Lecture 24

Wednesday, April 20, 2011
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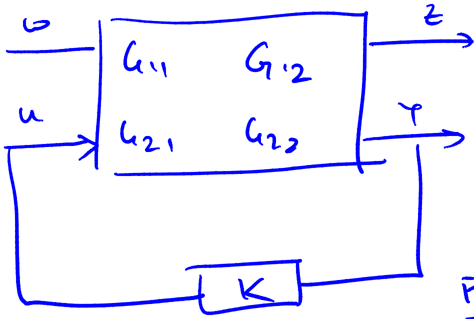


Fig 1:

- Let $G = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$ ✓ can always be assumed is stab. and detectable

- Suppose that the inherited realization of $G_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$ is also stabilizable and detectable

- The stability of the $(G-K)$ interconnection in Fig 1 is equivalent to the stability of the feedback interconnection of $(G_{22}-K)$ given by

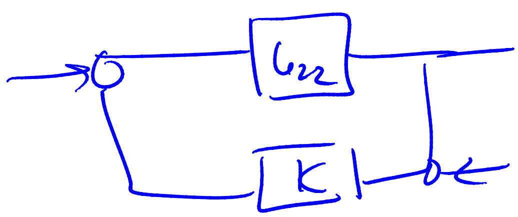


Fig 2

- From Youla parametrization all stabilizing controllers is given by

$$(Y-MQ)(X-NQ)^{-1}$$

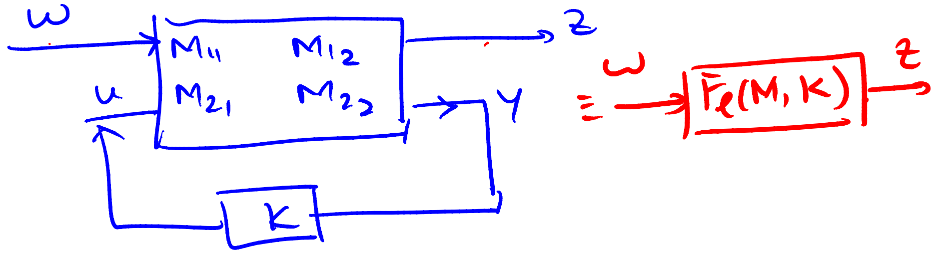
for fig 2: and therefore for Figure 1.

- Assuming that $\left(\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right) = G_{22}$ is stab. and detectable all ~~cl~~ closed-loop maps in figure 1 ~~are~~ can be written as

$\underline{\underline{H - UQV}}$ where Q is any stable parameter.

— x —

Lower linear fractional transformation:



$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$u = Ky$$

$$\left.\begin{aligned} z &= M_{11} w + M_{12} u \\ y &= M_{21} w + M_{22} u \\ u &= Ky \end{aligned}\right\} \quad y = M_{21} w + M_{22} K y$$

$$\Rightarrow (I - M_{22} K) y = M_{21} w$$

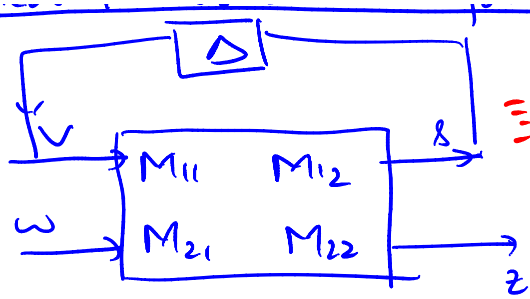
$$\Rightarrow y = (I - M_{22} K)^{-1} M_{21} w$$

$$\Rightarrow z = M_{11} w + M_{12} K (I - M_{22} K)^{-1} M_{21} w$$

$$\Rightarrow z = \underbrace{\left[M_{11} + M_{12} K (I - M_{22} K)^{-1} M_{21} \right]}_{F_l(M, K)} w.$$

$$z = F_l(M, K)\, w.$$

Upper Linear Fractional Transformation



$\equiv$

$$\begin{pmatrix} \delta \\ z \end{pmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} V \\ \omega \end{bmatrix} \left.\begin{matrix} \delta = M_{11}V + M_{12}\omega \\ z = M_{21}V + M_{22}\omega \end{matrix}\right.$$

$$V = \Delta\delta$$

$$\delta = M_{11}V + M_{12}\omega \Rightarrow \delta = M_{11}\Delta\delta + M_{12}\omega$$

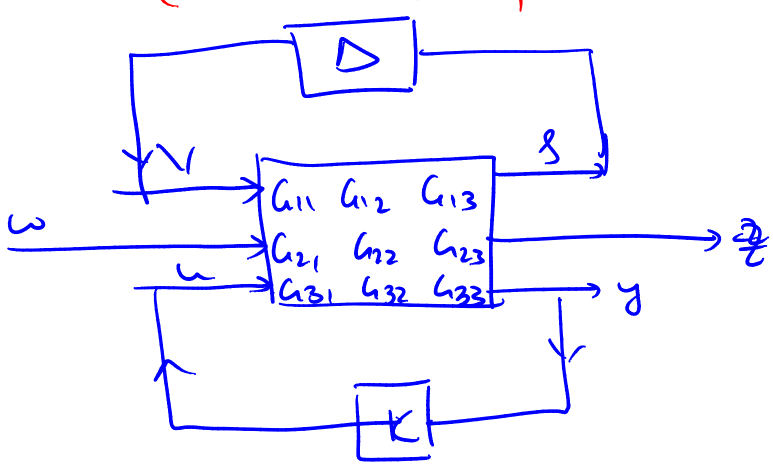
$$\Rightarrow \delta = (I - M_{11}\Delta)^{-1} M_{12}\omega.$$

$$z = M_{21}V + M_{22}\omega$$

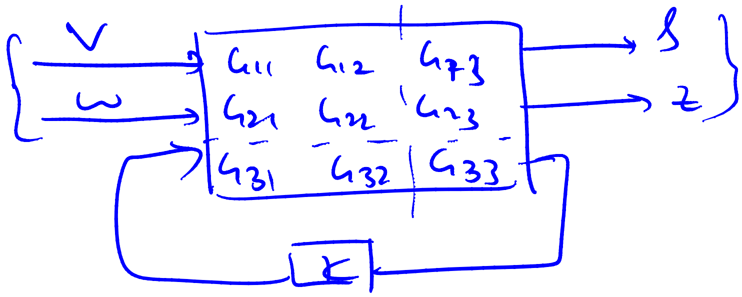
$$= M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}\omega + M_{22}\omega$$

$$= \underbrace{\left(M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}\right)}_{F_u(M,\Delta).}\omega$$

(G - K - Δ) framework

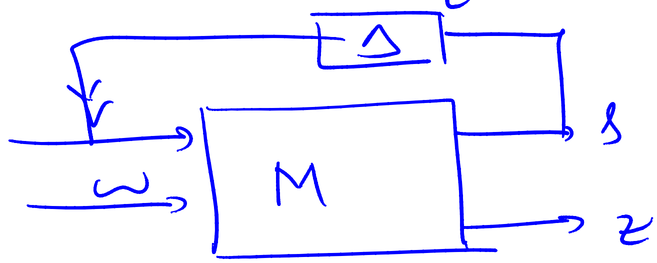


$$\begin{pmatrix} \delta \\ z \\ y \end{pmatrix} = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} V \\ \omega \\ u \end{bmatrix}$$



$\begin{pmatrix} v \\ w \end{pmatrix} \longmapsto \begin{pmatrix} \delta \\ z \end{pmatrix}$ is given by

$$\begin{pmatrix} \delta \\ z \end{pmatrix} = F_\ell(G, K) \cdot \begin{bmatrix} v \\ w \end{bmatrix}$$



$$M = F_\ell(G, K)$$

$\therefore$ the map from w to z is

$$F_u(M, \Delta)$$
$$= F_u[\, F_\ell(G,K), \Delta \,]$$

$\therefore$ The closed-loop map is given by

$$z = \Big\{ F_u[\, F_\ell(G,K), \Delta \,] \Big\} w.$$

—— x ——

The following properties of Fractional transformations are

1. Suppose C is an invertible matrix then

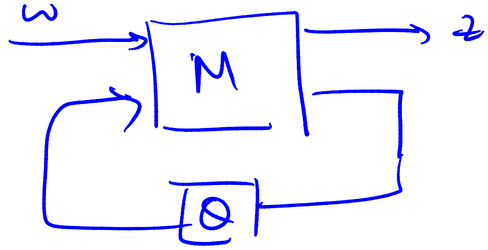
$$(i)\ (A+BQ)(C+DQ)^{-1} = F_\ell(M, Q)$$

$$(C+DQ)^{-1}(A+QB)^{-1} = F_u(N, Q)$$

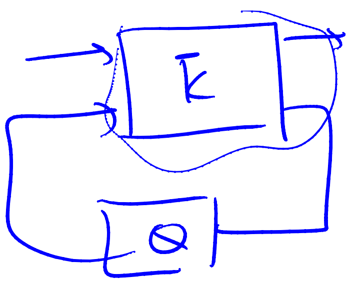
$= ((C+DQ)^{-1}(A+QB))^{-1} = F_u(N,Q)$

where $M = \begin{bmatrix} AC^{-1} & B - AC^{-1}D \\ C^{-1} & -C^{-1}D \end{bmatrix}$

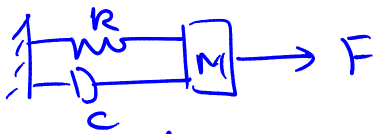
$N = \begin{bmatrix} C^{-1}A & C^{-1} \\ B - DC^{-1}A & -DC^{-1} \end{bmatrix}$



$w \mapsto z$ is given by $(A+BQ)(C+DQ^{-1})$



Parametric Uncertainty: An Example



- The system is a spring-mass-damper system with a force as an input to the mass.
- The dynamics is given by
$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = F/m.$$
- Let say that the parameters of the

model; k, m and c are all uncertain;
Suppose they are uncertain by 10%. Thus, the following hold

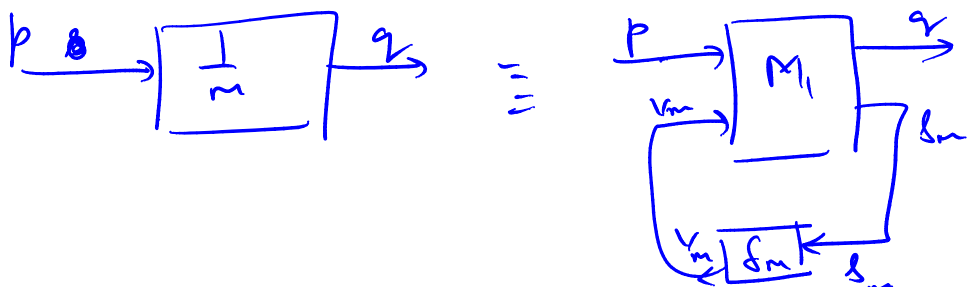
$$k = \bar{k}(1+0.1\delta_k)$$

$$c = \bar{c}(1+0.1\delta_c)$$

$$\frac{1}{m} = \frac{1}{\bar{m}(1+0.1\delta_m)}$$

$$\frac{1}{\bar{m}(1+0.1\delta_m)} \equiv F_\ell(M_1, \delta_m)$$ where

$$M_1 = \begin{bmatrix} \frac{1}{\bar{m}} & -\frac{0.1}{\bar{m}} \\ 1 & -0.1 \end{bmatrix}$$

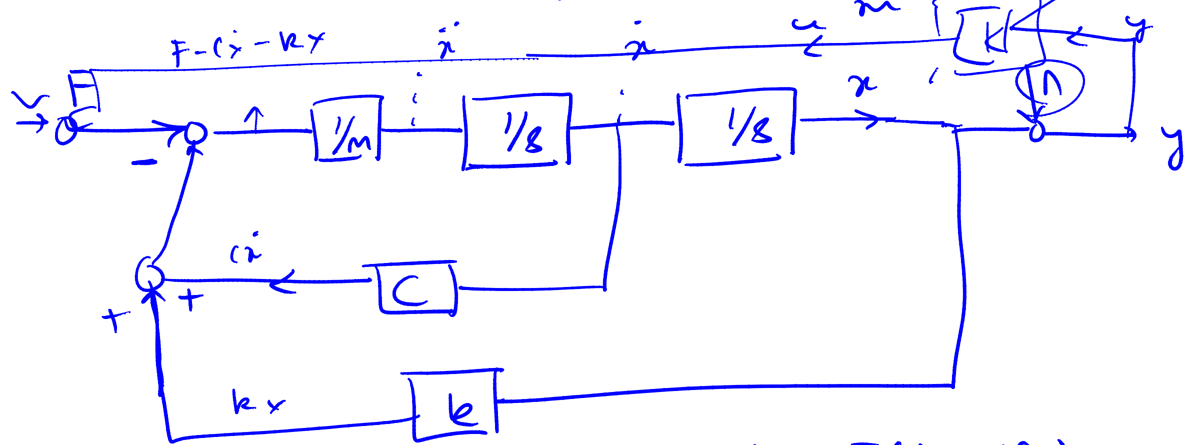


$$\begin{pmatrix} q \\ \delta_m \end{pmatrix} = \begin{bmatrix} \frac{1}{\bar{m}} & -\frac{0.1}{\bar{m}} \\ 1 & -0.1 \end{bmatrix} \begin{bmatrix} p \\ v_m \end{bmatrix}$$

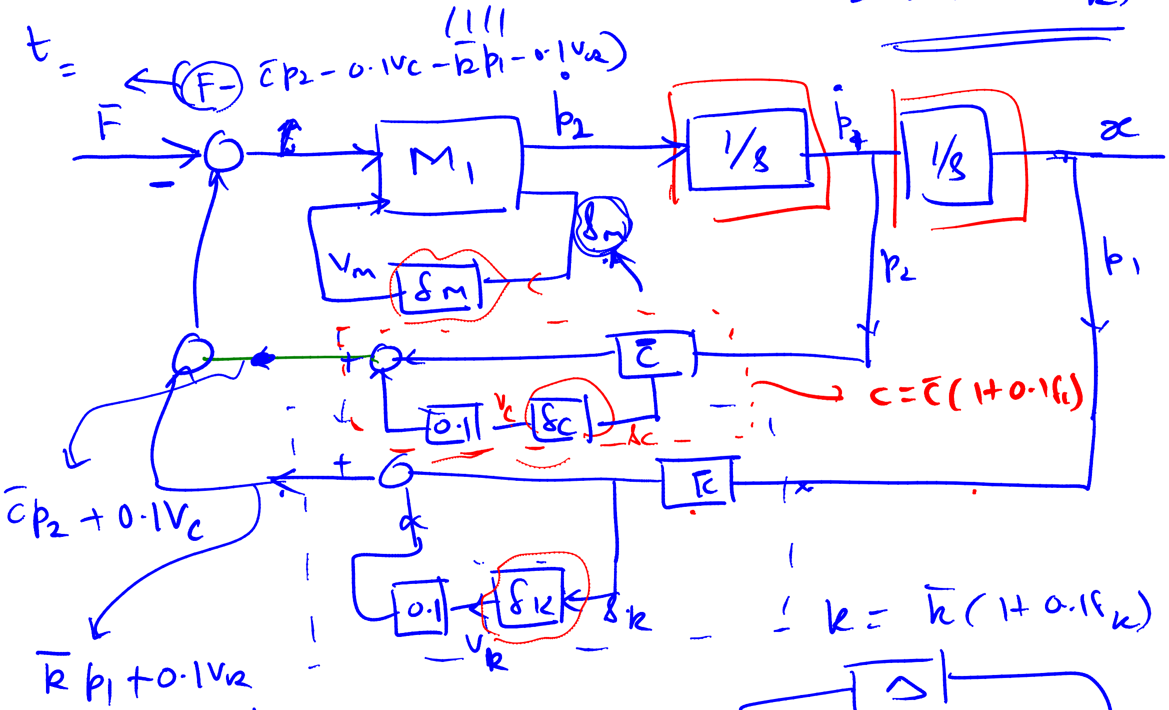
$$\Rightarrow \quad q = \frac{1}{\bar{m}} p - \frac{0.1}{\bar{m}} v_m$$

$$\delta_m = p - 0.1 v_m.$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F}{m} \; ; \quad \ddot{x} = \frac{F - c\dot{x} - kx}{m}$$

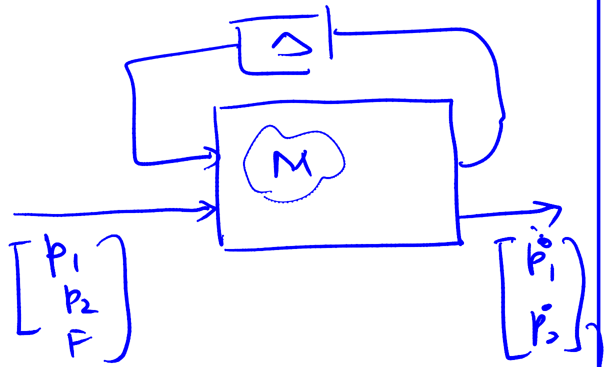


$$k = \bar{k}(1 + 0.1\delta_k)$$



$p_1 = x$

$p_2 = \dot{x}$

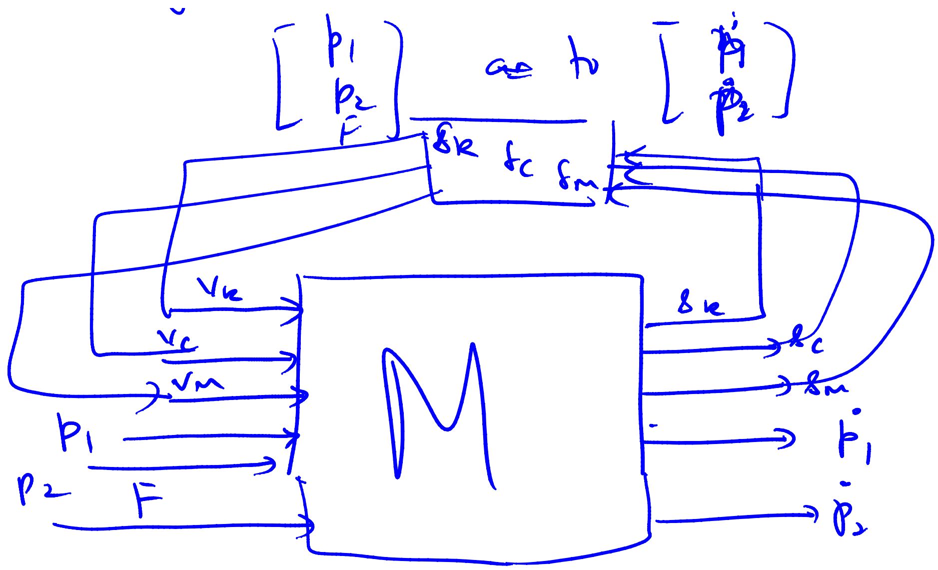


Find M such that

$F_u(M, \Delta)$ with $\Delta = \begin{bmatrix} \delta_k & 0 & 0 \\ 0 & \delta_c & 0 \\ 0 & 0 & \delta_m \end{bmatrix}$

gives the map from

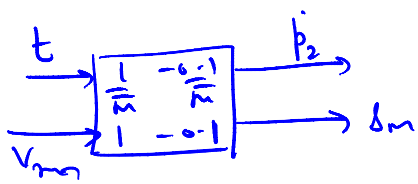
$\begin{bmatrix} p_1 \end{bmatrix}$ to $\begin{bmatrix} \dot{p} \end{bmatrix}$



$$\delta_k = \begin{bmatrix} 0 & 0 & 0 & \bar{k} & 0 & 0 \end{bmatrix} \begin{bmatrix} v_k \\ v_c \\ v_m \\ p_1 \\ p_2 \\ F \end{bmatrix}$$

$$\delta_c = \begin{bmatrix} 0 & 0 & 0 & 0 & \bar{c} & 0 \end{bmatrix} \begin{bmatrix} v_k \\ v_c \\ v_m \\ p_1 \\ p_2 \\ F \end{bmatrix}$$

$$\delta_m = \begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} v_k \\ v_c \\ v_m \\ p_1 \\ p_2 \\ F \end{bmatrix}$$



$$\begin{bmatrix} \dot{p}_2 \\ \delta_m \end{bmatrix} = \begin{bmatrix} \frac{1}{m} & -\frac{0.1}{m} \\ 1 & -0.1 \end{bmatrix} \begin{bmatrix} t \\ v_m \end{bmatrix}$$

$$\dot{p}_2 = \frac{1}{m} t - \frac{0.1}{m} v_m$$

$$\delta_m = t - 0.1 v_m.$$

$$= F - \bar{c}p_2 - 0.1v_c - \bar{k}p_1 - 0.1v_k) - 0.1V_m.$$

$$= F - \bar{c}p_2 - 0.1v_c - \bar{k}p_1 - 0.1v_k - 0.1V_m.$$

$$\delta_m = \begin{bmatrix} -0.1 & -0.1 & -0.1 & -\bar{k} & -\bar{c} & 1 \end{bmatrix} \begin{bmatrix} v_k \\ v_c \\ v_m \\ \vdots \end{bmatrix}$$

$$\dot p_1 = p_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_k \\ v_c \\ v_m \\ p_1 \\ p_2 \\ F \end{bmatrix}$$

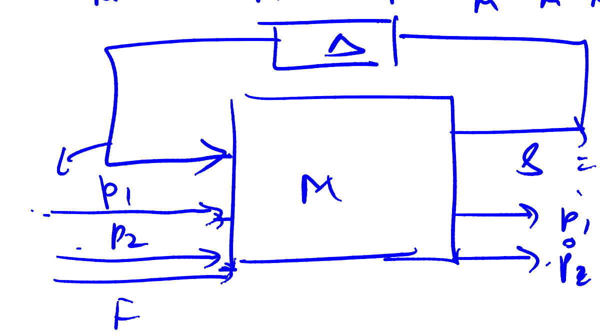
$$\dot p_2 = \frac{F - \bar c p_2 - 0.1 v_c - \bar k p_1 - 0.1 v_k}{\bar m} - \frac{0.1}{\bar m} v_m .$$

$$= \begin{bmatrix} \frac{-0.1}{\bar m} & \frac{-0.1}{\bar m} & \frac{-0.1}{\bar m} & -\frac{\bar k}{\bar m} & -\frac{\bar c}{\bar m} & \frac{1}{\bar m} \end{bmatrix} \begin{bmatrix} v_k \\ v_c \\ v_m \\ p_1 \\ p_2 \\ F \end{bmatrix}$$

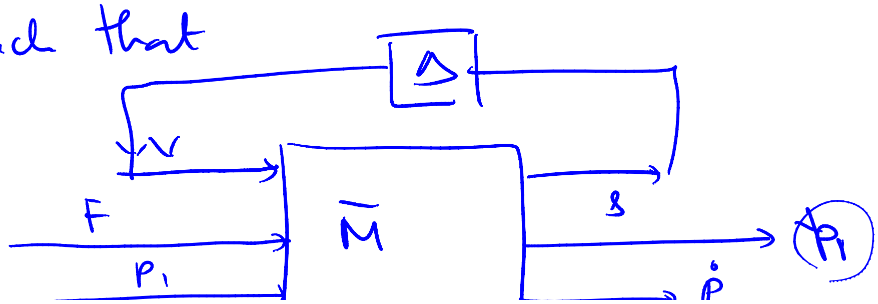
$$\begin{bmatrix} \delta_k \\ \delta_c \\ \delta_m \\ \dot p_1 \\ \dot p_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \bar k & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar c & 0 \\ -0.1 & -0.1 & -0.1 & -\bar k & -\bar c & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{-0.1}{\bar m} & \frac{-0.1}{\bar m} & \frac{-0.1}{\bar m} & \frac{-\bar k}{\bar m} & \frac{-\bar c}{\bar m} & \frac{1}{\bar m} \end{bmatrix} \begin{bmatrix} v_k \\ v_c \\ v_m \\ p_1 \\ p_2 \\ F \end{bmatrix}$$

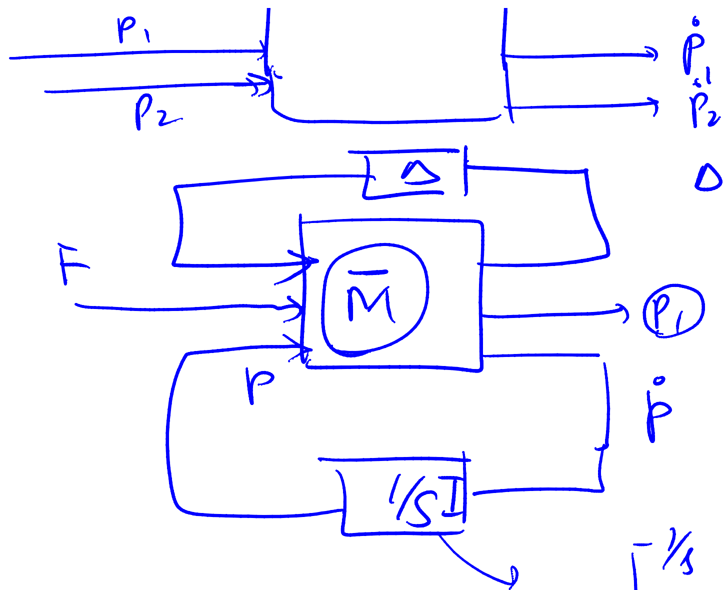
F, p_1, v_2

$$v = \begin{bmatrix} v_k \\ v_c \\ v_m \end{bmatrix} \qquad \delta = \begin{bmatrix} \delta_k \\ \delta_c \\ \delta_m \end{bmatrix}$$



Lets assume that $\bar M$ is the determined such that





$$\Delta = \begin{bmatrix}\delta_k & & 0\\ 0 & \delta_c & \\ & & \delta_m\end{bmatrix}$$

with $|\delta_k| \le 1$
$|\delta_c| \le 1$
$|\delta_m| \le 1$

$$\begin{bmatrix}1/s & 0\\ 0 & 1/s\end{bmatrix} = \frac{1}{s} I$$

$$p = \underbrace{\left\{F_u\left(F_\ell\left(\bar M, \tfrac{1}{s} I\right), \Delta\right)\right\}}\, F$$

$u = Ky + v$ where $y = p_1 + n$