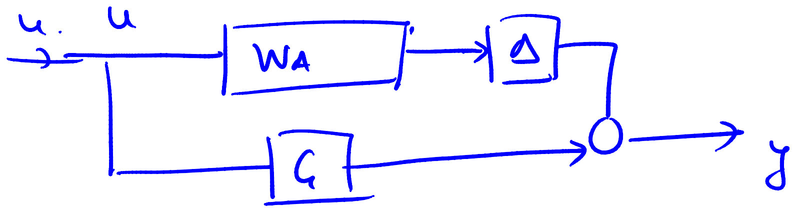
Lecture 14

Tuesday, March 08, 2011
8:12 AM

Robust stability and performance:

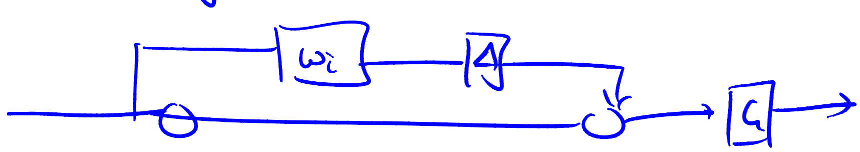
(1) Additive Uncertainty model



$$y = G + W_A \Delta.$$

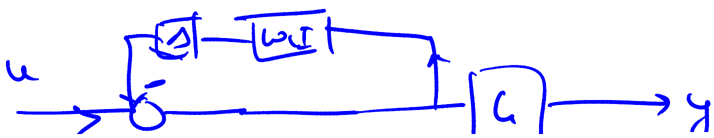
(2) Multiplicative uncertainty

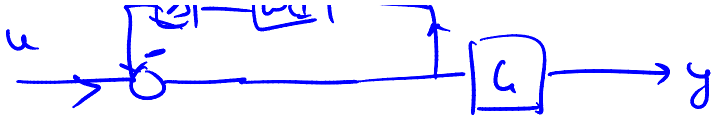
$$y = G(1 + w_i \Delta)$$



(3) Inverse multiplicative uncertainty

$$G(1 + w_{iI}\Delta)^{-1}$$





Assume that weights, ω_A, ω_i, ω_{iI} are all stable and minimumphase.

Δ is a stable transfer function in the sense that it belongs to H_∞.

H_∞ norm of a System is the induced L_2 norm:

Suppose Δ is analytic in the RHP.

$$\|\Delta\|_{H_\infty} = \sup_{Re(s)>0} |\Delta(s)| \longrightarrow \text{definition}$$

$$= \sup_{\omega>0} |\Delta(j\omega)| \longrightarrow \text{by the max modulus principle.}$$

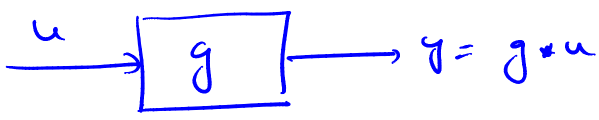
We will show now that if g is the impulse response of a stable transfer function G then

$$\|G\|_{H_\infty} = \sup_{\|u\|_2 \le 1} \|(g * u)\|_2$$

where $\|u\|_2 := \sqrt{\int_0^\infty (u(t))^2\, dt}$

$\downarrow$

energy of the time signal $u(t)$.



& $\|G\|_{H_\infty} = \sup \dfrac{\|y\|_2}{\|u\|_2}$.

Proof:
$$\sup_{\|u\|_2 \le 1} \|(g*u)\|_2^2 = \sup_{\|u\|_2 \le 1} \left| \int_0^\infty (g*u)^2(t)\, dt \right|$$

$$= \sup_{\|u\|_2 \le 1} \left| \int_0^\infty y(t)^2\, dt \right|$$

Using Parseval relation

Using Parseval relation $= \sup_{\|u\|_2 \le 1} \left| \int_0 y(t)\, dt \right|$

$$= \sup_{\|u\|_2 \le 1} \left| \int_0^{\infty} |Y(j\omega)|^2 d\omega \right|$$

where $Y(j\omega)$ is the fourier transform of $y(t)$.

$$= \sup_{\|u\|_2 \le 1} \left| \int_0^{\infty} G(j\omega)^2 |U(j\omega)|^2 d\omega \right|$$

$$\le \sup_{\|u\|_2 \le 1} \int_0^{\infty} \underbrace{|G(j\omega)|^2}\, \underbrace{|u(j\omega)|^2}\, d\omega$$

$$\le \sup_{\|u\|_2 \le 1} \int_0^{\infty} \underline{\left(\sup_{\omega \in \mathbb{R}} |G(j\omega)|\right)^2} |u(j\omega)|^2 d\omega$$

$$\le \sup_{\|u\|_2 \le 1} \left[\|G\|_{H_\infty}^2 \int_0^{\infty} |u(j\omega)|^2 d\omega \right]$$

$$= \|G\|_{H_\infty}^2 \underline{\sup_{\|u\|_2 \le 1} \boxed{\int_0^{\infty} |u(j\omega)|^2 d\omega}}$$

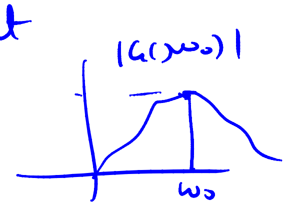
$$= \|G\|_{H_\infty}^2 \underline{\sup_{\|u\|_2 \le 1} \left(\|u\|_2^2 \right)}$$

$$\le \|G\|_{H_\infty}^2$$

$$\sup_{\|u\|_2 \le 1} \|y\|_2^2 \le \|G\|_{H_\infty}^2$$

$$\|G\|_{H_\infty} \ge \sup_{\|u\|_2 \le 1} \sqrt{\|g * u\|_2^2}$$

One can "choose" a u such that



Suppose $\|G\|_{H_\infty} = |G(j\omega_0)|$

$$\left| \int_0^{\infty} G(j\omega)^2 U(j\omega)^2 d\omega \right| \longrightarrow \left| \int^{\infty} G(j\omega_0)^2 |U(j\omega)|^2 d\omega \right.$$

$$\longrightarrow \left| \int_0^\infty G(j\omega_0)^2 \, |\phi(j\omega)|^2 \, d\omega \right|$$

$- \times -$

An important property of induced norms

$$\|G\|_{H_\infty} = \sup_{\|u\|_2 \le 1} \|Gu\|_2$$

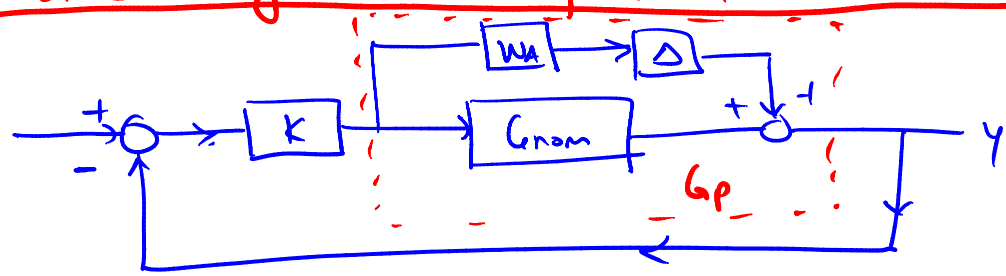
$$= \sup_{\|u\|_2 \neq 0} \frac{\|Gu\|_2}{\|u\|_2}$$

$$\underline{\|GH\|_{H_\infty} \le \|G\|_{H_\infty} \|H\|_{H_\infty}}$$

Submultiplicative property of induced norms.

$$\|(GHu)\| \le \|G\| \|H\| \|u\|$$

Robust stability Condition for Additive Uncertainty:



$$\Pi := \left\{ G(s) + W_A(s)\,\Delta(s) \;\middle|\; \|\Delta\|_{H_\infty} \le 1,\ \Delta \in H_\infty \right\}$$

When does the controller K stabilize all plants in Π? ; Assume that W_A is stable.

Assumption: We will assume that K stabilizes the interconnection with $\Delta = 0$.

Theorem: The controller K stabilizes all plants in Π if and only if

If and only if $\|W_A K S\|_{H_\infty} < 1$.

Proof: By assumption with $\Delta = 0$ the system is stable and suppose the number of encirclements of $(-1,0)$ by the Nyquist plot of $L = GK$ is N.

Now, Δ and W_A are stable and therefore the number of rhp poles of the perturbed

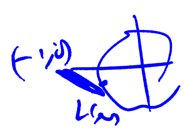
$$L_p = G_p K = (G + W_A \Delta) K = GK + W_A \Delta K$$

in relation to the rhp poles of $GK = L$

$$GK + W_A \Delta K \quad , \quad GK.$$

- As the nominal system is stabilized the rhp poles of K will remain as poles of GK.
- The W_A, Δ are stable $\therefore$ the rhp poles of $GK + W_A \Delta K$ have to be a subset of the rhp poles of the union of the rhp poles of G, K.

($\Leftarrow$) Suppose $\|W_A K S\|_{H_\infty} < 1$



$$\|W_A K S\|_{H_\infty} < 1 \Rightarrow \|W_A K S \Delta\|_{H_\infty}$$

$$\leq \|W_A K S\|_{H_\infty} \|\Delta\|_{H_\infty}$$

$$< 1 \quad \text{if } \|\Delta\|_{H_\infty} < 1$$

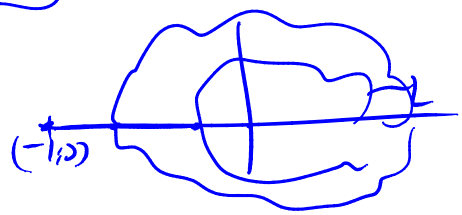
$$\Rightarrow \frac{|W_A K \Delta(j\omega)|}{|(1+L)(j\omega)|} < 1 \quad \forall \omega \in (0,\infty) \quad \|\Delta\|_{H_\infty} < 1.$$

$$\Rightarrow |W_A(j\omega) K(j\omega) \Delta(j\omega)| < |1 + L(j\omega)|$$

$L_p = \underset{\sim}{GK} + W_A \Delta K$ $\quad \forall \omega$ and $\|\Delta\|_{H_\infty} < 1$.

L

$\Rightarrow |L_p(j\omega) - L(j\omega)| < |1 + L(j\omega)|$



$\therefore$ The # of encirclements of the $(-1,0)$ point by the Nyquist of L_p remains the same as the # of encirclements of the $(-1,0)$ point by the Nyquist of the nominal L i.e.

The # of encirclements of $(-1,0)$ by Nyquist of L_p is N.

$\rightarrow$ The # of rhp poles of L_p $\leq$ # of rhp poles of L

$$P_p \leq P$$

$\rightarrow$ The nominal system L is stabilized by K

$\therefore$ $N = P$ (as $Z = N - P$ and $Z = 0$ for stability)

$\rightarrow$ Z_p be the rhp zeros of $(1 + L_p)$ then

$$Z_p = P_p - N = P_p - P \leq 0$$

$\therefore$ $P = P_p$ and $\underline{\underline{Z_p = 0}}$.

$\therefore$ The interconnection of K and L_p is stable and furthermore $P = P_p$

$\Rightarrow$ That there are $L_p = W_A \Delta K + GK$

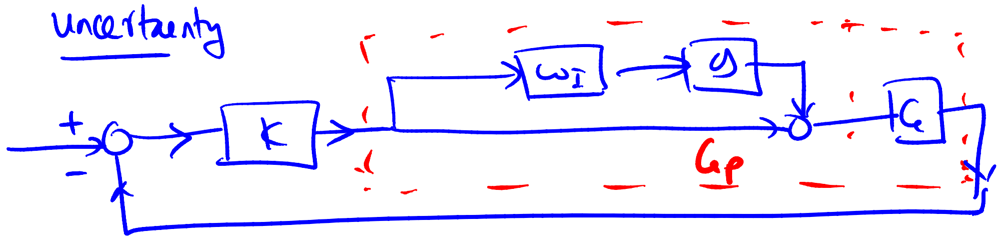
$\Rightarrow$ There are no unstable pole-zero cancellation

$\Rightarrow$ There are no unstable pole-zero cancellation while forming the product K and $(G + W_A \Delta)$

This proves that if $\|W_A K \Delta\|_{H_\infty} < 1$ then the feedback interconnection of any K that stabilizes G will also stabilize G_p.

(*) It can also be shown that a stable Δ with $\|\Delta\|_{H_\infty} \leq 1$ can be chosen such that the feedback interconnection is unstable with K and $G_p = G + W_A \Delta$.

Theorem: Robust stability condition for multiplicative uncertainty



$G_p = G(1 + W_I \Delta)$ where Δ is stable and $\|\Delta\|_{H_\infty} < 1$

(*) Assume that K stabilizes G (i.e. no unstable pole-zero cancellations in GK and # of encirclements of $(-1,0)$ given by $N = P$).
W_I is stable

Theorem: RS $\Leftrightarrow$ $\|W_I T\|_{H_\infty} < 1$.

Proof: Suppose $\|W_I T\|_{H_\infty} < 1 \Rightarrow \|W_I T \Delta\|_{H_\infty} \leq \|W_I T\|_{H_\infty} \|\Delta\|_{H_\infty}$

$$\underset{P}{<} 1 \quad \text{as } \|\Delta\|_{H_\infty} < 1$$

$$\Rightarrow |w_I T(j\omega)\, T(j\omega)\, \Delta(j\omega)| < 1 \text{ for all } \|\Delta\|_{H_\infty} < 1 \quad \forall\omega$$

$$\Rightarrow \frac{|(w_I G K \Delta)(j\omega)|}{|(1+GK)(j\omega)|} < 1 \quad \forall \|\Delta\|_{H_\infty} < 1 \quad \forall\omega$$

$$\Rightarrow |(L_p - L)(j\omega)| < |1 + L(j\omega)| \quad \forall \|\Delta\|_{H_\infty} \le 1 \quad \forall\omega$$

note $\quad L_p - L = G(1 + w_I \Delta) K - GK$

$$= (G w_I \Delta K)$$