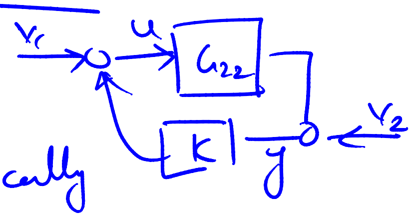
Stability for MIMO interconnection

The MIMO interconnection



of G_{22} and K is asymptotically stable (internally stable) if and only if

(i) There is no unstable pole-zero cancellation while forming the product

$n_L = n_{G_{22}} + n_K$. $\leftarrow L = G_{22}K$ $\left[\begin{array}{l} n_{G_{22}} = \text{the \# of rhp poles of } G_{22} \\ n_K = \text{the \# of rhp poles of } K \end{array} \right.$

(ii) $S = (I-L)^{-1}$ is a stable transfer-matrix

(i.e. all poles of S are in the lhp).

Theorem (Nyquist Criterion for MIMO Systems)

(i) $L = G_{22}K$ has no unstable pole-zero cancellation $\Rightarrow$ ($n_L = n_{G_{22}} + n_K$)

(ii) the Nyquist plot of $\det(I-L(s))$ make n_L counterclockwise encirclements of the origin

(iii) $\det(I-L(s))$ Nyquist plot does not touch the origin.

Proof: the realization of $L = G_{22}K$

$$G_{22} \stackrel{s}{=} \left[\begin{array}{c|c} A & B_2 \\ \hline C_2 & D_{22} \end{array} \right], \quad K \stackrel{s}{=} \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right]$$

that are stabilizable and detectable

$\therefore$ The inherited realization of $L = G_{22}K$

$$L \stackrel{s}{=} \left[\begin{array}{cc|c} A & B_2 C_K & B_2 D_K \\ 0 & A_K & B_K \\ \hline C_2 & D_{22} C_K & D_{22} D_K \end{array} \right]$$

and the inherited realization of $S=(I-L)^{-1}$

$$S = (I-L)^{-1} = \left[\begin{array}{c|c} A_S & B_S \\ \hline C_S & D_S \end{array}\right]$$

$$A_S = \begin{bmatrix} A & B_2 C_K \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 D_K \\ B_K \end{bmatrix} \left[I - D_{22} D_K\right]^{-1} \left[C_2 \;\; D_{22} C_K\right]$$

$$B_S = -\begin{bmatrix} B_2 D_K \\ B_K \end{bmatrix} \left[I - D_{22} D_K\right]^{-1}$$

$$C_S = \left[I - D_{22} D_K\right]^{-1} \left[C_2 \;\; D_{22} C_K\right]$$

$$D_S = (I - D_{22} D_K)^{-1}$$

$$G_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$$

$$G_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]$$

$$G_1: \quad \dot{x}_1 = A_1 x_1 + B_1 u, \qquad y_1 = C_1 x_1 + D_1 u$$

$$G_2: \quad \dot{x}_2 = A_2 x_2 + B_2 y_1, \qquad y = C_2 x_2 + C_2 y_1$$

- (A_L, B_L, C_L, D_L) is a stabilizable and detectable realization of L.

$\Updownarrow$

- (A_S, B_S, C_S, D_S) is a stabilizable and detectable realization of S.

$\Downarrow$

$S=(I-L)^{-1}$ is stable as a transfer matrix if and only if $\lambda(A_S)$ are in the lhp.

$\Updownarrow$

$\det(sI - A_S)$ has all zeros in the lhp.

$$\phi_{cl}(s) = \det(sI - A_L - B_L(I-D_L)^{-1} C_L)$$

$$\phi_{cl}(s) \det(I - D_L) = \phi_{ol}(s) \det(I - L(s)).$$

$\det(sI - A_2)$

$\det\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \det(P_{11})\det(P_{22} - P_{21}P_{11}^{-1}P_{12})$

$\Rightarrow \det(I - L(s)) = \dfrac{\Phi_{cl}(s)}{\Phi_{ol}(s)}$ (constant)

↓ rhp poles of L are P_{ol} in number.

$\Rightarrow$ for $\Phi_{cl}(s)$ to have no zeros in the rhp the Nyquist plot of $\det(I - L(s))$ has to encircle the origin P_{ol} times counterclockwise

The above provides a graphical means to evaluate the internal stability of a feedback interconnection of G_{22} and K.

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Parametrization of all Stabilizing Controllers.

(Multiple-input multiple-output systems).

→ Recall for SISO System G, a coprime representation of G is ~~that~~ of the form $G = \dfrac{n}{m}$ where n and m are stable transfer functions which have no common rhp zeros.

– $G = \dfrac{s-1}{(s+1)^2}$; $\dfrac{(s-1)/(s+1)^2}{1}$; $n = \dfrac{s-1}{(s+1)^2}$; $m = 1$.

$G = \dfrac{s-1}{s-2}$; $G = \dfrac{\frac{s-1}{s+1}}{\frac{s-2}{s+1}}$; $n = \dfrac{s-1}{s+1}$

$m = \dfrac{s-2}{s+1}$

$n = \dfrac{s-1}{(s+1)^2}$; $m = \dfrac{s-2}{(s+1)^2}$

↙ form a coprime representation of G.

given n and m being coprime, ∃ stable

→ given n and m being coprime, ∃ stable transfer function p and q s.t.

$$np - mq = 1. \rightarrow$$ Bezout identity, Diophantine Equation, Aryabhatta identity.

For matrices, there is a notion of right coprime and left coprime.

A transfer matrix T is said to have a doubly coprime factorization if ∃ $\tilde{X}, \tilde{Y}, \tilde{N}, \tilde{M}, M, N, Y, X$ all stable s.t.

$$\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = I = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$\tilde{X}M - \tilde{Y}N = I$$
$$\tilde{X}Y - \tilde{Y}X = 0$$
$$-\tilde{N}M + \tilde{M}N = 0$$
$$-\tilde{N}Y + \tilde{M}X = I$$

$\tilde{X}M - \tilde{Y}N = I$

$\tilde{X}Y = \tilde{Y}X$

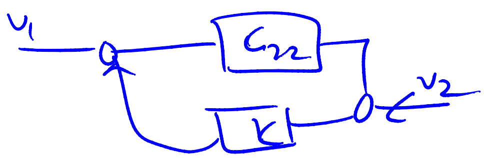
$\tilde{N}M = \tilde{M}N$

$(\tilde{M}, \tilde{N})$ are left coprime.

$$T = NM^{-1} = \tilde{M}^{-1}\tilde{N}.$$

"Every transfer matrix admits a dcf"

Theorem: Consider G_{22} and K being MIMO transfer matrices interconnected as



Suppose, G_{22} admits a dcf given by

$$\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = I$$

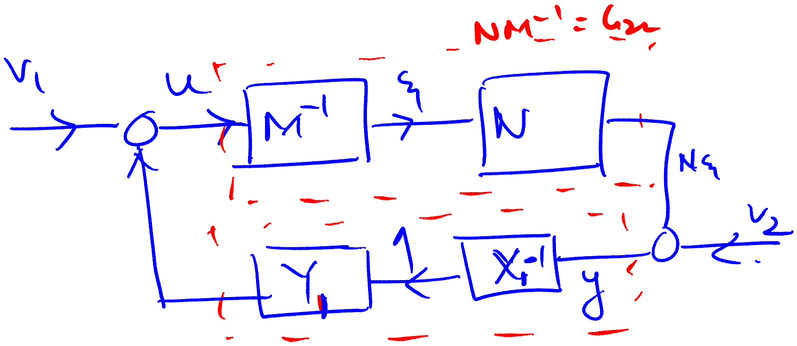
Suppose K has a rcf given by $K = Y_1 X_1^{-1}$

Suppose K has a rcf given by $K = Y_1 X_1^{-1}$

Then the $G_{22}-K$ interconnection is stable if and only if

$$\begin{pmatrix} M & Y_1 \\ N & X_1 \end{pmatrix}^{-1}$$ is a stable transfer matrix.

Proof:



$Y_1 X_1^{-1} \equiv K$.

Note that $M^{-1}, \ldots$

$$M^{-1}u = \xi \Rightarrow u = M\xi$$
$$X_1^{-1}y = \eta \Rightarrow y = X_1\eta$$

The map from (V_1, V_2) to the output $\begin{bmatrix} \xi \\ \eta \end{bmatrix}$ is given by

$$\left.\begin{aligned} u &= V_1 + Y_1\eta \\ y &= V_2 + N\xi \end{aligned}\right\} \Rightarrow \begin{aligned} M\xi &= V_1 + Y_1\eta \\ X_1\eta &= V_2 + N\xi. \end{aligned}$$

$$\Downarrow$$

$$\begin{pmatrix} M\xi - Y_1\eta \\ -N\xi + X_1\eta \end{pmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} M & -Y_1 \\ -N & X_1 \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$\therefore$ the map from $\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ to $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ is

$$\begin{bmatrix} M & -Y_1 \\ -N & X_1 \end{bmatrix} \quad \therefore \text{ the map}$$

$$\begin{bmatrix} \cdot & \cdot \\ -N & X_1 \end{bmatrix}$$

from $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ to $\begin{pmatrix} q \\ \eta \end{pmatrix}$ is $\begin{pmatrix} M & -Y_1 \\ -N & X_1 \end{pmatrix}^{-1}$

Suppose the map from $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ to $\begin{pmatrix} q \\ \eta \end{pmatrix}$ is stable

$\Updownarrow$ $\begin{pmatrix} M & -Y_1 \\ -N & X_1 \end{pmatrix}^{-1}$ is stable

$$\|q\| \le \alpha \|v_1\| + \beta \|v_2\|$$

$$\|\eta\| \le \gamma \|v_1\| + \delta \|v_2\|$$

$$u = Mq \Rightarrow \|u\| = \|Mq\|$$

$$\le \|M\| \, \|q\|$$

$$\le \|M\| \left[\alpha \|v_1\| + \beta \|v_2\|\right]$$

$$\|u\| \le \alpha \|M\| \|v_1\| + \beta \|M\| \|v_2\|$$

$$y = Nq + v_2$$

$$\Rightarrow \|y\| \le \|N\| \|q\| + \|v_2\|$$

$$\le \|N\| \alpha \|v_1\| + \|N\| \beta \|v_2\| + \|v_2\|$$

$\therefore$ the map from $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ to $\begin{pmatrix} u \\ y \end{pmatrix}$ is bounded $\therefore$ internal stability!

If $\begin{pmatrix} M & -Y_1 \\ -N & X_1 \end{pmatrix}^{-1}$ is stable $\Rightarrow$ internal stability

Suppose we have internal stability $\Rightarrow$

the map from $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{bmatrix} u \\ y \end{bmatrix}$ is stable map.

in other words $\exists$ constants $\alpha, \beta, \gamma, \delta$ s.t

$$\|u\| \le \alpha \|v_1\| + \beta \|v_2\|$$

$$\|y\| \le \gamma \|v_1\| + \delta \|v_2\|.$$

Note that $M^{-1}u = q \Rightarrow u = Mq$.

and $\tilde{X}M - \tilde{Y}N = I$

(multiplying both sides by q)

$$\tilde{X}Mq - \tilde{Y}Nq = q$$

$$\tilde{X}u - \tilde{Y}(y - v_2) = q$$

$$\therefore \quad q = \tilde{X}u - \tilde{Y}y + \tilde{Y}v_2$$

$$\|q\| \le \|\tilde{X}\|\,\|u\| + \|\tilde{Y}\|\,\|y\| + \|\tilde{Y}\|\,\|v_2\|.$$

$$\|q\| \le \|\tilde{X}\|\{\alpha\|v_1\| + \beta\|v_2\|\} + \|\tilde{Y}\|\{\gamma\|v_1\| + \delta\|v_2\|\} + \|\tilde{Y}\|\,\|v_2\|$$

$\therefore$ the map from $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to q$ is stable.

Illy the map from $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \to \eta$ is stable

$$\Downarrow$$

$$\begin{pmatrix} M & -Y_1 \\ -N & X_1 \end{pmatrix}^{-1} \text{ is stable.}$$

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Note that a dcf of G_{22} is given by

$$\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \cdot \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = I.$$

with $G_{22} = NM^{-1} = \tilde{M}^{-1}\tilde{N}$.

$$X, \tilde{Y}, \tilde{N}, \tilde{M} \in RH_\infty$$

$$\therefore \quad \begin{pmatrix} M & Y \\ N & X \end{pmatrix}^{-1} = \begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix} \text{ is stable}$$

$\therefore$ YX^{-1} is a stabilizing controller.

$$\begin{pmatrix} I & -Q \\ 0 & I \end{pmatrix}\begin{bmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{bmatrix}\begin{bmatrix} M & Y \\ N & X \end{bmatrix}\begin{pmatrix} I & Q \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & -Q \\ 0 & I \end{pmatrix}\begin{pmatrix} I & +Q \\ 0 & I \end{pmatrix}$$

$$= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$