Convex optimization in a Hilbert space setting.

Closed convex set: \( \forall \mathbf{x} \in \mathcal{X} \) is a convex set

\[ \forall \mathbf{k}_1, \mathbf{k}_2 \in \mathcal{L}, \quad 0 \leq \alpha \leq 1 \]

\[ \mathbf{k} = \alpha \mathbf{k}_1 + (1-\alpha) \mathbf{k}_2 \in \mathcal{L}. \]

Suppose \( \mathcal{M} \) is a subspace then

\[ \mu = \inf_{\mathbf{m} \in \mathcal{M}} \| \mathbf{x} - \mathbf{m} \| \]

we have shown that \( \exists \mathbf{m} \in \mathcal{M} \) s.t.

\[ \| \mathbf{x} - \mathbf{m} \| \leq \| \mathbf{x} - \mathbf{m} \| \quad \forall \mathbf{m} \in \mathcal{M} \]

and \( \mathbf{m} \) has the above property if and only if \( (\mathbf{x} - \mathbf{m}) \perp \mathcal{M} \).

Note that \( \mathcal{M} \) a subspace is also a convex set.
Suppose \( k \in K \) is such that \( \|x - k\| \leq \|x - k'\| + k \in K \).

Subspace \( M \)
\[
(x - m) \perp M.
\]
\[
\langle x - m, m - m' \rangle = 0
\]
\( \forall m \in M. \)

**Theorem:** Let \( K \) be a convex set and suppose \( x \in H \) and \( K \subseteq H \) where \( H \) is a Hilbert space. Then \( k \in K \) satisfies
\[
\|x - k\| \leq \|x - k'\| + k \in K
\]

if and only if
\[
\langle x - k, k - k' \rangle \leq 0 \quad k \in K.
\]

**Proof:** Suppose \( k \in K \) is such that
\[
\langle x - k, k - k' \rangle \leq 0 \quad k \in K
\]

Then,
\[
\|x - k\|^2 = \langle x - k, x - k \rangle = \langle x - k, x - k + k - k \rangle
\]
\[
= \langle x - k, x - k \rangle + \langle x - k, k - k \rangle
\]
\[
+ \langle k - k, x - k \rangle + \langle k - k, k - k \rangle
\]

\[
\quad + \langle k - k, k - k \rangle
\]
\[ \begin{align*}
&= \| x - k_0 \|^2 - 2 \langle x - k_0, k - k_0 \rangle \\
&\quad + \| k - k_0 \|^2 \\
\implies &\quad \| x - k \|^2 - \| x - k_0 \|^2 = \| k - k_0 \|^2 - 2 \langle x - k_0, k - k_0 \rangle \\
\implies &\quad \| x - k \|^2 > \| x - k_0 \|^2, \quad \forall \: k \in K.
\end{align*} \]

Suppose \( \exists \: k \in K \) s.t.
\[ \langle x - k_0, k - k_0 \rangle = \varepsilon > 0. \]

Then, note that
\[ k' = (1 - \alpha) k_0 + \alpha \: k \in K \quad \forall \: \alpha \in (0, 1) \]
\[ f(\alpha) = \| x - k' \|^2 = \langle x - (1 - \alpha) k_0 - \alpha k, x - (1 - \alpha) k_0 - \alpha k \rangle \]
\[ = \langle x - k_0 + \alpha (k_0 - k), x - k_0 + \alpha (k_0 - k) \rangle \]
\[ = \langle x - k_0, x - k_0 \rangle + \alpha \langle k_0 - k, k_0 - k \rangle \]
\[ + \alpha \langle k_0 - k, x - k_0 \rangle + \alpha^2 \langle k_0 - k, k_0 - k \rangle \]
\[ = \| x - k_0 \|^2 - 2 \alpha \langle x - k_0, k - k_0 \rangle \\
\quad + \alpha^2 \| k - k_0 \|^2 \]
\[ f(\alpha) = \| x - k_0 \|^2 - 2 \alpha \varepsilon + \alpha^2 \| k - k_0 \|^2 \]
\[ \frac{d f(\alpha)}{d \alpha} \bigg|_{\alpha = 0} = -2 \varepsilon. \]

\[ \therefore \quad 0 < \alpha < 1 \quad \text{s.t.} \quad f(\alpha) < f(0) \]
\[ \Rightarrow \quad 0 < \alpha < 1 \quad \text{s.t.} \quad \| x - k \|^2 < \| x - k_0 \|^2 \]
Thus, if \( k \in K \) s.t. \( \|x - k\| < \|x - k_0\| \)
\( \iff k_0 \) is not the closest to \( x \) in \( K \).

**Theorem:** Consider the problem

\[
\mu = \inf_{k \in K} \|x - k\|
\]

where \( x \in H \), \( K \subset H \) is a **Hilbert Space** and \( K \) a **closed convex subset of** \( H \).

Then, if \( k_0 \in K \) s.t.

\[
\|x - k_0\| = \mu
\]

such a \( k_0 \) satisfies

\[
\langle x - k_0, k - k_0 \rangle \leq 0 \quad \forall k \in K.
\]

**Proof:**

\[
\mu = \inf_{k \in K} \|x - k\|
\]

Then given any \( \eta > 0 \) \( \exists k_0 \in K \) s.t.

\[
0 \leq \|x - k_0\| \leq \mu + \eta.
\]

It's clear that \( \lim_{n \to \infty} \|x - k_n\| = \mu \).

Therefore, \( \mu \) Parallelization identity provides the following.

\[
\|x - k_0\| = \|x - k_0\|^2 + \|x - k_n + (x - k_n)\|^2
\]
\[ = 2 \| x - k \|^2 + 2 \| x - k_f \|^2 \]

\[ \implies \| k_i - k \|^2 = 2 \| x - k \|^2 + 2 \| x - k_f \|^2 - 2 \| 2 x - (k_f + k_i) \|^2 \]

\[ = 2 \| x - k \|^2 + 2 \| x - k_f \|^2 - 4 \| x - \frac{k_i + k_f}{2} \|^2 \]

Given \( \varepsilon > 0 \), choose \( N > 0 \) s.t. \( \varepsilon < \frac{\varepsilon}{N} \). Then if \( i, j > N \)

\[ \| k_i - k_j \|^2 \leq 2 \left( \mu^2 + \varepsilon^2 \right) + 2 \left( \mu^2 + \varepsilon^2 \right) - 4 \mu^2 \]

\[ = \varepsilon^2 \]

\[ \implies \| k_i - k_f \| \leq \varepsilon \]

So, \([k_i]\) is a Cauchy sequence.

\( k_i \) is a Cauchy sequence in a Hilbert space \( H \). So, if \( k_0 \in H \) such that \( \| k_i - k_0 \| \to 0 \) as \( i \to \infty \).

Now, \( k_0 \in K \) because \( k_i \in K \) and \( K \) is closed.

\[ \| x - k_0 \| = \| x - k_0 + k - k_0 \| \]

\[ = \| x - k + k_0 - k_0 \| \]

\[ \leq \| x - k \| + \| k_0 - k_0 \| \]

\[ = \| x - k \| \leq \mu \]
\[\|x - k\|_1 \leq \frac{1}{\mu}\]

\[\|x - k\|_1 = \frac{1}{\mu}\]

(\(\|x - k\|_1 \geq \frac{1}{\mu}\) for some \(k_0 \in K\)).

we have shown that \(\exists k_0 \in K\) s.t.
\[\|x - k\|_1 \leq \|x - k\|_1 + \|k - k_0\|_1\]

\(<x - k_0, k - k_0> \leq 0 \quad \forall k \in K\).

Suppose \(\exists k' \in K\) s.t.
\[\|x - k'\|_1 \leq \|x - k_0\|_1 + \|k - k_0\|_1\]

\[k(\alpha) = \alpha k_1 + (1 - \alpha) k_0\]

\[\|x - k(\alpha)\|_1^2 = \|x - \alpha k_1 + (1 - \alpha) k_0\|_1^2\]

\[\|x - k(\alpha)\|_1^2 = \|x - (\alpha k_1 + (1 - \alpha) k_0)\|_1^2\]

\[\|x - k_1\|_1^2 = \|x - k_1 + k_0 - k_1\|_1^2\]

\[= \|x - k_0\|_1^2 + \|k_1 - k_0\|_1^2 + 2\langle x - k_0, k_1 - k_0\rangle\]

\[\mu^2 = \mu^2 + \|k_1 - k_0\|_1^2\]

\[-2 \langle x - k_0, k_1 - k_0\rangle\]

\[0 = \|k_1 - k_0\|_1^2 + 2\langle x - k_0, k_1 - k_0\rangle\]

\[\Rightarrow \varepsilon > 0, \quad k_1 = k_0\]

\[\Rightarrow \varepsilon = 0, \quad k_1 = k_0\]

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\[\Rightarrow \varepsilon = 0, \quad k_1 = k_0\]
uniqueness does hold

\[ c_2 \parallel c_1 \]

\[ 2n - \nu, k - k_0 \leq 0 \quad \forall k \in k. \]
Then $L$ is characterized by $x^* = (m_1, m_2)$ for some $m_1, m_2$ such that

$$L = \{ x \in \mathbb{R}^2 \mid \langle x, x^* \rangle = c \}$$

In other words

$$L = \{ x \in \mathbb{R}^2 \mid \langle (x_1, x_2), (m_1, m_2) \rangle = c \}$$

$$= \{ x \in \mathbb{R}^2 \mid m_1 x_1 + m_2 x_2 = c \}$$

$$L = \{ x \in \mathbb{R}^2 \mid \langle x, x^* \rangle = c \}$$

we will assume for the line $L$ shown that $m_1 > 0, m_2 > 0$.

Half Spaces: $x_2$
Then \( A = \{ x \in \mathbb{R}^2 \mid \langle x, m \rangle \leq c \} \)
\( B = \{ x \in \mathbb{R}^2 \mid \langle x, m \rangle \geq c \} \)

\(-\)ve half space \( (x_1, x_2) \)
\( +\)ve half space

Suppose \( \mathbf{x} = (x_1, x_2) \in A \)

Then \( x_2' > x_2 \)

\[ m_1 x_1 + m_2 x_2' \leq m_1 x_1 + m_2 x_2 = c \]

\[ m_1 x_1 + m_2 x_2 \leq c \]