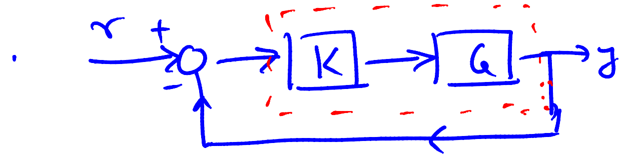
Lecture 5

Tuesday, February 01, 2011
8:13 AM



1. Frequency domain Specifications: $|S(j\omega)|$ small in $[0, \omega_B]$ leads to $|L|$ being large in $[0, \omega_B]$
   - $|T(j\omega)|$ small in $[\omega_{BT}, \infty)$ leads to $|L|$ being small in $[\omega_{BT}, \infty)$

   S- Sensitivity transfer function $= \frac{1}{1+L}$

   T- Complimentary Sensitivity t.f. $= \frac{L}{1+L}$

2. Internal model principle
   - $|L|$ should have -20 db/decade slope in the DC region for being Type I
   - " " " -40 db/decade slope in the DC region for Type II
   - -60 db/decade slope for Type III in the DC region

$$K_p = \lim_{s \to 0} |L(s)|$$

$$K_v = \lim_{s \to} |s L(s)| \; ; \; K_a = \lim_{s \to 0} |s^2 L(s)|$$

Step Tracking error $= \frac{1}{1+K_p}$

ramp " " $= \frac{1}{K_v}$

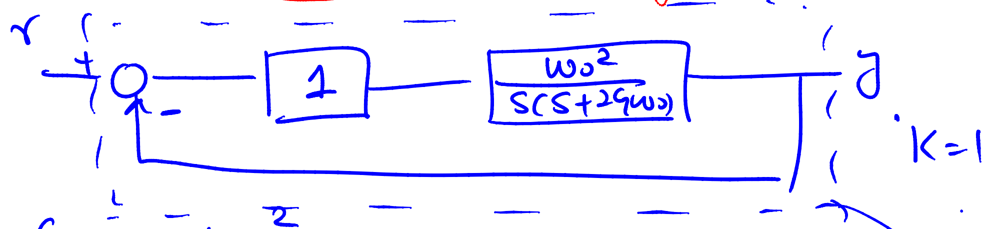
t^2 " " $= 1/K_a$.

3. Time domain Specifications

These are specified with respect to the step response of the system.

* Look at a prototype Second order example.

Second order Prototype



$$G = \frac{\omega_0^2}{s(s+2\zeta\omega_0)} \quad ; \qquad 0 < \zeta < 1$$

→ The closed-loop map from $r \mapsto y$ is

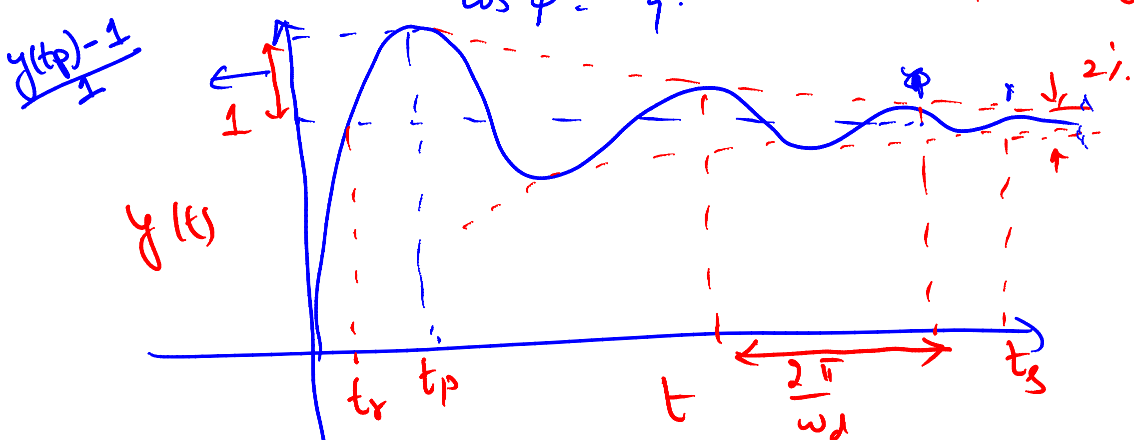
$$\frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

Step Response of the closed-loop System has the following form

$$\boxed{y(t) = 1\left[1 - \frac{e^{-\zeta\omega_0 t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi)\right]}$$

where $\omega_d := \omega_0\sqrt{1-\zeta^2}$ ↘ damped natural frequency

$\cos\phi = \zeta.$



– t_r = rise time; the first time the output hits the value 1; $t_r = \dfrac{\pi - \cos^{-1}(\zeta)}{\omega_0\sqrt{1-\zeta^2}}$

– t_p: is the time when the maximum is reached

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$$t_p = \frac{\pi}{\omega_0\sqrt{1-\zeta^2}} \quad \text{(peak time)}$$

- M_p = Maximum fraction overshoot $= y(t_p) - 1$

$$= e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

→ t_s = Settling time is the time after which $|y(t)|$ lies with 2% of the steady state

$$t_s = \frac{4}{\zeta\omega_0}.$$

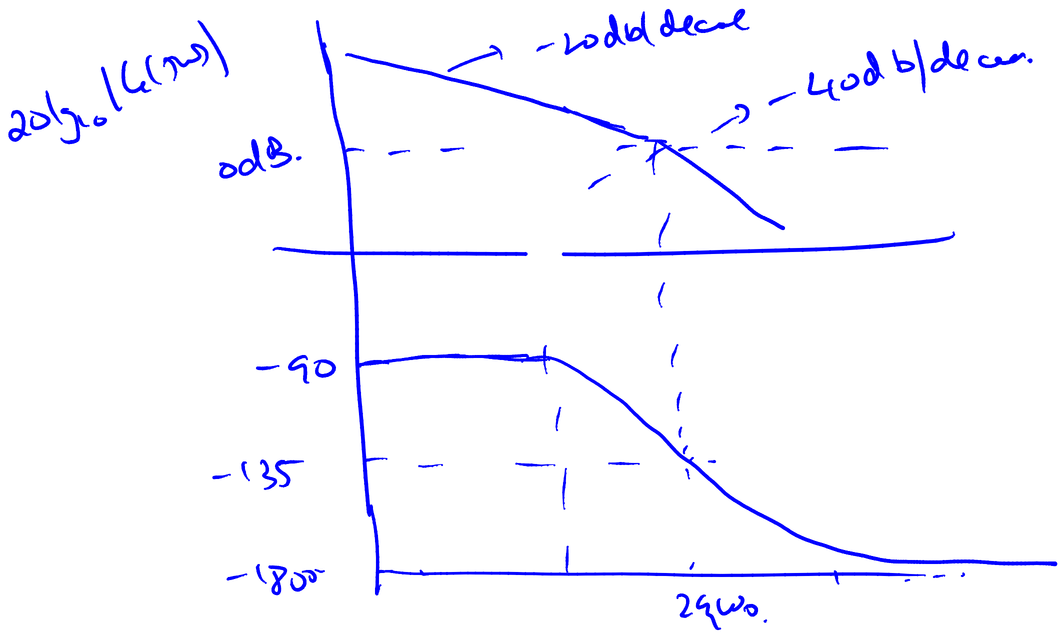
What is desired is that t_r, t_s to be small and M_p to be small.

~~Frequency response~~

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}; \qquad \omega_r = \omega_0(1-2\zeta^2).$$

Open-loop response and its connection to closed-loop behavior

$$G = \frac{\omega_0^2}{s(s+2\zeta\omega_0)}$$



ω_{gc} can be obtained by solving

$$|G(j\omega_{gc})| = 1$$

this yields

$$\omega_{gc} = \omega_0 \left[\sqrt{1+4\zeta^4} - 2\zeta^2\right]^{1/2}$$

$$\begin{aligned} &\text{ⓐ}\ \zeta = 0; & \omega_{gc} &= \omega_0 \\ &\zeta = 1 & \omega_{gc} &= \sqrt{0.24}\ \omega_0 \\ &\zeta = 0.707 & \omega_{gc} &= \sqrt{0.41}\ \omega_0. \\ &\zeta = 0.5 & \omega_{gc} &= \sqrt{0.62}\ \omega_0. \end{aligned}$$

A good rule of thumb is that

$$\omega_0 = 1.4\,\omega_{gc} \approx (1.2 - 1.4)\,\omega_{gc}$$

$G_a = \dfrac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$; the 3dB point is

obtain by $|G_a(j\omega)|^2 = 1/2$

this yields $\omega_{BW} = \omega_0 \sqrt{1 - 2\zeta^2 + \sqrt{2 + 4\zeta^4 - 4\zeta^2}}$

$$\boxed{\omega_{BW} \approx (1.2 - 1.6)\,\omega_{gc}.}$$

gain crossover of L.

Bandwidth of the closed-loop system

Even though the relation of ω_{gc} (open-loop characteristic) and ω_{BW} (closed-loop characteristic) hold exactly for 2nd order prototype this relation guides the intuition for a general L that is not neccessarily 2nd order prototype

→ Phase Margin for the 2nd order prototype:

$PM = \pi + \angle G(j\omega_{gc})$

$= \frac{\pi}{2} - \text{atan}\left[\frac{\left(\sqrt{1+4\zeta^4} - 2\zeta^2\right)^{1/2}}{2\zeta}\right]$

$\zeta = 0.5$; $PM = 51°$

$\zeta = 0.6$ $PM = 59°$

$\zeta = 0.7$; $PM = 65°$

$\zeta = 1$; $PM = 76°$.

A good rule of thumb is

$\boxed{PM = 100\,\zeta}$ in degrees.

① The maximum fraction overshoot

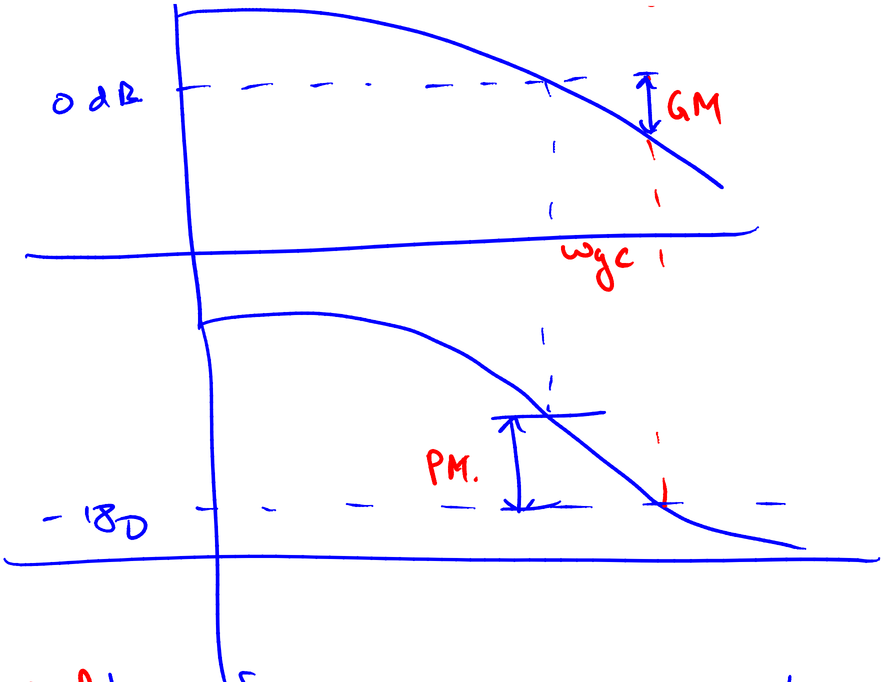
$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$.

depends only on ζ.

If M_p is specified then essentially on ζ and therefore on PM.

This relationships are exact for a second order system. Nevertheless they guide our intuition for a general system.

④ Lets assume an open loop bode plot (not neccessarly a second order prototype)



Closed loop Specs	Open loop
1. Bandwidth ω_{BW}	$(1.2 - 1.6)\,\omega_{gc} = \omega_{BW}$
2. PM (in degrees) $= 100\zeta$	can be read from plot above.
3. GM	Can be read from the bode plot
4. $M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$	$\zeta = \frac{\text{PM in deg}}{100}$
5. $t_s = \frac{4}{\zeta\omega_0}$	$\omega_0 = 1.4\,\omega_{gc}$
6. $t_r = \frac{\pi - \cos^{-1}\zeta}{\omega_0(\sqrt{1-\zeta^2})}$	$\omega_0 = 1.4\,\omega_{gc}$, $\zeta = \frac{\text{PM in deg}}{100}$
7. Steady state behavior	determined by the behavior of $L(j\omega)$ near $\omega = 0$.

Controller design

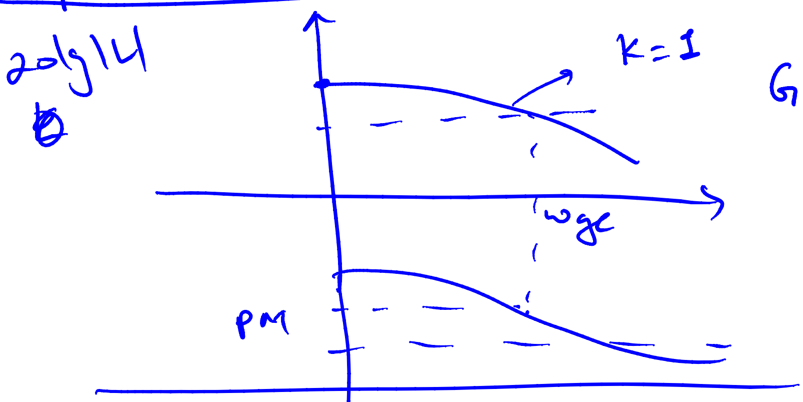
Specs on closed loop $\longrightarrow$ Specs on L (loop shape of L)

$\downarrow$

design Controller K such that

$GK = L$

Proportional Controller:



$e^{-\pi\zeta/\sqrt{1-\zeta^2}}$

Suppose the Spec is

$M_p \leq 0.16$.

$e^{-\pi\zeta/\sqrt{1-\zeta^2}} \leq 0.16$

$\zeta \geq \zeta_d$

- First translate $M_p \leq 0.16$ to

a $PM \geq PM_d$.

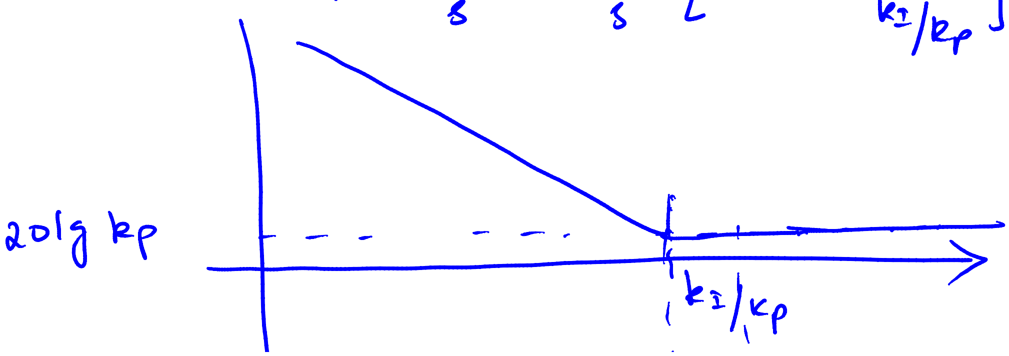
Find ω_{gcd} s.t. $\angle G(j\omega_{gcd}) + \pi = PM_d$

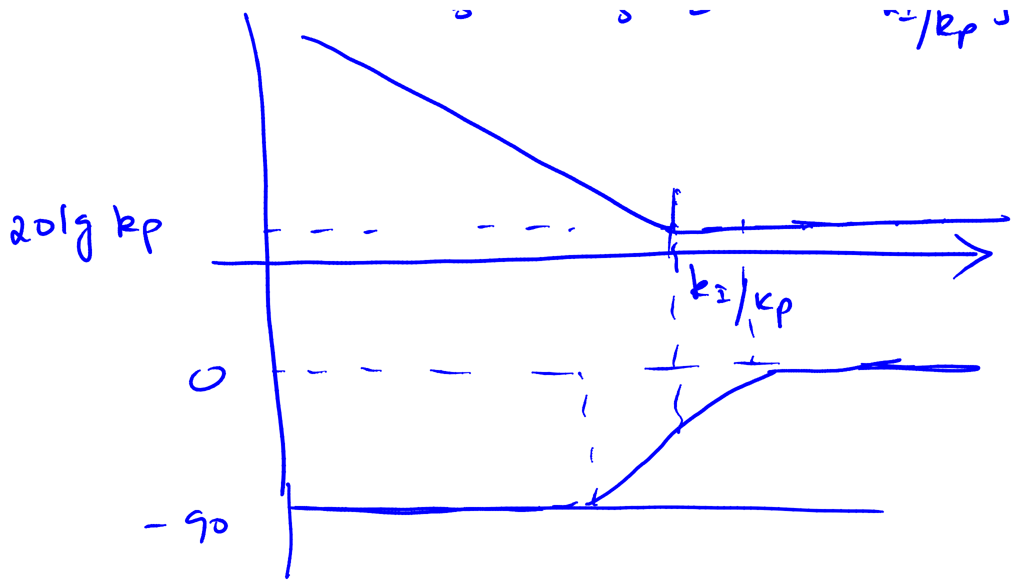
and choose K such that

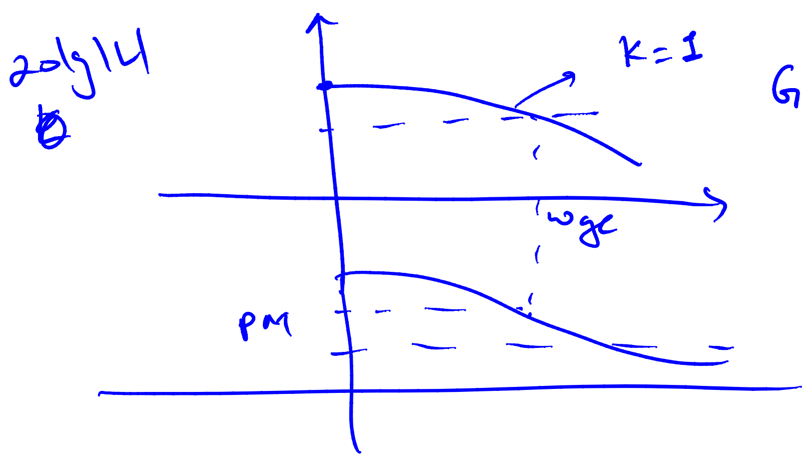
$$|K\, G(j\omega_{gcd})| = 1$$

Proportional Integral Controller

$$K = k_p + \frac{k_I}{s} = \frac{k_I}{s}\left[1 + \frac{s}{k_I/k_p}\right].$$







$M_p \le 0.16\% \longrightarrow PM_d$.