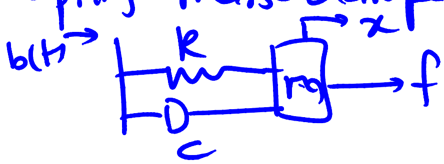
Lecture 16

Thursday, March 24, 2011
8:14 AM

(*) Systems are often described in terms of input-output ~~w.~~ transfer function or in terms of actual realizations

- input-output systems implicitly assumes an "input" signal and an output signal.
- A ~~phy~~ realized physical is usually a bunch of ODE, pdes.

Example: Spring-mass-damper system



$$m\ddot{x} = f + k(b-x) - c\dot{x}$$

$$\Rightarrow m\ddot{x} + kx + c\dot{x} = f + kb(t).$$

$$y = x \ ;$$

$$Y = \left[\frac{f(s)}{ms^2+k+cs} + \frac{k\, b(s)}{ms^2+k+cs}\right].$$

$$Y = \boxed{\frac{1}{ms^2+cs+k}}\, f(s) + \frac{k}{ms^2+cs+k}\, b(s)$$

$$Y = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \dfrac{f(s)}{ms^2+cs+k} \\ \dfrac{s f(s)}{ms^2+cs+k} \end{bmatrix}$$ with for the input

— x —

(?) (a) Given a transfer function what is

⑦ (a) Given a transfer function what is the "minimal" physical realization of the transfer function?

State-Space approach:

$$\ddot{p} + \dot{p} + kp = f \; ; \quad x = \begin{bmatrix} p \\ \dot{p} \end{bmatrix}$$

$$\dot{x} = Ax + Bf \; ; \quad A = \begin{bmatrix} 0 & 1 \\ -k & -1 \end{bmatrix} ; \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

inputs.

Here the notion of stability is asymptotic stability: when does the solution $x(t) \to 0$ as $t \to \infty$ for any initial condition $x(0)$ with the input set to zero.

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) \, d\tau$$

(Variation of parameters formula)

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \cdots$$

$A \in \mathbb{R}^{n \times n}$

The initial condition response

$$x(t) = e^{At} x(0).$$

For a.s. $x(t) \to 0$ as $t \to \infty$ for any $x(0)$

$\Leftrightarrow$ $\lambda_i(A)$ is in the Lhp for eigenvalues λ_i of A.

$$\dot{x} = Ax + Bu \quad ; \quad a.s \Leftrightarrow \lambda_i(A) \in LHP$$
$$y = Cx + Du \quad ; \quad y \text{ is the output equation.}$$

output (y) input (u)

Lets assume initial conditions are zero
$x(0) = 0$

$$sX(s) - X(0) = AX(s) + BU(s)$$

$$\Rightarrow \quad (sI - A)X(s) = X(0) + BU(s)$$

$$\Rightarrow \quad X(s) = (sI-A)^{-1}X(0) + (sI-A)^{-1}BU(s).$$

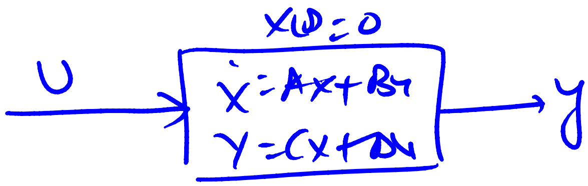
$X(0) \equiv 0$

$$X(s) = (sI-A)^{-1}BU(s)$$

$$Y(s) = CX(s) + DU(s)$$
$$= C(sI-A)^{-1}BU(s) + DU(s)$$

$$\therefore \quad Y(s) = \left[C(sI-A)^{-1}B + D\right]U(s).$$

Summary: Given a physical realization and identified inputs and outputs we have obtained a transfer function relating input and the output.



$$U \longrightarrow \boxed{C(SI-A)^{-1}B + D} \longrightarrow y$$

The input-output transfer function matrix is stable iff all poles of all elements are in the strict lhp.

→ When are the two notions of stability equivalent?

→ If A is a.s then any input-output transfer function will be stable

<u>Pf:</u> $Y = (C(SI-A)^{-1}B + D)\, U$

$$G = \frac{C\ \mathrm{Adj}[SI-A]}{\det(SI-A)} B + D \ ; \quad P^{-1} = \frac{\mathrm{Adj}(P)}{\det(P)}$$

$$= \frac{\{C\ \mathrm{Adj}(SI-A)\, B + D \det(SI-A)}{\det(SI-A)} = \frac{n(s)}{d(s)}$$

where $n = C\,\mathrm{Adj}(SI-A)B + D\det(SI-A)$

$d = \det(SI-A)$

If A has eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_n$

$$\det(SI-A) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)$$

$$G = \left(\frac{C\,\mathrm{Adj}(SI-A)B + D\det(SI-A)}{(s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n)} \right)$$

It follows that as $\lambda_i \in$ lhp, G has all poles in the lhp.

$\therefore$ a.s $\Rightarrow$ i.o. stability

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It is easily possible that $G(s)$ is stable in the i-o sense but A is not stable.

→ these involve $\because$

$C\,\mathrm{Adj}(sI-A)B + D\det(sI-A)$
having common unstable factors with
$\det(sI-A)$.

i.o. stability $\not\Rightarrow$ a.s.
(with a given choice of input and output)

2. When does $C(sI-A)^{-1}B+D$ stable as a transfer matrix $\Leftrightarrow$ A is a.s.?

~~Pf~~: ~~Sol~~ Answer: (C,A) has to be ~~obs~~ detectable and (B,A) has to be stabilizable.

→ Observability:

(Control input is known)

$$x(t) = e^{At}x(0) + \left(\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau\right)$$

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

$$\underbrace{\left[y(t) - C\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau - Du(t)\right]}_{m(t)} = Ce^{At}x(0).$$

$$m(t) = Ce^{At}x(0)$$

$d\ \ m\ \ \ \ \ \ c\ A\ \ At\ \ x(0)$

$$\frac{dm(t)}{dt} = CAe^{At}x(0)$$

$$\frac{d^2m(t)}{dt} = CA^2e^{At}x(0)$$

$$\vdots$$

$$\frac{d^{n-1}m(t)}{dt} = CA^{n-1}e^{At}x(0).$$

$$\frac{d^{n}m(t)}{dt} = CA^{n}e^{At}x(0)$$

$$\vdots$$

$$\begin{bmatrix} m(t) \\ m^{(1)}(t) \\ \vdots \\ m^{(n-1)}(t) \end{bmatrix} = \begin{bmatrix} Ce^{At} \\ CAe^{At} \\ \vdots \\ CA^{n-1}e^{At} \end{bmatrix} \Bigg|_{t=0} x(0)$$

$$\begin{bmatrix} m(0) \\ m^{(1)}(0) \\ \vdots \\ m^{(n-1)}(0) \end{bmatrix} = \underbrace{\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{\mathcal{O}} x(0)$$

$$x(0) = \mathcal{O}^{-1}\begin{bmatrix} m(0) \\ \vdots \\ m^{(n-1)}(0) \end{bmatrix}$$

$$x(t) = e^{At}\mathcal{O}^{-1}\begin{bmatrix} m(0) \\ \vdots \\ m^{(n-1)}(0) \end{bmatrix} + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d(\tau)$$

Cayley Hamilton thm gives the necessity

$$\dot{x} = Ax + Bu, \qquad u = Kx + v$$

$$\underline{\dot{x} = Ax + BKx + Bv}$$

$= (A + BK)x + Bv$

Controllability

$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$

$x_d = x(t_f)$

$x(t) = x_d$

$(x_d - e^{At_f}x(0))$

$\int_0^{t_f} e^{A(t_f-\tau)} B \int^{t_f} B^* e^{A^*(t_f-\tau)} (x_d - e^{At_f}x(0)) d\tau$

(FF^{-1})