Robust Control: HW 8

Problem 1: [On system zeros] Let $G(s)$ be a $p \times m$ transfer matrix and let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization of $G(s)$.

Definition 1 *$z_0 \in \mathcal{C}$ is an invariant zero of the system realization if it satisfies*

$$\begin{pmatrix} A - z_0 I & B \\ C & D \end{pmatrix} < normalrank \begin{pmatrix} A - sI & B \\ C & D \end{pmatrix}$$

1. Prove that $G(s)$ has full-column (row) normal rank if and only if $\begin{pmatrix} A - sI & B \\ C & D \end{pmatrix}$ has a full-column (row) normal rank.

   Thus z_0 is a zero of the transfer matrix $G(s)$ if and only if its an invariant zero.

2. Suppose $u(t) = u_0 e^{\lambda t}$ with $\lambda \in \mathcal{C}$ and $u_0 \in \mathcal{C}^m$ is a constant vector. Suppose $u(t)$ is a input to G where λ is not a pole of $G(s)$. Show that the output of the system with input $u(t)$ and with initial state $(\lambda I - A)^{-1} B u_0$ is $y(t) = G(\lambda) u_0 e^{\lambda t}$, for all $t \geq 0$. In particular if λ is a zero of $G(s)$ then $y(t) = 0$.

Problem 2:[On Internal Stability]

1. Let $P = \begin{pmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s+1} \end{pmatrix}$ and $K = \begin{pmatrix} \frac{1-s}{s+1} & -1 \\ 0 & -1 \end{pmatrix}$ in a positive feedback interconnection. Evaluate $det(I - PK)$ and find the poles of this transfer function. Are there any rhp poles? Can you conclude that the closed-loop system internally stable? If yes prove otherwise disprove.

2. Let $G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$ be the generalized plant with G_{22} the map from the control input u to the measured output y. Let $G_{11} = \frac{1}{s-1}$, $G_{12} = G_{21} = 0$ and $G_{22} = \frac{1}{(s-2)(s-3)}$. Is this generalized plant stabilizable through u?

Problem 3: [On Coprime Factorization]

1. Suppose $K = UV^{-1}$ is a right coprime factorization of K that internally stabilizes $G = NM^{-1}$ in a positive feedback interconnection. Prove that all stabilizing controllers can be parameterized as

$$K' = (U + MQ)(V + NQ)^{-1} \text{ with } Q \text{ stable.}$$

2. Let $P = \frac{1}{s-1}$. Obtain a parametrization of all controllers K that internally stabilize P in a positive feedback interconnection.

3. (Internal Model Control) Consider Figure **??** that shows a plant P and P_0 is a model that is used to implement a controller (the dotted outline shows the controller). These types of controllers are very common in process control.

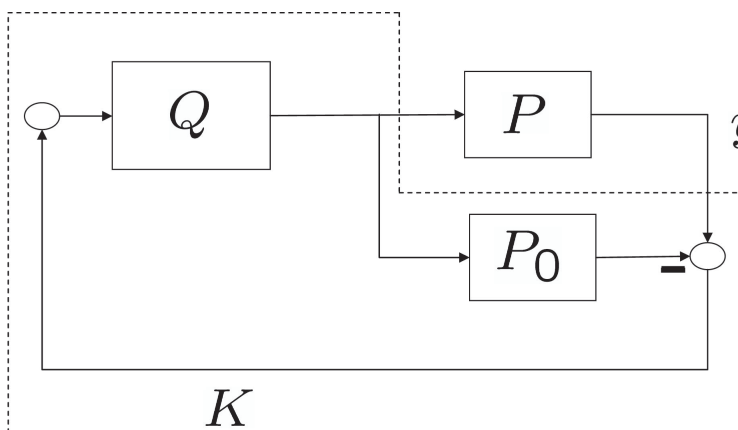

Figure 1:

- Show that is the closed-loop system is internally stable then necessarily P and P_0 cannot have any common unstable poles.
- If $P_0 = P$ (that is we have an exact model) then show that the system is stable for any stable Q. Show that this configuration gives a parametrization of all stabilizing controllers of P_0.

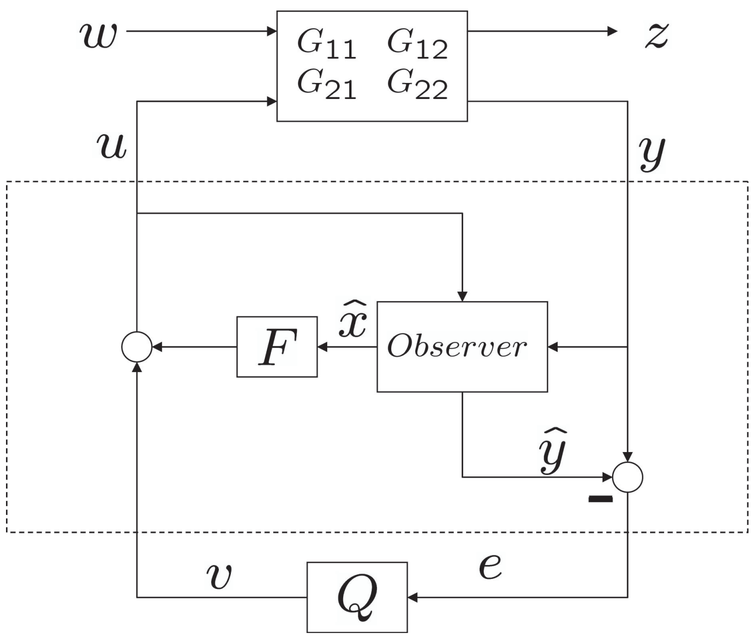

Figure 2:

4. (Parametrization Via Observer Based Feedback) Consider Figure **??** that shows a Generalized plant G with G_{22} being the map from the control input to the measured output y. Let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ be a minimal realization of G with the inherited realization of G_{22} being $\left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ (assume that the inherited realization is stabilizable and detectcable). Thus G_{22} is described by the dynamics

$$\begin{aligned} \dot{x} &= Ax + B_2 u \\ y &= Cx + D_{22} u \end{aligned}$$

Consider a observer that is described by the dynamics

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + B_2 u - L(y - \hat{y}) = (A + LC_2)\hat{x} + B_2 u - Ly \\ \hat{y} &= C\hat{x} + D_{22} u \end{aligned}$$

where L is chosen such that $A + LC_2$ has all eigenvalues in the left half plane. Suppose F is such that $A + B_2 F$ has all eigenvalues in the left half plane. Let $u = F\hat{x} + v$.

(a) Obtain a realization of the controller (called the observer based controller) above .

Hint: Note that the controller maps y to u.

(b) Show that with the above controller implementation the interconnection is stable. *Hint: Compute the closed-loop A matrix and show that it is Hurwitz (all eigenvalues in the left half plane).*

(c) Show that the map from v to e is zero.

Let $M,\ N,\ Y,\ X$ and $\tilde{M},\ \tilde{N},\ \tilde{Y},\ \tilde{X}$ be such that

$$\begin{pmatrix} M & Y \\ N & X \end{pmatrix} \begin{pmatrix} -v \\ e \end{pmatrix} = \begin{pmatrix} u \\ y \end{pmatrix},$$

$$\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} -v \\ e \end{pmatrix}.$$

(a) Show that a realization for $\begin{pmatrix} M & Y \\ N & X \end{pmatrix}$ is

$$\left[\begin{array}{l|rr} A + B_2F & -B_2 & -L \\ \hline F & -I & 0 \\ C_2 + D_{22}F & -D_{22} & -I \end{array}\right],$$

Hint: Use Figure **??** *and the plant, observer realization to obtain the result.*

(b) Show that a realization for $\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix}$ is

$$\left[\begin{array}{c|cc} A + LC_2 & (B_2 + LD_{22}) & -L \\ \hline F & -I & 0 \\ -C_2 & -D_{22} & -I \end{array}\right].$$

Hint: Use Figure **??** *and the plant, observer realization to obtain the result.*

(c) Show that $\begin{pmatrix} M & Y \\ N & X \end{pmatrix}$ and $\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix}$ forms a doubly coprime factorization of the plant $G_{22} = NM^{-1} = \tilde{M}^{-1}\tilde{N}$.

(d) Show that all stabilizing controllers can be generated by letting $v = Qe$ with Q stable.

(e) Let J be the map from y and v to u and e. Obtain a realization of this map.

(f) Show that all stabilizing controllers can be obtained as

$$K(Q) = \mathcal{F}_\ell(J, Q),\ \ Q \text{ stable.}$$

Show this relationship schematically with the generalized plant.

(g) Let $G_{11} = \frac{1}{s+1}$, $G_{12} = \begin{pmatrix} \frac{1}{s+1} & \frac{1}{s+2} \end{pmatrix}$, $G_{21} = \begin{pmatrix} \frac{1}{s+1} \\ \frac{1}{s+1} \end{pmatrix}$ and $G_{22} = \begin{pmatrix} \frac{1}{s-1} & \frac{1}{s-2} \\ \frac{2}{s} & \frac{1}{s+2} \end{pmatrix}$. Obtain the Youla parametrization of all stabilizing controllers. Obtain all closed loop maps achievable through stabilizing controllers in the form of $H - UQV$.