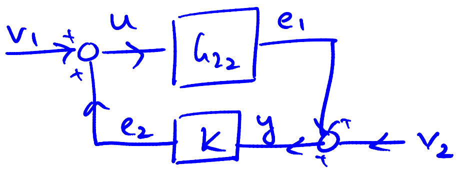
Lecture20

Tuesday, April 05, 2011
8:05 AM



G_{22} : $\dot{x}_{22} = A x_{22} + B_2 u$
$e_1 = C_2 x_{22} + D_{22} u$

$G_{22} \equiv \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]$

K : $\dot{x}_K = A_K x_K + B_K y$
$e_2 = C_K x_K + D_K y$

$K \equiv \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$

→ We assume that the above realizations of G_{22} and K are stabilizable and detectable.

Input - output stability:

→ The above interconnection is I-O stable if the map (transfer matrix) from $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} u \\ y \end{pmatrix}$ is a stable map (it is analytic in the Rhp).

$$H(G_{22}, K) : \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \longmapsto \begin{pmatrix} u \\ y \end{pmatrix}$$

$$H(G_{22}, K) \equiv \begin{bmatrix} I & -K \\ -G_{22} & I \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} (I - KG_{22})^{-1} & (I - KG_{22})^{-1} K \\ (I - G_{22}K)^{-1} G_{22} & (I - G_{22}K)^{-1} \end{bmatrix} \quad ?$$

I-O stability $\Leftrightarrow$ $(I - G_{22}K)^{-1}$, $(I - KG_{22})^{-1} K$, $(I - G_{22}K)^{-1} G_{22}$, $(I - KG_{22})^{-1}$

I-O stability $\Leftrightarrow$ " all poles of the above transfer matrices have to be in the Lhp".

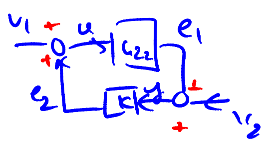
→ Asymptotic stability:

Given any initial condition $(x_{G_{22}}(0), x_K(0))$ with $v_1 = v_2 = 0$ we have that $(x_K(t), x_{G_{22}}(t)) \longmapsto 0$

as $t \to \infty$.

→ <u>Theorem</u>: The <u>inherited realization</u> of the interconnection map $H(G_{22}, K)$ is stabilizable and detectable if $\left(\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right)$ and $\left(\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right)$ are stabilizable and detectable. Therefore, the asymptotic stability of the interconnection is equivalent to the I-O stability of the interconnection.

<u>Stability Theorem for MIMO System</u>:



<u>Theorem</u>: Let $n_{G_{22}}$ and n_K be the number of rhp poles of G_{22} and K. Then the interconnection is internally stable (asymptotically stable) if and only if the following conditions are satisfied.

(i) The number of rhp poles of $L = G_{22}K$ is

$$n_{G_{22}} + n_K \quad \text{(no unstable pole-zero cancellation).}$$

(ii) The matrix transfer function $(I - G_{22}K)^{-1}$ is stable

<u>Proof</u>:

$$\left.\begin{array}{l} u = v_1 + Ky \\ y = v_2 + G_{22}u \end{array}\right\} \Rightarrow \begin{array}{l} u - Ky = v_1 \\ y - G_{22}u = v_2 \end{array}$$

$$\begin{bmatrix} I & -K \\ -G_{22} & I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

$$G_{22}: \begin{array}{l} \dot{x}_{22} = A x_{22} + B_2 u \\ e_1 = C_2 x_{22} + D_{22} u \end{array} \quad \Bigg| \quad K: \begin{array}{l} \dot{x}_K = A_K x_K + B_K y \\ e_2 = C_K x_K + D_K y \end{array}$$

$$v_1 = u - e_2 = u - C_K x_K - D_K y$$
$$v_2 = y - e_1 = y - C_2 x_{22} - D_{22} u.$$

$$\dot{x}_{22} = A x_{22} + B_2 u,$$

$$\dot{x}_{22} = A x_{22} + B_2 u$$
$$\dot{x}_k = A_k x_k + B_k y$$
$$v_1 = +0 x_{22} - C_k x_k + u - D_k y$$
$$v_2 = -C_2 x_{22} - 0 x_k - D_{22} u + y.$$

$$x \equiv \begin{bmatrix} x_{22} \\ x_k \end{bmatrix}; \quad \dot{x} = \begin{bmatrix} A & 0 \\ 0 & A_k \end{bmatrix} \begin{bmatrix} x_{22} \\ x_k \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_k \end{bmatrix} \underbrace{\begin{bmatrix} u \\ y \end{bmatrix}}_{\text{input}}$$

$$\underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\text{output}} = \begin{bmatrix} 0 & -C_k \\ -C_2 & 0 \end{bmatrix} \begin{bmatrix} x_{22} \\ x_k \end{bmatrix} + \begin{bmatrix} I & -D_k \\ -D_{22} & I \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}.$$

$\therefore$ A realization of the map $T: \begin{bmatrix} u \\ y \end{bmatrix} \mapsto \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

is given by

$$T = \left[\begin{array}{c|c} \overbrace{\begin{pmatrix} A & 0 \\ 0 & A_k \end{pmatrix}}^{A_1} & \overbrace{\begin{pmatrix} B_2 & 0 \\ 0 & B_k \end{pmatrix}}^{B_1} \\ \hline \underbrace{\begin{pmatrix} 0 & -C_k \\ -C_2 & 0 \end{pmatrix}}_{C_1} & \underbrace{\begin{pmatrix} I & -D_k \\ -D_{22} & I \end{pmatrix}}_{D_1} \end{array}\right]$$

Note that $H(G_{22}, K) := T^{-1} : \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \mapsto \begin{bmatrix} u \\ y \end{bmatrix}.$

A realization for T^{-1} is given by

$$T^{-1} = \left[\begin{array}{c|c} \overbrace{A_1 + B_1 D_1^{-1} C_1}^{\bar{A}} & \overbrace{-B_1 D_1^{-1}}^{\bar{B}} \\ \hline \underbrace{-D_1^{-1} C_1}_{\bar{C}} & \underbrace{D_1^{-1}}_{\bar{D}} \end{array}\right]$$

$$\bar{A} = \begin{bmatrix} A & B_2 C_k \\ 0 & A_k \end{bmatrix} + \begin{bmatrix} B_2 D_k \\ B_k \end{bmatrix} [I - D_{22} D_k]^{-1} (C_2 \quad D_{22} C_k)$$

$$\bar{D} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} B_k \\ I \end{bmatrix} (I - D_{22} D_k)^{-1} [D_{22} \quad I].$$

The following are equivalent

(i) $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is stabilizable and detectable

(ii) (A, B_2, C_2, D_{22}) and (A_k, B_k, C_k, D_k) are stabilizable and detectable. (HW7)

(Sketch of the proof:)

Suppose $(\bar{A}, \bar{C})$ is not detectable.

$\Rightarrow \exists \lambda \in$ Rhp and $x \neq 0$ s.t. $\bar{A}x = \lambda x$
and $\bar{C}x = 0$; let $x = \begin{bmatrix} x_G \\ x_K \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} A & B_2 C_K \\ 0 & A_K \end{bmatrix} \begin{bmatrix} x_G \\ x_K \end{bmatrix} = \lambda \begin{bmatrix} x_G \\ x_K \end{bmatrix}$$

and $$D^{-1} \begin{bmatrix} 0 & -C_K \\ -C_2 & 0 \end{bmatrix} \begin{bmatrix} x_G \\ x_K \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A x_G + B_2 C_K x_K \\ A_K x_K \end{bmatrix} = \begin{pmatrix} \lambda x_G \\ \lambda x_K \end{pmatrix}$$

$C_K x_K = 0$

$C_2 x_G = 0.$

$\Rightarrow$ $A x_G = \lambda x_G$; $A_K x_K = \lambda x_K$
$C_2 x_G = 0$; $C_K x_K = 0$

→ both x_G and x_K cannot be zero.

→ This implies that at least one of the pairs (A, C_2) or (A_K, C_K) is not detectable.

— x —

Summary: $\text{eig}(\bar{A}) \in \text{Lhp} \Leftrightarrow H(G_{22}, K)$ is stable.

— x —

We will show that $(I - G_{22}K)^{-1}$ has a inherited realization which has the A matrix equal to $\bar{A}$

$$G_{22} = \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array}\right]; \quad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

$(I - G_{22}K)^{-1}$?

→ The realization for $L = G_{22}K$ is

$$A_L = \begin{bmatrix} A & B_2 C_K \\ 0 & A_K \end{bmatrix}; \quad B_L = \begin{bmatrix} B_2 D_K \\ B_K \end{bmatrix}$$

$$C_L = \begin{bmatrix} C_2 & D_{22} C_K \end{bmatrix}; \quad D_L = D_{22} D_K.$$

→ What is the realization of $(I - G_{22}K)$?

$$\left[\begin{array}{c|c} A_L & -B_L \\ \hline C_L & I - D_L \end{array}\right]$$

→ $(I - G_{22}K)^{-1} = \left[\begin{array}{c|c} A_S & B_S \\ \hline C_S & D_S \end{array}\right]$

$$A_S = \begin{pmatrix} A & B_2 C_K \\ 0 & A_K \end{pmatrix} + \begin{pmatrix} B_2 D_K \\ B_K \end{pmatrix} (I - D_{22} D_K)^{-1} (C_2 \;\; D_{22} C_K)$$

$$B_S = -\begin{bmatrix} B_2 D_K \\ B_K \end{bmatrix} (I - D_{22} D_K)^{-1}$$

$$C_S = (I - D_{22} D_K)^{-1} (C_2 \;\; D_{22} C_K)$$

$$D_S = (I - D_{22} D_K)^{-1}$$

$A_S = \bar{A}$

The following are equivalent

(i) (A_L, B_L, C_L, D_L) is a stabilizable and detectable realization of L

(ii) (A_S, B_S, C_S, D_S) is a stabilizable and detectable realization of S.

→ Summary (1) $\left(\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right)$ is a realization of $H(G_{22}, K)$ that is stab. and detectable.

∴ $I=0$ I-O stability of $H(G_{22}, K) \equiv \lambda(\bar{A}) \in$ lhp.

(ii) $\left(\begin{array}{c|c} A_S & B_S \\ \hline C_S & D_S \end{array}\right)$ is a stabilizable and detectable realization if there are no unstable pole-zero cancellations in forming L.

(iii) $A_S = \bar{A}$

(⇐) Assume no unstable pole-zero cancellation while forming $L = G_{22}K$

S is a stable transfer matrix

$\left(\begin{array}{c|c} A_S & B_S \\ \hline C_S & D_S \end{array}\right)$ is a stabilizable and detectable realization

$\lambda(A_S) \in$ lhp $\Leftrightarrow$ S is stable

$\circledast$ $\lambda(\bar{A}) \in$ lhp $\Leftrightarrow$ S is stable

$\circledast$ $(\Rightarrow)$ Suppose $\lambda(\bar{A}) \in$ lhp.

$\Rightarrow$ $\lambda(A_S) \in$ lhp. $\rightarrow$ S is stable

$\Rightarrow$ (A_S, B_S, C_S, D_S) is stabilizable and detectable

$\Downarrow$

(A_L, B_L, C_L, D_L) is stab. and detectable

$\Downarrow$

no pole zero cancellation in forming L.