# Lecture 18

Thursday, March 31, 2011
8:13 AM

$$\dot{x} = Ax + Bu \; ; \quad \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$
$$y = Cx + Du$$

## Stabilizability:

1. $(A, B)$ is said to be a stabilizable pair if there exists a matrix $K$ such that

$$\lambda_i(A+BK) \in \mathbb{C}_{lhp} \quad \text{for eigenvalues } \lambda_i \text{ of } A+BK.$$

2. $u = +K\overset{\text{full state feedback}}{x} + v$

$$\left.\begin{array}{l} \dot{x} = Ax + BKx + Bv \\ y = Cx + Du \end{array}\right\} \quad \begin{array}{l} \dot{x} = (A+BK)x + Bv \\ y = Cx + Du \end{array}$$

3. Controllability iff $\Leftrightarrow$ $\exists$ $K$ such that $\lambda_i(A+BK)$ can be placed anywhere in the complex plane.

$\Leftrightarrow$ $[B \;\; AB \; \cdots \; A^{n-1}B]$ has full row rank.

$\Leftrightarrow$ $\forall$ left eigenvector $x$ of $A$ (i.e. $x^*A = \lambda x^*$) $x^*B \neq 0$.

$$\boxed{\begin{array}{l} \text{Suppose } \exists\, x^* \text{ s.t. } \\ \quad x^*A = \lambda x^* \\ \text{and } x^*B = 0 \text{ then} \\ x^*[B \;\; AB \cdots A^{n-1}B] \\ = [x^*B \;\; x^*AB \;\cdots\; x^*A^{n-1}B] \\ = [0 \quad 0 \;\cdots\; 0] \\ \therefore [B \cdots A^{n-1}B] \\ \text{does not have full row rank} \end{array}}$$

4. Stabilizability: $\Leftrightarrow$ given $x^*$ s.t. $x^*A = \lambda x^*$, then $x^*B \neq 0$ if $\}$

$Re(\lambda) \geq 0.$

Note that $(A,B)$ controllable $\Leftrightarrow$ $(A^T, C^T)$ observable.

(*) Observability $\Leftrightarrow$ $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has full column rank

(*) $\Leftrightarrow$ $\nexists$ $\forall$ $x \neq 0$ s.t. $Ax = \lambda x$; $Cx \neq 0$.

$$(\overset{A}{A^T} + \overset{B}{C^T}\overset{K}{L^T}) = (A+LC)$$

(*) Detectability: given $A$ and $C$

$\exists$ $L$ s.t. $\underbrace{(A+LC)}$ ~~~~ has all eigenvalues in the strict lhp

$\Leftrightarrow$ (*) $\{\forall\ x \neq 0$ s.t. $Ax = \lambda x$ and $Re(\lambda) \geq 0$; we have $Cx \neq 0\}$

Pf: $\Rightarrow$ $\Leftarrow$

Suppose $\exists$ $x \neq 0$ $Ax = \lambda x$ and $Cx = 0$ then

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x = \begin{bmatrix} Cx \\ CAx \\ \vdots \\ CA^{n-1}x \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ does not have full column rank.

Theorem: Suppose $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

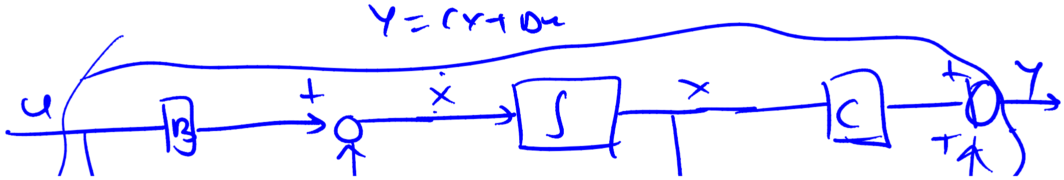
$G$: transfer matrix; $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$: realization of the transfer.

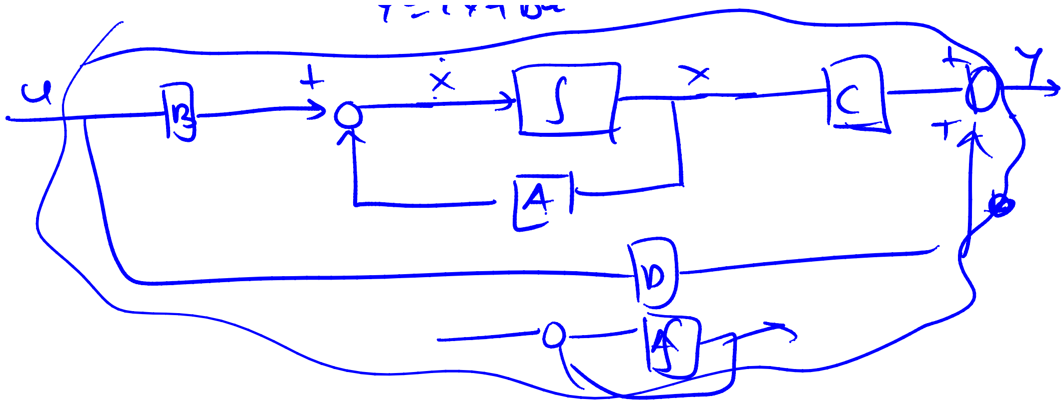
$$G(s) = C(sI - A)^{-1}B + D.$$

The order of the transfer ~~fn~~ matrix function $G$ is $n$ $\Leftrightarrow$ $(A, B, C, D)$ is a controllable and observable realization.

$\dot{x} = Ax + Bu$

$y = Cx + Du$






$$G(s) = C(sI-A)^{-1}B+D = \frac{C\,\mathrm{Adj}(sI-A)B + D\det(sI-A)}{\det(sI-A)}$$

(i) (A,B,C,D) is controllable and observable

$\Updownarrow$

the realization is minimal. (cannot find another $(\tilde{A},\tilde{B},\tilde{C},\tilde{D})$ with $\dim(\tilde{A}) < \dim(A)$ s.t. $C(sI-A)^{-1}B+D = \tilde{C}(sI-\tilde{A})^{-1}\tilde{B}+\tilde{D}$).

$\Updownarrow$

There are no pole-zero cancellation in forming $G(s)$ from $C(sI-A)^{-1}B+D$.

(ii) ~~(A,B,C,D)~~ $G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ involves no unstable pole zero cancellation $\Leftrightarrow$ (A,B,C) is a detectable and a stabilizable realization.

— If $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is a stabilizable and detectable realization then

$G(s) = C(sI-A)^{-1}B+D$ does not involve any unstable pole-zero cancellation

$$= \frac{C\,\mathrm{Adj}(sI-A)B + D\det(sI-A)}{\det(sI-A)} = \frac{n(s)}{d(s)}$$

n and d do not ~~have~~ any unstable common factors.

$\Rightarrow$ $\quad G(s)$ is a stable transfer matrix

$\Updownarrow$

A has all eigenvalues in lhp.

**Summarize:**

given $G(s) = C(sI - A)^{-1}B + D$ ; $(A, B, C)$ stabilizable and detectable

then $G(s)$ stable (all poles in lhp)

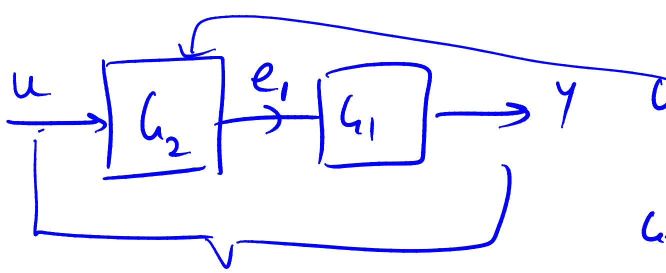
$\Updownarrow$

A is stable.

"I-O stability $\Leftrightarrow$ asymptotic stability".

## Properties of State Space Realizations:

$\rightarrow$ Suppose $G_1 \equiv \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ ; $G_2 \equiv \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$




$$G_1(s) = \frac{(s+1)}{(s+2)(s+3)}$$

$$G_2(s) = \frac{(s+2)(s+3)}{(s+1)(s+4)}$$

$$G_1 G_2 = \frac{\cancel{s+1}}{(\cancel{s+2})(\cancel{s+3})} \frac{(\cancel{s+2})(\cancel{s+3})}{(\cancel{s+1})(s+4)}$$

$$= \frac{1}{(s+4)}$$

$G_2 \equiv$ $\quad \dot{x}_2 = A_2 x_2 + B_2 u$

$\qquad e_1 = C_2 x_2 + D_2 u$

$G_1 \equiv$ $\dot{x}_1 = A_1 x_1 + B_1 e_1$

$y = C_1 x_1 + D_1 e_1$

$\dot{x}_2 = A_2 x_2 + B_2 u$

$e_1 = C_2 x_2 + D_2 u$

$\dot{x}_1 = A_1 x_1 + B_1 C_2 x_2 + B_1 D_2 u$

$\dot{x}_2 = A_2 x_2 + B_2 u$

$$\dot{x}_1 = A_1 x_1 + B_1(C_2 x_2 + D_2 u) = $$
$$= A_1 x_1 + B_1 C_2 x_2 + B_1 D_2 u$$

$$y = C_1 x_1 + D_1(C_2 x_2 + D_2 u)$$
$$= C_1 x_1 + D_1 C_2 x_2 + D_1 D_2 u$$

$$y = C_1 x_1 + D_1 C_2 x_2 + D_1 D_2 u$$

$$\dot{x} = \overbrace{\begin{bmatrix} A_1 & B_1 C_2 \\ 0 & A_2 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 D_2 \\ B_2 \end{bmatrix}}_{B} u$$

$$y = \underbrace{\begin{bmatrix} C_1 & D_1 C_2 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{D_1 D_2}_{D} u$$

Inherited realization is

$$\left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array}\right]$$