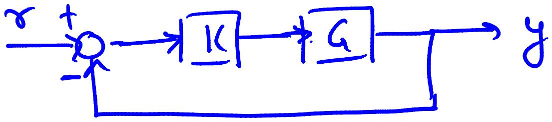
Lecture 9

Thursday, February 17, 2011
8:10 AM



- **Sensitivity S**

  The specifications on the sensitivity transfer function can be imposed by the condition

  $\|W_p S\|_{H_\infty} \leq 1$; where W_p is the performance weight

- **Complimentary Sensitivity T**

  The specs on T can be imposed by

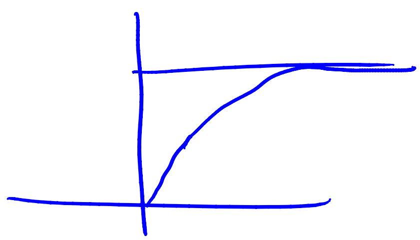
  $\| W_T T \|_{H_\infty} \leq 1$; W_T is the noise rejection weight

- **Weight to avoid actuator saturation:**

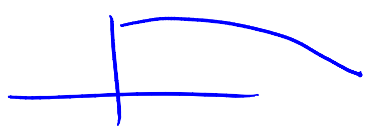
  $\| W_u KS \|_{H_\infty} \leq 1$; W_u is the weight on KS.

Shapes.

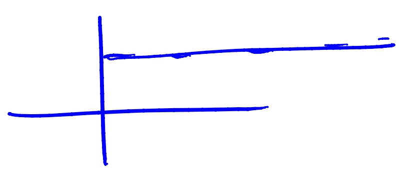
$\frac{1}{W_p}$ =



$\frac{1}{W_T}$ =



$\frac{1}{W_u}$ =



H_∞ problem solves the following problem:

$$\min_{K \text{ Stabilizing}} \left\| \begin{matrix} W_S S(K) \\ W_T T(K) \\ W_u KS \end{matrix} \right\|_{H_\infty} = \gamma$$

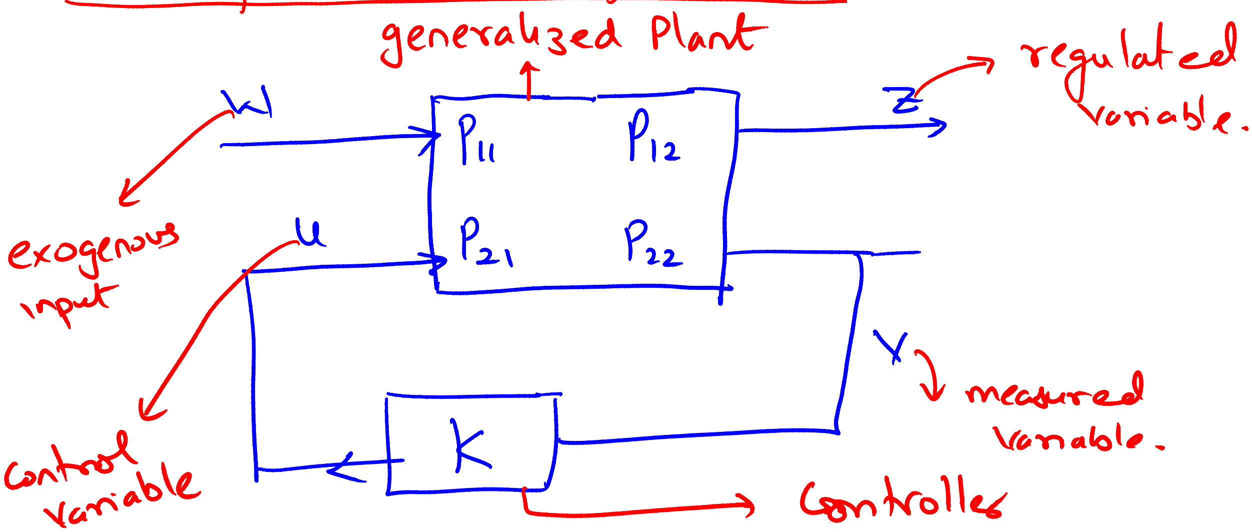
$$\|f\|_{H_\infty} = \sup_{\omega \in \mathbb{R}} \bar{\sigma}(f(j\omega))$$

$\| W_u K S \|_{H_\infty}$

K is such that the feedback interconnection is stable! — Stacked H_∞ problem

$$\gamma < 1 \not\Leftrightarrow \|W_S S\|_{H_\infty} < 1 \; ; \; \|W_T T\| < 1$$
$$\|W_u K S\|_{H_\infty} < 1$$

Linear fractional Transformations:



$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

and
$$\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \quad \text{-- (1)}$$

$$u = Kv \quad \text{-- -- (2)}$$

$$z = P_{11} w + P_{12} u$$
$$v = P_{21} w + P_{22} u$$
$$u = Kv$$
$$= K[P_{21} w + P_{22} u]$$
$$= K P_{21} w + K P_{22} u$$
$$\Rightarrow (I - K P_{22}) u = K P_{21} w$$

$$\Rightarrow \quad u = (I - KP_{22})^{-1} K P_{21}\, w$$

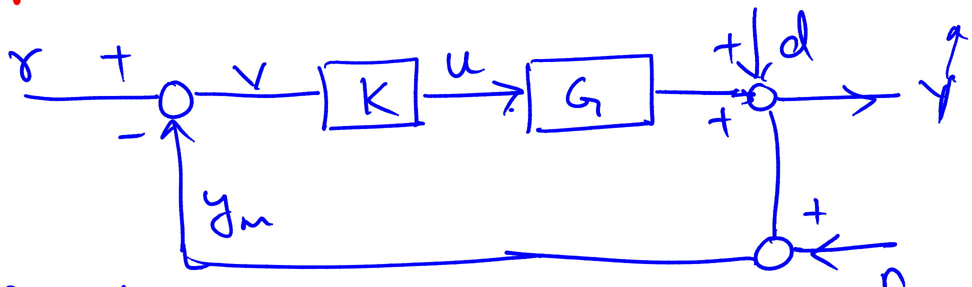
$$z = P_{11} w + P_{12} (I - KP_{22})^{-1} K P_{21}\, w$$

$$= \underbrace{\left[P_{11} + P_{12} (I - KP_{22})^{-1} K P_{21} \right]}_{= F_\ell (P, K)} w.$$

lower linear fractional transformation of P and K.

$y = d + Gu$

Example:



Regulated Variable

- Suppose we want to regulate the error

$$e := y - r.$$

$$\therefore \quad z = y - r$$

Exogenous inputs : $w := \begin{bmatrix} r \\ n \\ d \end{bmatrix}$; all the external inputs to the feedback interconnection

Determining the generalized Plant given that

$z = y - r$; and $w = \begin{bmatrix} r \\ n \\ d \end{bmatrix}$ find the

matrix P such that

$$\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

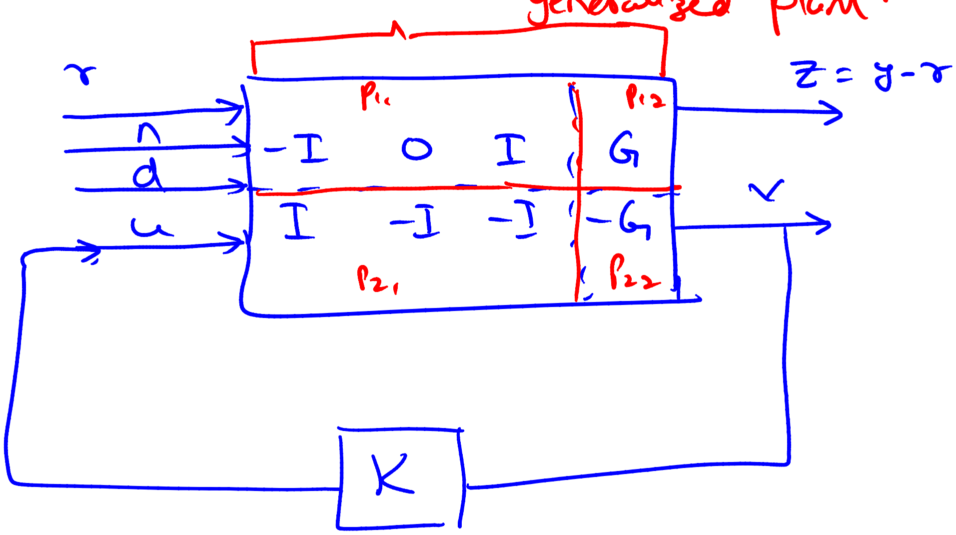
$$= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \begin{pmatrix} r \\ n \\ d \end{pmatrix} \\ u \end{bmatrix}$$

$$z = y - r = d + Gu - r = [\underbrace{-I \;\; 0 \;\; I}_{P_{11}} \;\; \underbrace{G}_{P_{12}}] \begin{pmatrix} r \\ n \\ d \\ u \end{pmatrix}$$

$$v = r - y_m = r - (y+n)$$
$$= r - y - n = r - Gu - d - n$$
$$= [I \;\; -I \;\; -I \;\; -G] \begin{pmatrix} r \\ n \\ d \\ u \end{pmatrix}$$

$$\therefore \quad P = \begin{bmatrix} \overbrace{-I \quad 0 \quad I}^{P_{11}} & \overbrace{G}^{P_{12}} \\ \underbrace{I \quad -I \quad -I}_{P_{21}} & \underbrace{-G}_{P_{22}} \end{bmatrix}$$

Generalized plant.



$$\min_{K \text{ stabilizing}} \; \| F_l(P, K) \|_{\mathcal{H}_\infty}$$

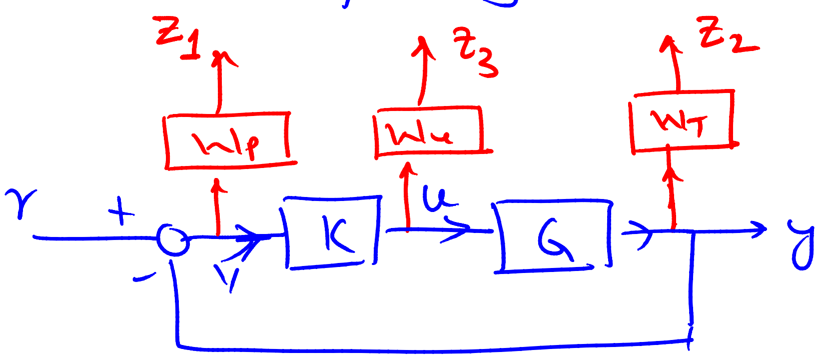
closed-loop transfer function

from $\quad w$ to z

The generalized plant for the stacked H_∞ problem:

For the stacked H_∞ problem the closed-loop maps are W_pS, $W_T T$ and $W_u KS$.

Consider the following closed-loop System

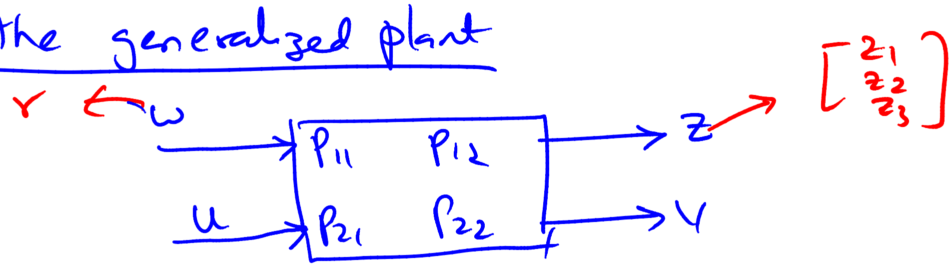


- Let the exogenous input w be r
- Let the regulated variable $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

Then the closed-loop map from w to z

$$w \mapsto z = \begin{bmatrix} W_p S \\ W_T T \\ W_u KS \end{bmatrix}$$

<u>Now the generalized plant</u>



$$z_1 = W_p(r - Gu)$$

$$= W_p r - W_p G u = \begin{bmatrix} W_p & -W_p G \end{bmatrix} \begin{bmatrix} r \\ u \end{bmatrix}$$

$$z_2 = W_T y = W_T G u = \begin{bmatrix} 0 & W_T G \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$z_3 = W_u u = \begin{bmatrix} 0 & W_u \end{bmatrix} \begin{bmatrix} w \\ \end{bmatrix}$$

$$z_3 = W_u u = [0 \quad W_u] \begin{bmatrix} w \\ u \end{bmatrix}$$

$$v = r - y = r - Gu = [I \quad -G] \begin{bmatrix} w \\ u \end{bmatrix}$$

$$\begin{bmatrix} z \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} \begin{bmatrix} W_p & -W_p G \\ 0 & W_T G \\ 0 & W_u \end{bmatrix} \\ [I \quad -G] \end{bmatrix}}_{\text{generalized plant for the stacked } H_\infty \text{ problem } P} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$F_\ell(P, K) = \begin{bmatrix} W_p S \\ W_T T \\ W_u KS \end{bmatrix}$$

<u>Another interpretation of the H_∞ norm of an input output system ϕ</u>

ϕ is a transfer function that is stable.

$$\max_{w \neq 0} \frac{\|\phi w\|_2^2}{\|w\|_2} = \|\phi\|_{H_\infty}$$

(*) Let the impulse response of ϕ be h.

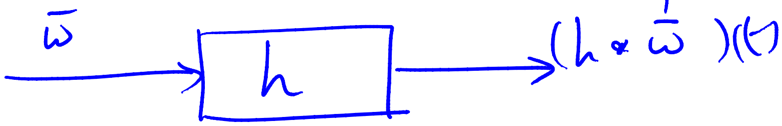
- the output of the system ϕ due to an input w is
  ϕw in the laplace domain
  $(h * \bar{w})(t)$ in the time domain

$$\phi = -\mathcal{L}h$$

$$\omega = \mathcal{L}\bar{\omega}$$

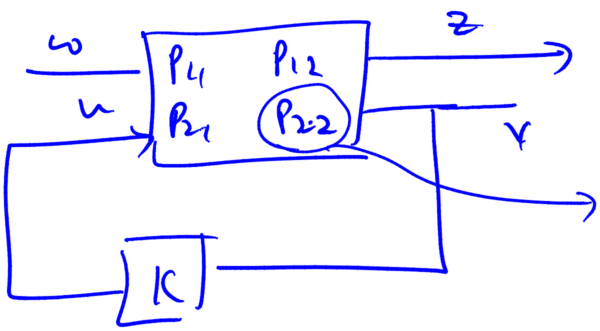
$$\phi\omega = \mathcal{L}\, h * \bar{\omega}$$

$$\max_{\|\bar{\omega}\|_2 \neq 0} \frac{\|(h * \bar{\omega})\|_2}{\|\bar{\omega}\|_2}$$



∴ The H_∞ norm of a system captures the energy amplification of the system.

& "H_∞ norm of a system is the induced l_2 norm".



P_{22} is the actual plant – G.

—— x ——

$$\min_{K \text{ stabilizing}} \| F_\ell(P, K) \|_{H_\infty}$$

$$\underbrace{[P_{11} + P_{12}(I - K P_{22})^{-1} K P_{21}]}_{''}$$

$= (P\ K)$

$$\min_{K \text{ stabilizing}} \left\| P_{11} + P_{12}\underbrace{\overbrace{(I - KP_{22})^{-1}}^{\otimes}}\, K\, P_{21} \right\|_{H_\infty}$$

ⓐ Suppose $P_{22} \approx -G$ is stable.

SISO, G, and SISO, K

$$K(I - KP_{22})^{-1}$$

$P_{22} = -G$.

Suppose we define $K(I - KP_{22})^{-1} =: Q$

- If K is stabilizing controller the Q is stable

(If G is stable) then K is a stabilizing controller if and only if $Q = K(I + KG)^{-1}$ is stable.

$G = \dfrac{n_G}{d_G}$; $K = \dfrac{n_K}{d_K}$ be coprime

$$Q = \frac{n_K}{d_K} \frac{1}{1 + \frac{n_G n_K}{d_G d_K}} = \frac{n_K d_G \cancel{d_K}}{\cancel{d_K}(d_G d_K + n_G n_K)}$$

$$= \frac{n_K d_G}{(d_G d_K + n_G n_K)}$$

Q is stable and G is stable then interconnection can be unstable only if the unstable root of $(d_G d_K + n_G n_K)$ is cancelled by the unstable root of n_K

$\therefore\ n_K(s_0) = 0 \Rightarrow (d_G d_K)(s_0) = 0$

$\Rightarrow d_K(s_0) = 0$

$\Rightarrow \quad n_K(s_0) = d_-(s_0) = 0 \quad \Rightarrow$

$$\min_{K \text{ stabilizing}} \| P_{11} + P_{12} \underbrace{(I - G_{22}K)^{-1} K}_{\downarrow} P_{21} \|_{H_\infty}$$

$$\min_{Q \text{ stable}} \| P_{11} + P_{12}\, Q\, P_{21} \|_{H_\infty}$$