Lecture 25

Thursday, April 21, 2011
8:16 AM

Linear fractional transformations (LFT).

1. A Spring-mass-damper System with uncertain parameters; how to cast this in a ~~ ~~ upper LFT.

Robust Stability of MIMO Systems:

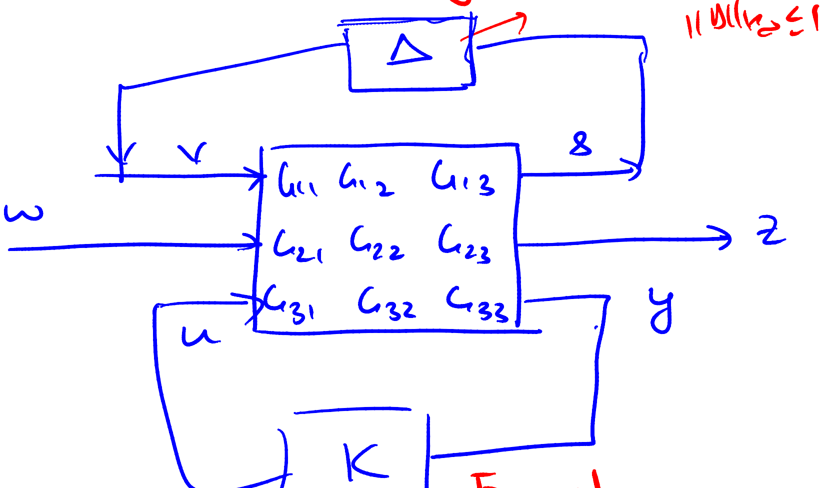


Figure 1

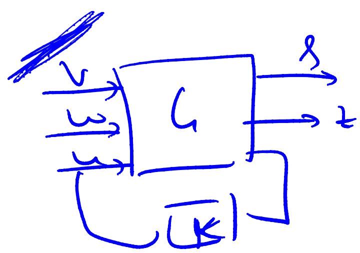
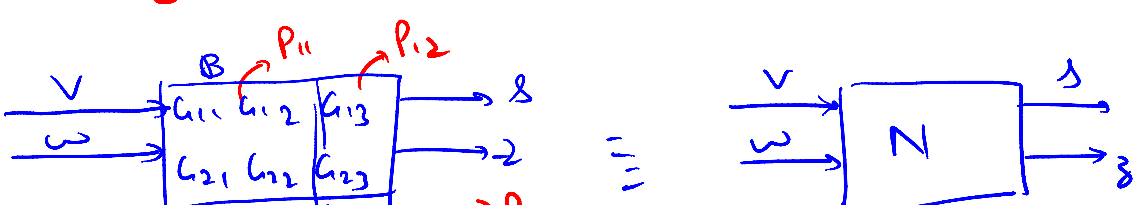
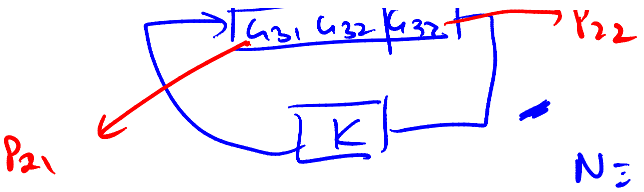


Figure 2

Nominal Stability: The $G-K-\Delta$ interconnection in Figure 1 above is nominally stable if the $G-K$ interconnection in Figure 2 is internally stable.





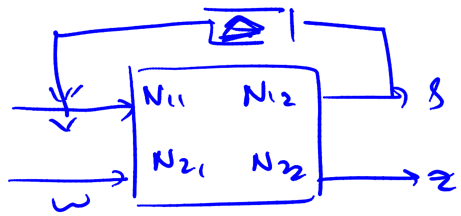
$N = F_\ell(G, K)$.

$N = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}$

⊛ If NS is there then N is a stable transfer matrix

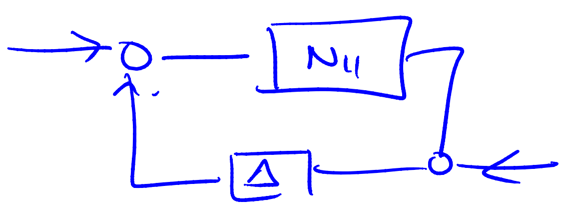
Lets assume NS

⊛



⊛ Clearly any inherited realization of N_{11} from a stab. and detectable realization of N would be stab. and detectable as N is stable

⊛ The stability of the N – Δ interconnection is equivalent to the stability of N_{11} – Δ interconnection



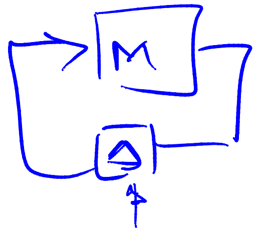
Definition: The G-K-Δ interconnection is said to be robustly stable if for all $\Delta \in B\Delta_{LTI}$ the interconnection is internally stable.

RS $\Leftrightarrow$ (1) (G-K-Δ) is Nominally stable

(2) N_{11} – Δ interconnection is stable $\forall\ \Delta \in B\Delta_{LTI}$

$M = N_{11}$ $\qquad \underline{B\Delta_{LTI} = \{ \Delta \in RH_\infty \mid \|\Delta\|_{H_\infty} \leq 1 \}}.$

"unstructured Δ"



Theorem:

① M stable
② Δ stable.

then

$\underline{\underline{\det(I - M(j\omega)\, \Delta(j\omega)) \neq 0}}$ $\quad \forall\, \Delta \in B\Delta_{LTI}$ is equivalent to $\underline{\underline{RS}}$.

→ Suppose we have RS ⇒ for all $\Delta \in B\Delta_{LTI}$ the interconnection is internally stable.

∴ The $\det(I - M(j\omega)\Delta(j\omega))$ does not touch the origin or encircle the origin.

∴ $\det(I - M(j\omega)\Delta(j\omega)) \neq 0$. $\forall \omega$
$\forall\, \Delta \in B\Delta_{LTI}$

→ Suppose $\exists\, \Delta \in B\Delta_{LTI}$ such that the M-Δ configuration is unstable.

⇒ $\exists$ a Δ with $\|\Delta\|_{H_\infty} \leq 1$ and $\det(I - M\Delta)$ encircles the origin or touches the $\underline{\underline{\text{origin}}}$

→ If $\det(I - M(j\omega)\Delta(j\omega))$ touches the origin for some $\omega \in \mathbb{R}$ then we are done

→ Suppose not then the Nyquist plot of $\det(I - M(s)\Delta(s))$

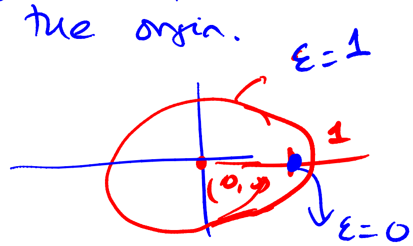
encircles the origin

$\rightarrow \quad f(\varepsilon, s) = \det(I - M(s)\,\varepsilon\,\Delta(s))$

$f(\varepsilon, s) = \det(I) = 1 \quad \text{if } \varepsilon = 0$

$f(1, s) = \det(I - \varepsilon M(s)\,\Delta(s))$
which encircles the origin.

$\exists\, \varepsilon' \in [0,1]$ such that
$f(\varepsilon', s)$ touches the origin.



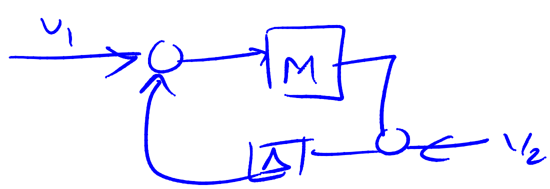
$\rightarrow \det(I - M\varepsilon'\Delta(s))$ touches the origin

$\rightarrow \|\varepsilon'\Delta(s)\|_{H_\infty} \le |\varepsilon'|\,\|\Delta\|_{H_\infty} < 1$

$\Delta' = \varepsilon'\Delta(s)$ then

$\det(I - M(j\omega)\,\Delta'(j\omega)) = 0$ for some ω

Suppose $\quad B\Delta_{LTI} = \{\Delta \in RH_\infty \quad \|\Delta\|_{H_\infty} \le 1\}$



If $\|M\|_{H_\infty} < 1$ then RS otherwise not

Lemma: $\quad \max\limits_{\bar\sigma(B) \le 1} \rho(AB) = \max\limits_{\bar\sigma(B) \le 1} \bar\sigma(AB) = \bar\sigma(A).$

where $\bar\sigma(C)$ is the largest singular value of the matrix C.

Proof: given any matrix A it admits a decomposition
$A = U\Sigma V^*$ (SVD) $\quad A \in \mathbb{R}^{m\times n}$
where $U^*U = V^*V = I$.

and Σ has only diagonal ~~nonzero~~ entries

$$\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & 0 \end{bmatrix}$$

The largest entry is the maximum Singular value of A.

- ρ is the Spectral radius of the matrix = the max $|\lambda|$ where λ is an eigenvalue of A.

(a) It is true that $\rho(C) \leq \bar{\sigma}(C)$

if $|\lambda| = \rho(C)$ for an eigenvalue λ of C then

$\exists\, x$ s.t. $Cx = \lambda x$. and $\|x\| = 1$

clearly $\dfrac{\|Cx\|_2}{\|x\|_2} = |\lambda|$

$$\therefore \underbrace{\max_{y \neq 0} \frac{\|Cy\|_2}{\|y\|_2}}_{} \geq |\lambda|$$

$$\Rightarrow \bar{\sigma}(C) \geq |\lambda| = \rho(C)$$

$$\rho(AB) \leq \bar{\sigma}(AB)$$

$$\max_{\bar{\sigma}(B) \leq 1} \rho(AB) \leq \max_{\bar{\sigma}(B) \leq 1} \bar{\sigma}(AB)$$

$\rightarrow$ $B = VU^*$; $A = U \Sigma V^*$

$AB = U \Sigma V^* V U^* = U \Sigma U^*$

$$AB = U\Sigma \underbrace{V^* V}_{I} U^* = U \Sigma U^*$$

and $$\rho(AB) = \rho(U\Sigma U^*) = \rho(\Sigma) = \sigma_1$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \qquad\qquad\qquad = \bar{\sigma}(A)$$

$$U^{-1}$$

$$\bar{\sigma}(B) = \bar{\sigma}(VU^*) = 1$$

$$\rho(AB) = \bar{\sigma}(A) \text{ for some } \bar{\sigma}(B) \le 1$$

$$\Rightarrow \max_{\bar{\sigma}(B) \le 1} \rho(AB) \geq \bar{\sigma}(A)$$

$$\Rightarrow \max_{\bar{\sigma}(B) \le 1} \rho(AB) = \max_{\bar{\sigma}(B) \le 1} \bar{\sigma}(AB) = \bar{\sigma}(A).$$