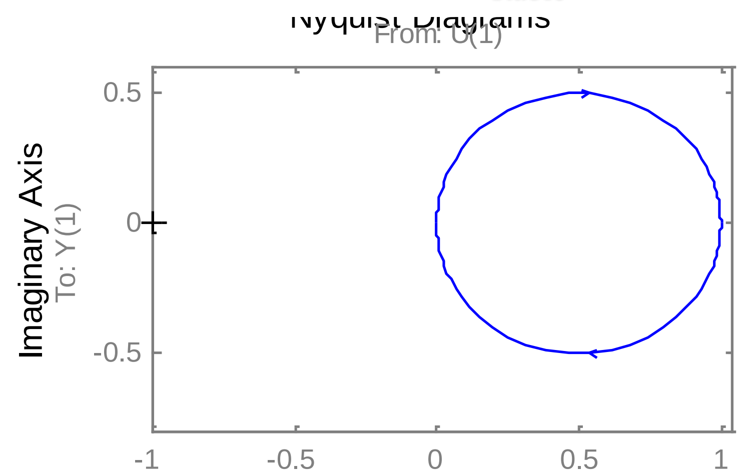
- Example: $G(s) = \frac{1}{s+1}$
- Nyquist plot: (MATLAB)
  - ⋆ $\#$ encirclements $= N = 0$
  - ⋆ $\#$ of Poles in RHP $= P = 0$
  - ⋆ $Z = N + P = 0$ implies stable!
- Matlab Command

```
G=tf(1,[1 1]);
nyquist(G);
```

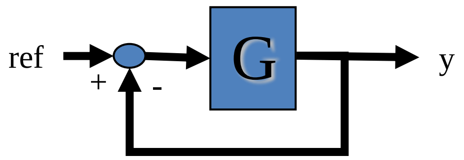
- Note that MATLAB plots the -1 point for you

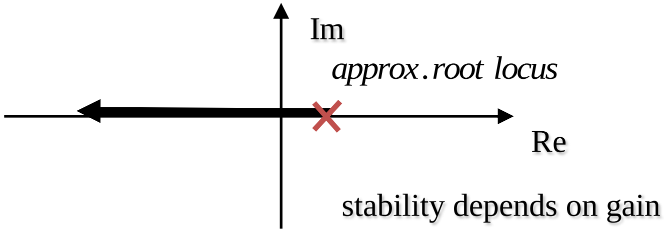






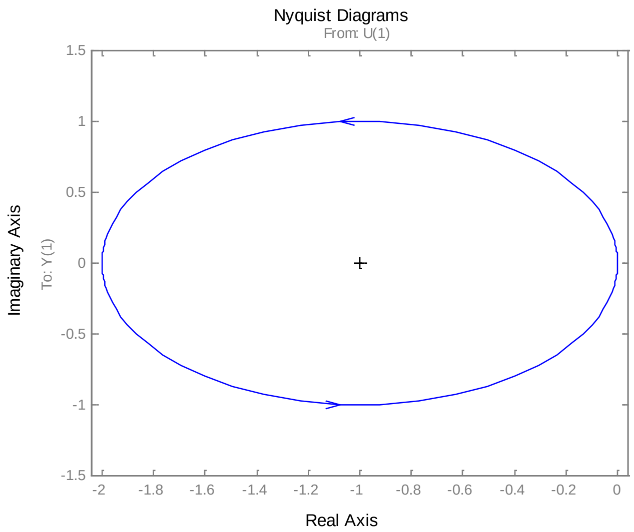
Example: Unstable system





- Example: $G(s) = \frac{2}{s-1}$
- Nyquist plot: (MATLAB)
  - ⋆ # encirclements $= N = -1$
  - ⋆ # of Poles in RHP $= P = 1$
  - ⋆ $Z = N + P = 0$ implies stable!
- Matlab Command

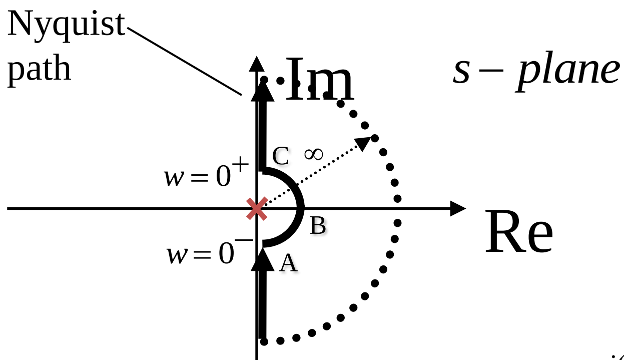
```
G=tf(2,[1 -1]);
nyquist(G);
```



- If gain is reduced, $N = 0$, and therefore $z = 1 \Rightarrow$ system is unstable (as expected)

$j\omega$ axis zeros and poles

- The s-plane contour that encircles the right-half plane is called the Nyquist path. This path follows $j\omega$-axis:
  - ⋆ We previously considered case with no $j\omega$-axis poles
- What if there are $j\omega$-axis poles?
  - ⋆ Nyquist path must be modified to not pass through them we add a jog around pole, usually a semi-circle of infinitely small radius

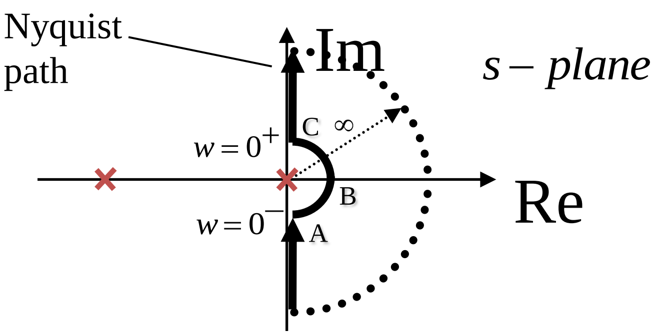


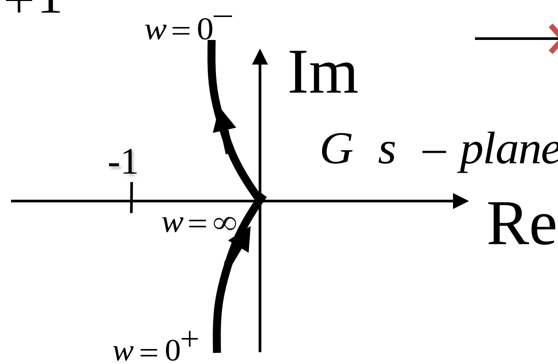
$$s = \varepsilon \cdot e^{j\theta} \Rightarrow G(\varepsilon \cdot e^{j\theta})H(\varepsilon \cdot e^{j\theta}) = \frac{K(1+\varepsilon \cdot e^{j\theta})\ldots}{\varepsilon \cdot e^{j\theta}(1+\varepsilon \cdot e^{j\theta})\ldots} \approx \frac{K}{\varepsilon \cdot e^{j\theta}} = \frac{K}{\varepsilon} \cdot e^{-j\theta}$$

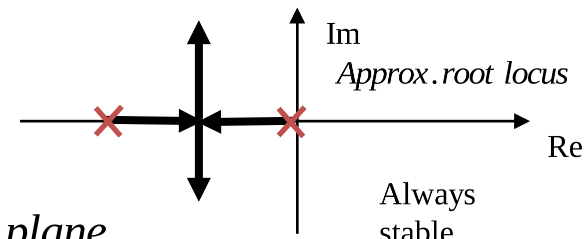
- The semi-circle maps to an infinite- radius semi-circle in the $G(s)H(s)$ plane
- Following example will illustrate

Example 1

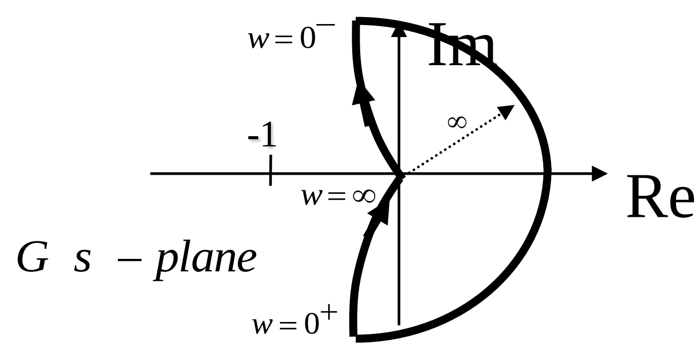
$$G(s)H(s)=\frac{K}{s(Ts+1)}$$





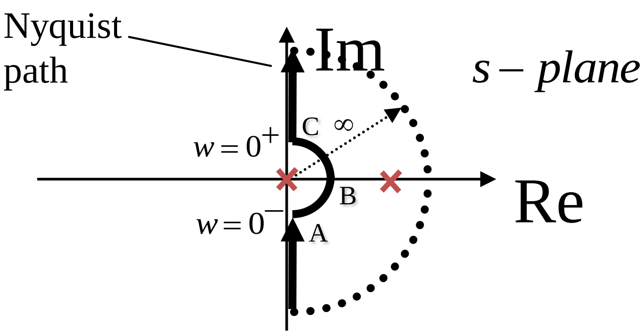


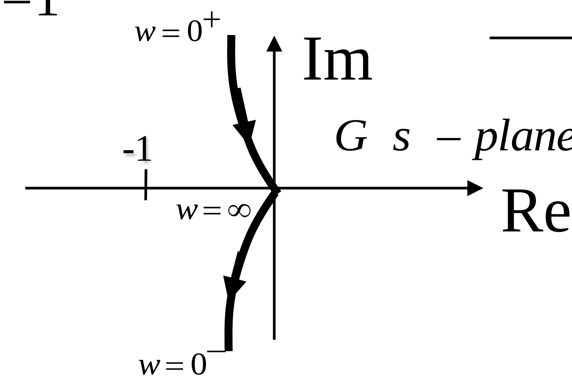
- Problem: How does plot close (MATLAB doesnt help)?
  - ⋆ Look at phase of GH as go around pole it goes from:
    - – $+90$ degrees at A ($\omega = 0-$) (Remember: Poles contribute negative phase)
    - – 0 degrees at B
    - – -90 degrees at $C(\omega = 0+)$
  - ⋆ In this case, plot must close in CW direction
    - – $N = 0, P = 0, Z = 0$ implies stable!

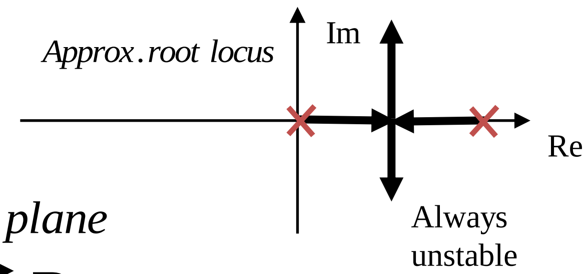


Example 2

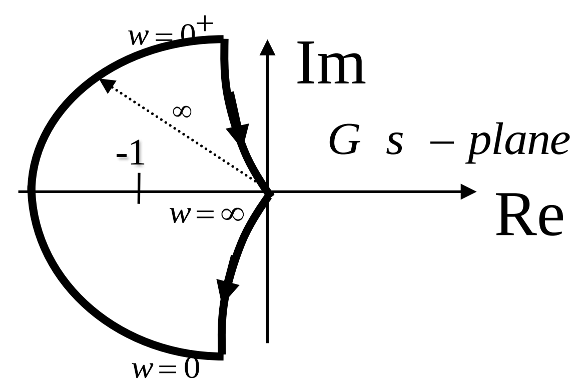
$$G(s)H(s) = \frac{K}{s(Ts-1)}$$



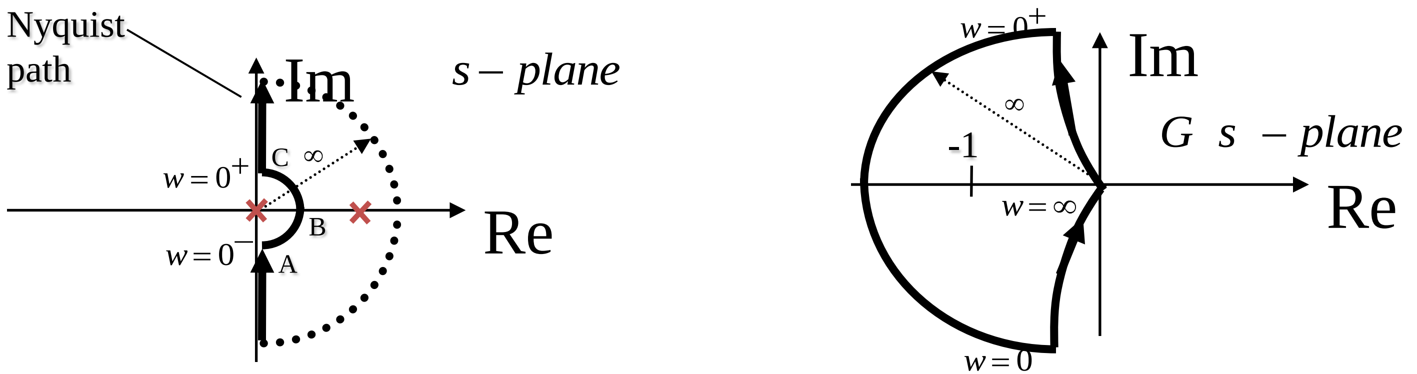




- Problem: How does plot close?
  - ⋆ Look at phase of GH as go around pole … it goes from:
    - – $90 - 180 = -90$ degrees at A $(\omega = 0-)$
    - – $0 - 180 = -180$ degrees at B
    - – $0 - 270 = -270$ degrees at $C(\omega = 0+)$
  - ⋆ In this case, plot must close in CW direction
    - – $N = 1, P = 1, Z = 2$ implies unstable!



Generalized Nyquist Stability

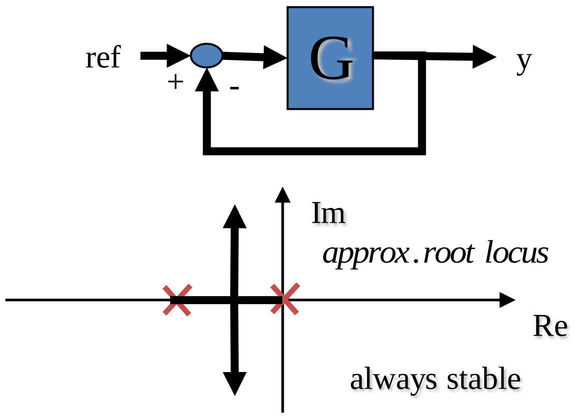


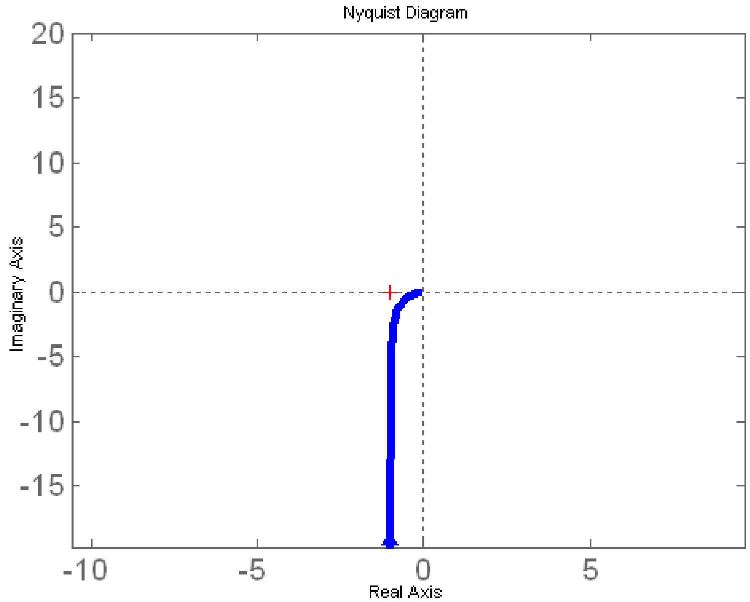
- If the OL transfer fn. $G(s)H(s)$ has k poles in the RHP, then for stability the $G(s)H(s)$ locus must encircle the -1 point k times in the CCW direction as a representative point s traces the modified Nyquist path in the CW direction
- See in Ogata for good examples

Example

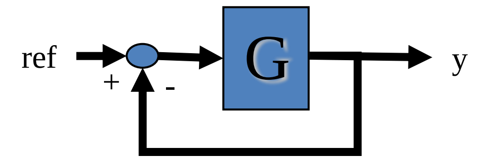
- $G(s) = \frac{1}{s(s+1)}$
- Nyquist plot: (MATLAB)
  - ⋆ # encirclements: $N = 0$
  - ⋆ # of Poles in RHP: $P = 0$
  - ⋆ $Z = N + P = 0$ implies stable!
- MATLAB commands:

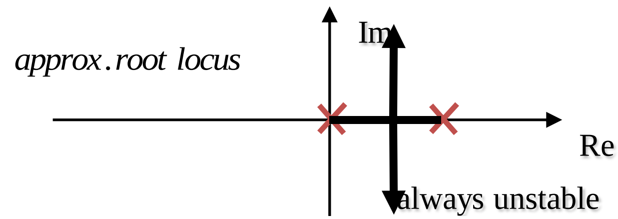
```
sys=tf([1],[1 1 0]);
nyquist(sys)
```

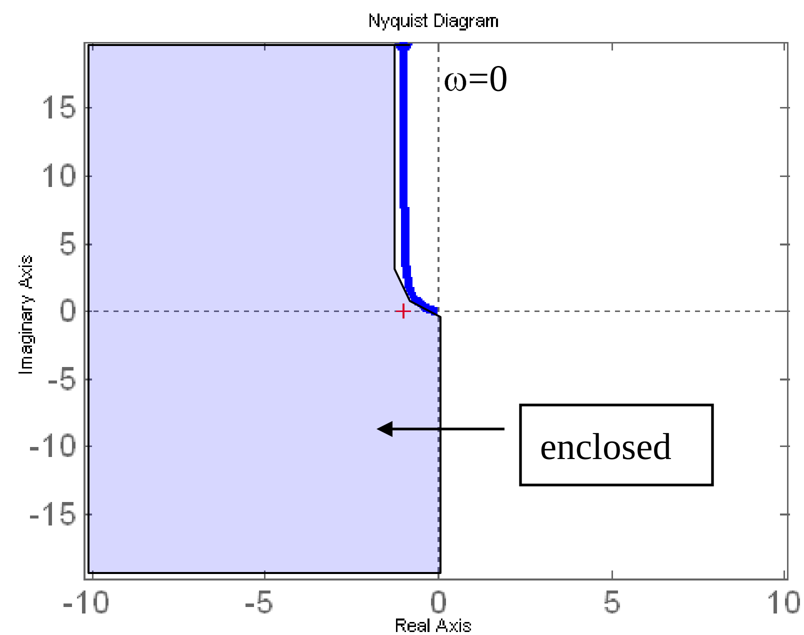




Example







- $G(s) = \frac{1}{s(s-1)}$
- Nyquist plot: (MATLAB)
  - ⋆ # encirclements: $N = 1$
  - ⋆ # of Poles in RHP: $P = 1$
  - ⋆ $Z = N + P = 2$ implies unstable!
- MATLAB commands:

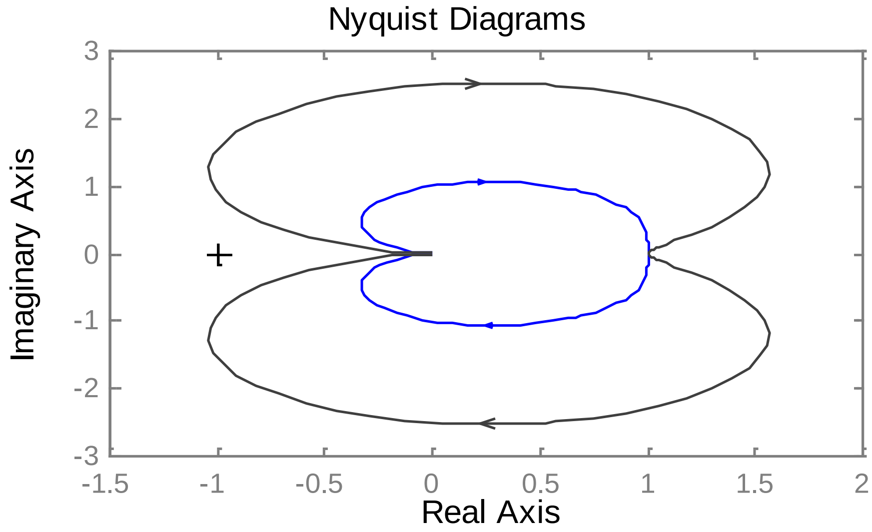
```
 sys=tf([1],[1 -1 0]);
nyquist(sys)
```

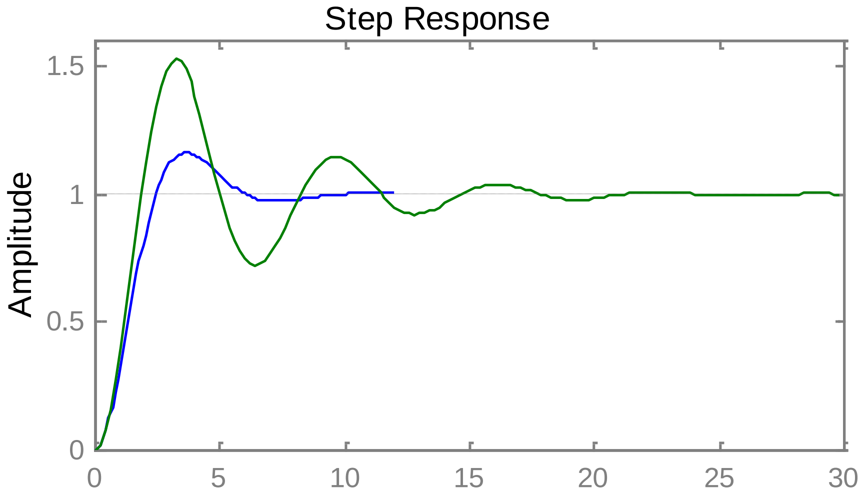
Possible Scenarios for Stability

- Three possible Nyquist Stability Scenarios:
  1. No encirclement of -1
     - ⋆ System is stable if there are no poles of $G(s)H(s)$ in RHP
     - ⋆ Otherwise unstable
  2. CCW encirclement of -1
     - ⋆ System is stable if # of CCW encirclements = # poles of $G(s)H(s)$ in RHP
     - ⋆ Otherwise unstable
  3. CW encirclement of -1
     - ⋆ Unstable system

Relative Stability

- In general, the closer the Nyquist plot is to the -1 point, the less stable the system will be:





Gain and Phase Margins

- Remember familiar notions of Gain and Phase margin
- Gain Margin - How much gain can be changed before instability

$$G.M. = \frac{1}{\left|GH(jw_\pi)\right|} = -20\log_{10}\left|GH(jw_\pi)\right|$$

frequency where $\angle GH = -180^\circ$

- Phase Margin - How much phase lag can be added before instability

$$\phi_{PM} = 180^\circ + \angle GH(jw_1) \quad degrees$$

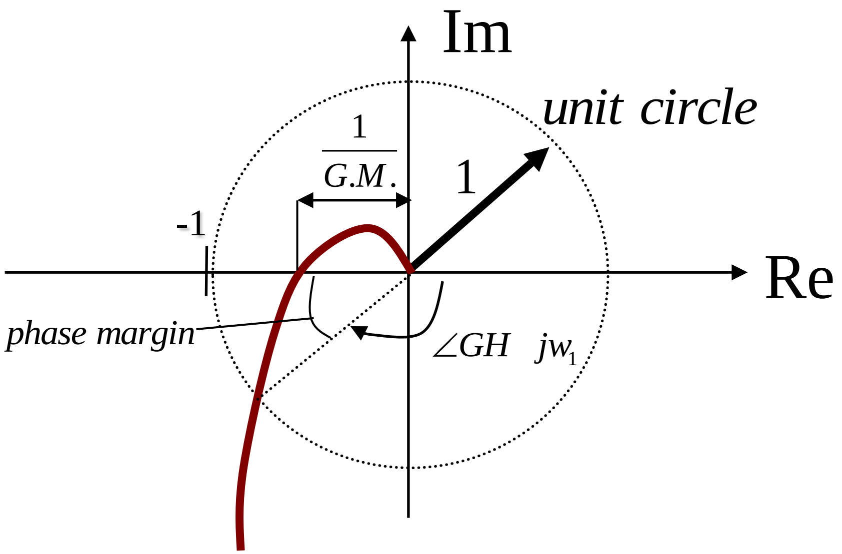
frequency where $\left|GH\right| = 1$

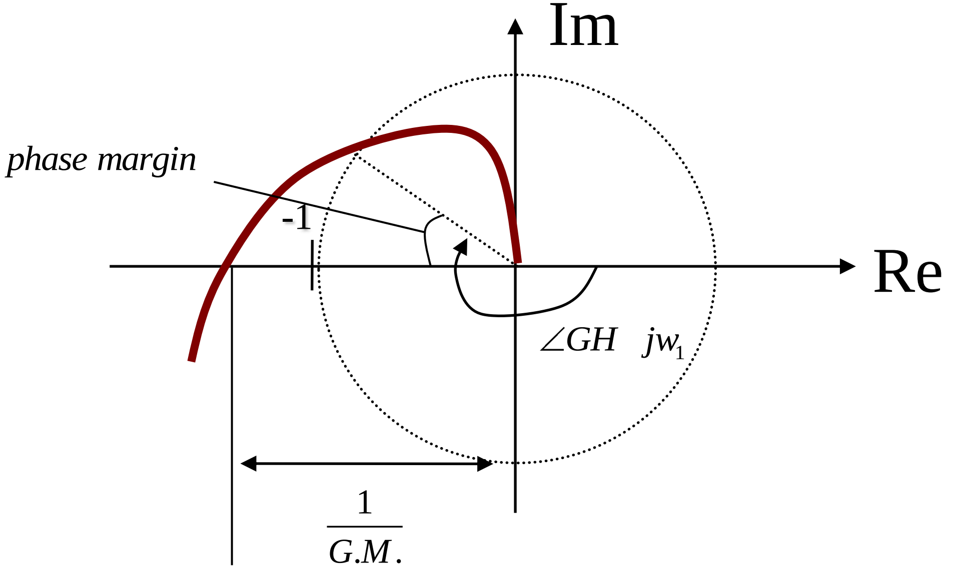
Gain and Phase Margins

Stable Systems

Unstable Systems

$$G(s) - plane$$





Purpose of Margins

- The gain and phase margins of a system are an indication of how close the system is to instability
  - ⋆ Good Design Targets:
    - PM: (Damping of .3 to .7 for 2nd order system: $30^o < PM < 70^o$)
    - GM: (Can account for gain uncertainty of a factor of 2: $GM > 6\ dB$)