Lecture 15

Thursday, March 10, 2011
8:14 AM

Robust Stability and Robust Performance for Single input Single output Systems.

* For additive uncertainty where $\|\Delta\|_{H_\infty} \le 1$

$$G_p = G + w_A \Delta \quad \text{the ;}$$

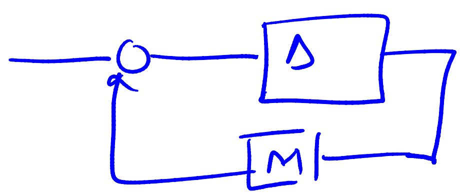
robust stability $\Leftrightarrow$ $\|w_A K S\|_{H_\infty} < 1$

* For multiplicative uncertainty where

$$G_p = G(1 + w_I \Delta) \quad ; \quad \|\Delta\|_{H_\infty} \le 1$$

robust stability $\Leftrightarrow$ $\|w_I T\|_{H_\infty} < 1$.

<u>Small gain Theorem:</u>



Suppose M and Δ are stable transfer functions then the feedback interconnection in the figure is stable if and only if

$$\|M\|_{H_\infty} \|\Delta\|_{H_\infty} < 1.$$

<u>Proof:</u> Note that for stability of the interconnection There should no pole-zero cancellation while forming the

product $M\Delta$ and furthermore

$$|1 + M\Delta(j\omega)| > 0$$

$\therefore$ stability $\iff$ $|1 + M(j\omega)\Delta(j\omega)| > 0 \quad \forall \omega \in R^+$

Note that

$$|1 + M(j\omega)\Delta(j\omega)| \geq \big|1 - |M(j\omega)||\Delta(j\omega)|\big|$$

$$\geq \big|1 - \|M\|_{H_\infty}\|\Delta\|_{H_\infty}\big|$$

$\therefore$ if $\|M\|_{H_\infty}\|\Delta\|_{H_\infty} < 1$ then

$|1 + M(j\omega)\Delta(j\omega)| > 0 \quad \forall \omega \in R^+$

$\therefore$ Stability of the interconnection.

This prove the sufficiency of the stability condition.

Suppose $\|M\|_{H_\infty} > 1$ then $\exists\ \omega_0$ such that

$\Rightarrow \quad |M(j\omega_0)| > 1$

Let $\quad M(j\omega_0) = |M(j\omega_0)| e^{j\angle M(j\omega_0)}$

Let $\quad \underline{\underline{\Delta(j\omega_0)}} = k \dfrac{j\omega_0 - \alpha}{j\omega_0 + \alpha}$

choose α, k such that

$\angle \Delta(j\omega_0) M(j\omega_0) = -1$. $\quad |\Delta(j\omega_0)M(j\omega_0)| = -1$

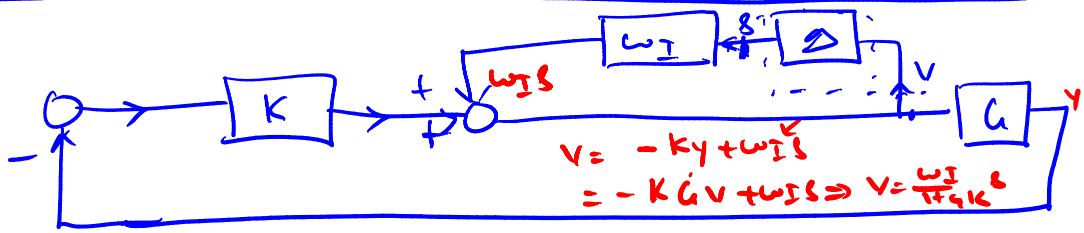
$1 + M(j\omega_0)\Delta(j\omega_0) = 0$

$$\boxed{\Delta(s) = k \frac{s - \alpha}{s + \alpha}} \quad ; \quad -\alpha < 0$$

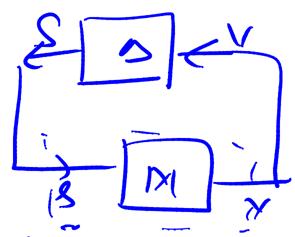
$\|\Delta(s)\| \|M\|_{H_\infty} = 1$

$$|1 + M(j\omega)\Delta(j\omega)| = 0$$

Robust Stability Condition for inverse Multiplicative Uncertainty



$$G_p = G(1+w_I\Delta)^{-1} \text{ with } \Delta \text{ stable}$$
$$\text{and } \|\Delta\|_{H_\infty} < 1$$



Let's assume that K stabilizes the interconnection with $\Delta = 0$.

The transfer function from δ to v assume Δ is ~~no~~ removed from the feedback interconnection is

$\therefore$ The interconnection can also be seen as a $M-\Delta$ interconnection with

$$M = \frac{w_I}{1+GK} = w_I S.$$

$\therefore$ The ~~I~~ interconnection is stable $\Leftrightarrow$

$$\|M\|_{H_\infty} < 1 \Leftrightarrow \|w_I S\|_{H_\infty} < 1.$$

Robust Performance in the SISO Case

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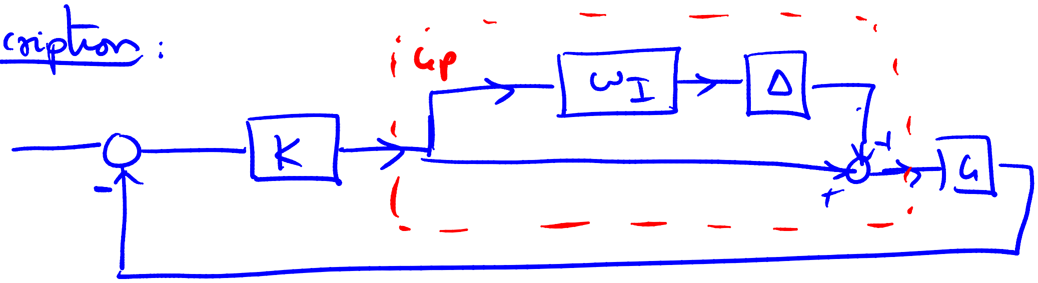
Robust Performance for multiplicative uncertainty description:



$$G_p = G(1+w_I\Delta)\,; \qquad \|\Delta\|_{H_\infty}<1$$

Δ is stable.

(*) Lets assume there is a tracking requirements imposed by a performance weight W_p.

$\therefore$ We need $\|W_p S_p\|_{H_\infty} < 1$

$$S_p = \frac{1}{1+G_pK} = \frac{1}{1+G(1+w_I\Delta)K}$$

$$\forall \|\Delta\|_{H_\infty} < 1.$$

(*) RS $\Leftrightarrow$ $\|w_I T\|_{H_\infty} < 1$ ✓

Lemma: RP $\Leftrightarrow$

(1) $\|w_I T\|_{H_\infty} < 1$ (2) $\left\|\dfrac{w_p S}{1+w_I T\Delta}\right\|_{H_\infty} < 1$

$\forall \Delta$ stable $\|\Delta\|_{H_\infty} < 1$.

Proof: RP $\Leftrightarrow$ $\underbrace{\overbrace{\|w_I T\|_{H_\infty} < 1}^{\Updownarrow RS}}_{RS}$, $\underbrace{\|w_p S_p\|_{H_\infty} < 1}$

$$\|W_p S_p\|_{H_\infty} < 1 \iff \left| W_p \frac{1}{1+G(1+W_I\Delta)K} \right| < 1 \quad \forall \omega$$

$$\iff \left| W_p \frac{\frac{1}{1+GK}}{\frac{1+GK}{1+GK} + \frac{W_I GK\Delta}{1+GK}} \right| < 1$$

$$\iff \left| W_p \frac{S}{1+W_I T\Delta} \right| < 1$$

$$\iff \left\| W_p \frac{S}{1+W_I T\Delta} \right\|_{H_\infty} < 1. \quad \forall \Delta \in \quad \|\Delta\|_{H_\infty} \le 1.$$

Thm

RP $\iff$ $\| |W_p S| + |W_I T| \|_{H_\infty} < 1.$

Suppose $\| |W_p S| + |W_I T| \| < 1$

Then $|W_I T| < 1 \quad \forall \omega$

$$\Rightarrow \|W_I T\|_{H_\infty} < 1$$

$\uparrow$ RS

$$|W_p S| + |W_I T| < 1 \quad \forall \omega$$

$|\Delta(j\omega)| < 1$

$$\Rightarrow |W_p S| < 1 - |W_I T| < 1 - |W_I T||\Delta|$$

$$\therefore |W_p S| < |1 + W_I T\Delta|$$

$$\Rightarrow \frac{|W_p S|}{|1+W_I T\Delta|} < 1 \quad \forall (\Delta(j\omega)) < 1$$

$$\iff \text{RP} \quad \|W_p S_p\|_{H_\infty} < 1 \quad \forall \Delta \text{ with } \|\Delta\|_{H_\infty} < 1$$

— × —

Suppose $\exists\, \omega_0$ s.t

$$|W_p S| + |W_I T| > 1$$

$$\Rightarrow \quad |W_p(j\omega_0) S(j\omega_0)| + |W_I(j\omega_0) T(j\omega_0)| > 1$$

Case 1: $|W_I(j\omega_0) T(j\omega_0)| > 1$

$$\Downarrow$$

No RS $\Rightarrow$ no RP

WLOG assume $|W_I(j\omega_0) T(j\omega_0)| < 1$

$$|W_p(j\omega_0) S(j\omega_0)| > \boxed{1 - |W_I T(j\omega_0) T(j\omega_0)|}$$

Choose Δ stable s.t $\underline{W_I T \Delta(j\omega_0)}$

$\|\Delta\|_{H_\infty} < 1$ and

and $\measuredangle(j\omega_0) W_I T(j\omega_0) = -|W_I T \Delta$

$\left|\dfrac{W_p S}{1 + T W_I \Delta}\right| < 1$

$$\therefore \left|\frac{W_p(j\omega_0) S(j\omega_0)}{1 + T W_I \Delta}\right| > \frac{|W_p(j\omega_0) S(j\omega_0)|}{(1 + |T W_I \Delta|)}$$

$$= \frac{|W_p(j\omega_0) S(j\omega_0)|}{1 - (T W_I \Delta)(j\omega_0)}$$

$$> 1.$$

— x —