Lead Controller

Monday, November 23, 2009
12:21 AM

$$G(s) = k \frac{Ts+1}{\alpha Ts + 1} \quad ; \quad 0 < \alpha < 1 ; \; T > 0 ; \; k > 0$$

$$G(j\omega) = k \frac{j\omega T + 1}{\alpha j\omega T + 1} = k \frac{(1 + j\omega T)(1 - j\alpha\omega T)}{1 + \alpha^2 T^2 \omega^2}$$

$$= k \frac{[1 + j(\omega T - \alpha\omega T) + \alpha\omega^2 T^2}{1 + \alpha^2 T^2 \omega^2}$$

$$= k \left\{ \left[\frac{1 + \alpha\omega^2 T^2}{1 + \alpha^2 T^2 \omega^2} + \frac{j\omega T(1-\alpha)}{1 + \alpha^2 T^2 \omega^2} \right] \right\}$$

$$\Rightarrow \text{Real}(G(j\omega)) = k \frac{1 + \alpha\omega^2 T^2}{1 + \alpha^2 T^2 \omega^2}$$

$$\text{Img}(G(j\omega)) = k \frac{\omega T(1-\alpha)}{1 + \alpha^2 T^2 \omega^2}$$

Let $x \hat{=} k \frac{(1 + \alpha\omega^2 T^2)}{1 + \alpha^2 T^2 \omega^2}$

$y \hat{=} k \frac{\omega T(1-\alpha)}{1 + \alpha^2 T^2 \omega^2}$

$$a = \frac{k}{2\alpha}(1+\alpha) \quad ; \quad r = \frac{k(1-\alpha)}{2\alpha}$$

Then it can be shown that

$$(x-a)^2 + y^2 = r^2$$

Bode

Sunday, November 29, 2009
10:54 PM

The Bode of

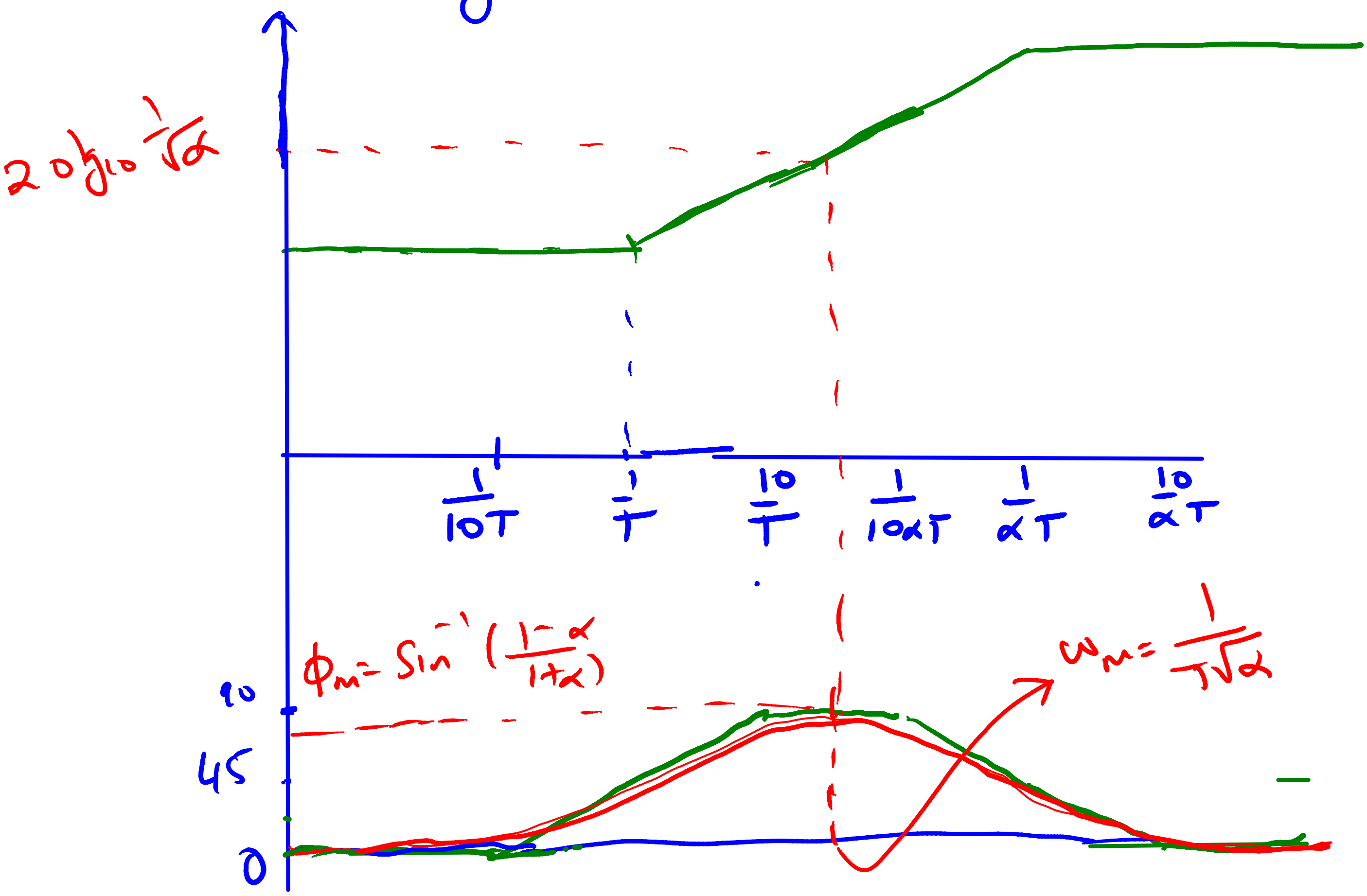
$$\frac{Ts+1}{\alpha Ts+1}\ ;\ 0<\alpha<1$$

$$① \quad \frac{Ts+1}{\alpha Ts+1} = \frac{\frac{s}{T^{-1}}+1}{\frac{s}{(\alpha T)^{-1}}+1}$$

has break frequencies at $\frac{1}{T}$ and $\frac{1}{\alpha T}$ with a zero at $1/T$ and a pole at $1/\alpha T$

As $\alpha<1$; $\frac{1}{\alpha T} > 1/T$

The Bode is given below

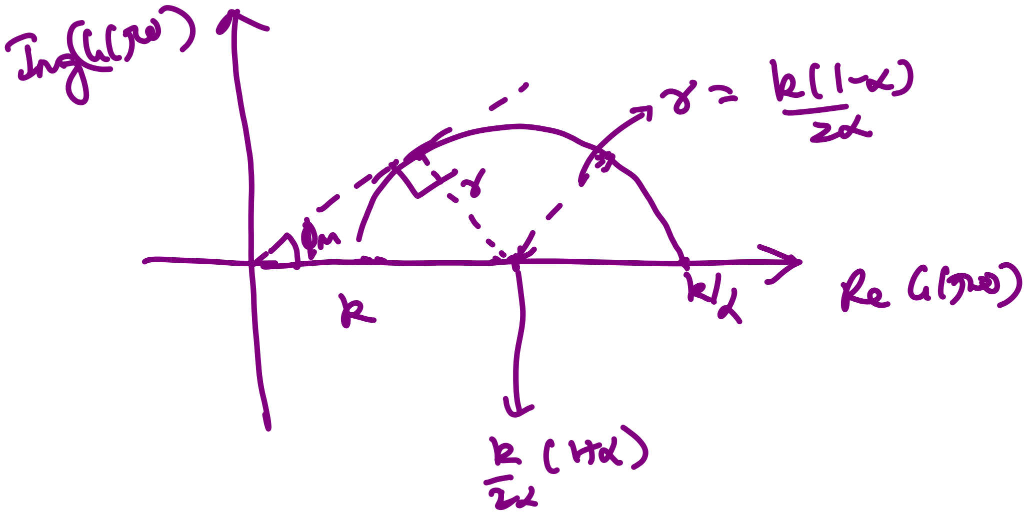


Nyquist of a lead controller

Monday, November 23, 2009
12:24 AM

Therefore the Nyquist plot of

$$G(s) = k\,\frac{Ts+1}{\alpha Ts+1}$$ is given by



maximum phase is ϕ_m then

$$\sin\phi_m = \frac{\gamma}{a} = \frac{1-\alpha}{1+\alpha}.$$

Note also that when $\omega = \omega_m = \frac{1}{T\sqrt{\alpha}}$ then

$$Re\ G(j\omega_m) = k\,\frac{(1+\alpha\omega_m^2T^2)}{1+\alpha^2T^2\omega_m^2} = \frac{k\,2}{1+\alpha}$$

$$Img\ G(j\omega_m) = k\,\frac{\omega_m T(1-\alpha)}{1+\alpha^2T^2\omega_m^2} = \frac{k}{\sqrt{\alpha}}\,\frac{(1-\alpha)}{1+\alpha}$$

Maximum phase of a lead term

Monday, November 23, 2009
12:25 AM

and

$$\sin \angle G(j\omega_m) = \frac{\frac{k}{\sqrt{\alpha}} \frac{(1-\alpha)}{1+\alpha}}{\sqrt{\frac{\cancel{2}4k^2}{(1+\alpha)^2} + \frac{k^2}{\alpha}\frac{(1-\alpha)^2}{(1+\alpha)^2}}}$$

$$= \frac{\frac{k}{\sqrt{\alpha}} \frac{(1-\alpha)}{1+\alpha}}{\frac{k}{(1+\alpha)\sqrt{\alpha}} \sqrt{\cancel{4}4\alpha + 1+\alpha^2 - 2\alpha}}$$

$$= \frac{1-\alpha}{\sqrt{(1+\alpha)^2}} = \frac{1-\alpha}{1+\alpha}$$

$\therefore$ The frequency at which maximum phase is achieved is $\omega_m = \frac{1}{T\sqrt{\alpha}}$

The <u>magnitude at this frequency</u> ω_m is

$$\sqrt{\frac{(2k)^2}{(1+\alpha)^2} + \frac{k^2(1-\alpha)^2}{\alpha(1+\alpha)^2}} = \frac{k}{(1+\alpha)\sqrt{\alpha}} \sqrt{4\alpha+(1-\alpha)^2}$$

$$= \frac{k}{\sqrt{\alpha}}.$$

Summary

Monday, November 23, 2009
12:26 AM

Summary: The term

$$G(s) = k \frac{Ts+1}{\alpha T s + 1} \qquad \begin{array}{c} 0<\alpha<1 \\ T>0 \\ k>0 \end{array}$$

is called a lead term.

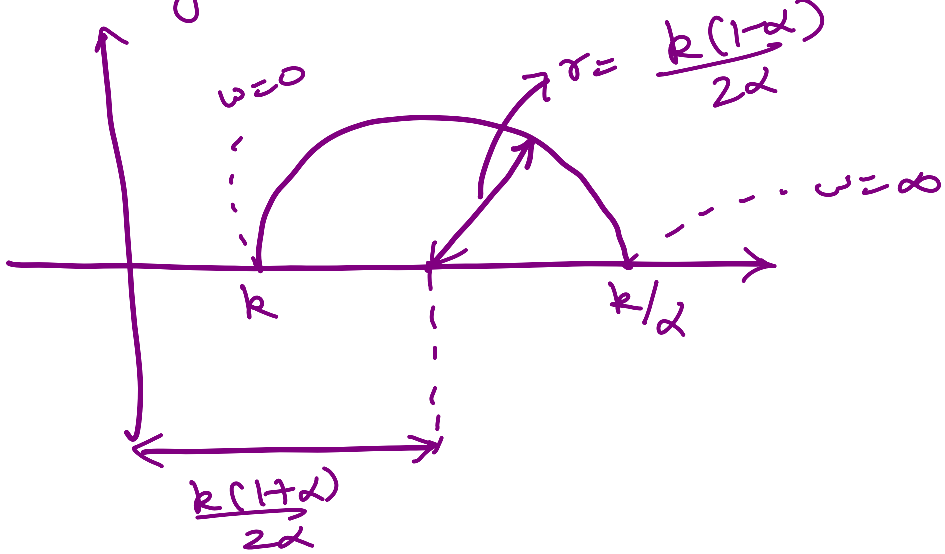
- The phase of $G(j\omega) \geq 0$ for $\omega \in [0, \infty)$
- The maximum phase of $G(j\omega)$ is when $\omega = \omega_m = \frac{1}{T\sqrt{\alpha}}$ with $|G(j\omega_m)| = \frac{k}{\sqrt{\alpha}}$ and

max phase $\angle G(j\omega_m) =: \phi_m$ Satisfies

$$\sin \phi_m = \frac{1-\alpha}{1+\alpha}$$

The Nyquist of $G(s) = k \frac{Ts+1}{\alpha T s + 1}$

is given by



Lead Controller design

Monday, November 23, 2009
12:30 AM

Lead Compensator Design (steps)

Step 1: Choose k to satisfy static error constants (K_v)

Step 2: Using this k, draw a Bode diagram of $G_1(s) = kG(s)$, and evaluate the phase margin

Step 3: Determine the necessary phase angle needed to meet design specs.

Step 4: Let the extra phase needed be ϕ_{extra}. Then the phase that the controller should provide is given by $\phi_m = \phi_{extra} + (6^o - 10^o)$. Determine α from $\sin\phi_m = \frac{1-\alpha}{1+\alpha}$.

Step 5: Find the frequency ω_c where $|G_1(j\omega_c)| = -20\log\left(\frac{1}{\sqrt{\alpha}}\right)$. ω_c is the new gain cross over frequency. We design ω_m to be equal to ω_c. Therefore $\frac{1}{T\sqrt{\alpha}} = \omega_c$. Therefore $T = \frac{1}{\omega_c\sqrt{\alpha}}$

Step 6: compensator is given by $G_c(s) = k\frac{Ts+1}{\alpha Ts+1}$

Step 7: Check the design. If it is not satisfactory, one may have to iterate.

Example

Monday, November 23, 2009
12:31 AM

Example

- For a given plant $G(s) = \frac{4}{s(s+2)}$ design a lead controller so that the resulting unity feedback closed loop system has $GM > 10$, $PM > 50^o$ and $K_v = 20$.

Design Steps:

Step 1: Choose k to satisfy static error constants (K_v)

⋆

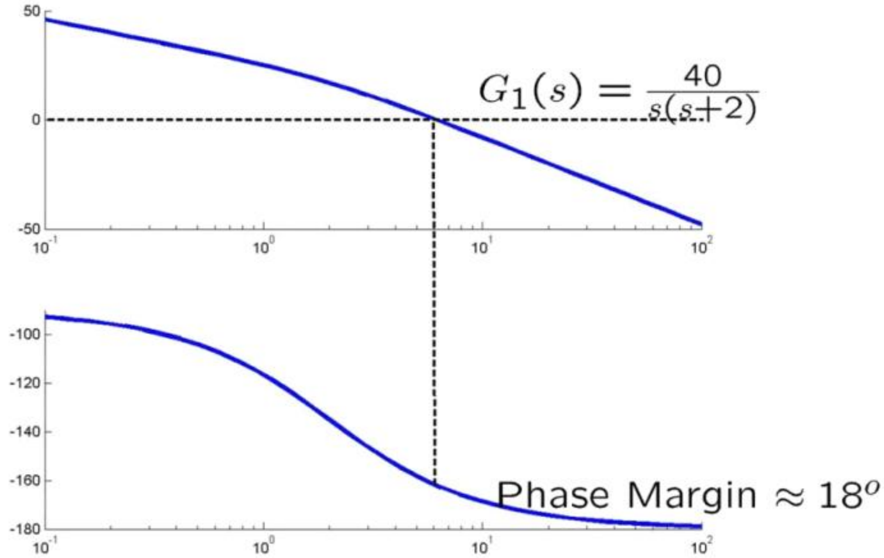
$$\begin{aligned} & K_v = 20 \Rightarrow \lim_{s\to 0} sG_c(s)G(s) = 0 \\ \Rightarrow\ & \lim_{s\to 0} s\left(k\frac{Ts+1}{\alpha Ts+1}\right)\left(\frac{4}{s(s+2)}\right) = 20 \Rightarrow 2k = 20 \\ \Rightarrow\ & k = 10 \end{aligned}$$

Step 2-3

Monday, November 23, 2009
12:32 AM

Example

Step 2: Draw Bode diagram of $G_1(s) = kG(s)$ and find PM



Step 3: $\phi_{extra} = 50 - 18 = 32^o$, Therefore $\phi_m \approx 32 + 6 = 38^o$

Steps 4-6

Monday, November 23, 2009
12:34 AM

Example

Step 4: Determine α from $\sin\phi_m = \frac{1-\alpha}{1+\alpha}$, i.e.,

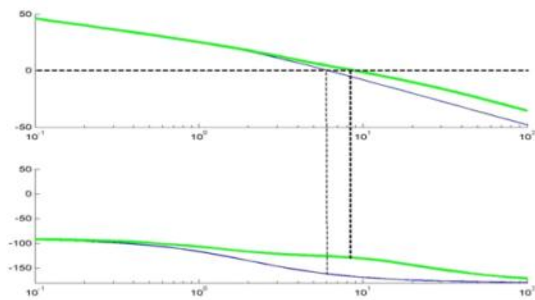
$\sin(38^o) = \frac{1-\alpha}{1+\alpha} \Rightarrow \alpha = 0.24$

Step 5: Find the frequency ω_c where $|G_1(j\omega_c)| = -20\log\left(\frac{1}{\sqrt{\alpha}}\right) = -6.2\ dB$. From bode plot this occurs at $\omega_c = 9\ rad/s$. Then $T = \frac{1}{\omega_c\sqrt{\alpha}} = 0.227$

Step 6: Therefore

$$G_c(s) = k\frac{Ts+1}{\alpha Ts+1} = 10\ \frac{0.227s+1}{0.054s+1}$$

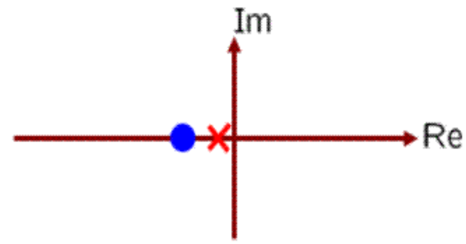
Step 7: valid design: $PM = 50.7^o$, $GM = \infty$



Lag Controller

Friday, November 20, 2009
3:25 PM

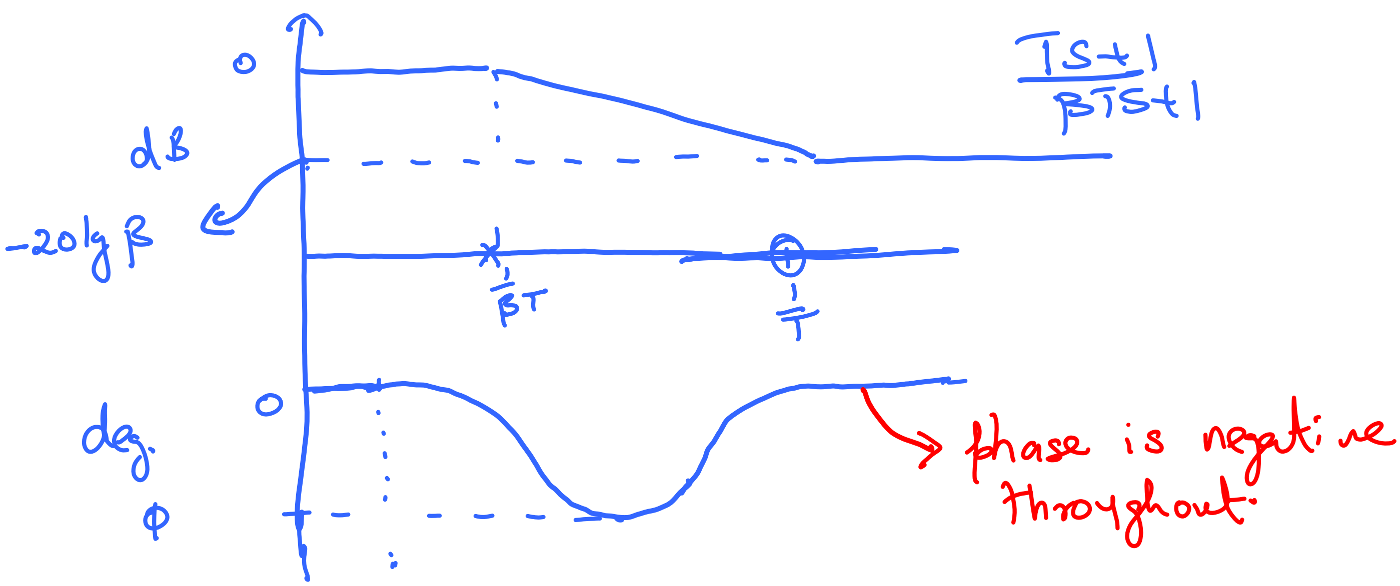
Frequency Domain Design (Lag Compensator)



- General form of a Lag Compensator:

$$G_c(s) = k\frac{Ts+1}{\beta Ts+1} \quad \beta > 1$$

- The main use of the lag compensator is to drag the O.L. magnitude down so as to provide sufficient phase margin
- Compare to the lead, which pushed the O.L. phase plot up to get correct phase margin
- Both share similar structure, but note that different order of poles of zeros. A lead controller acts similar to a PD controller, a lag controller acts similar to a PI controller



Lag Compensator Design (steps)

Step 1: Choose k to satisfy static error constants (K_v)

Step 2: Using this k, draw a Bode diagram of $G_1(s) = kG(s)$, and determine the required phase margin. Required PM = PM specified $+10^o$. Find the frequency ω_c where $\angle(G_1(j\omega_c))$ is equal to required PM. ω_c is the new gain cross over frequency.

Step 3: Choose the corner frequency of the zero

- ⋆ We want to change the magnitude plot without changing the phase plot at the new crossover frequency
- ⋆ Therefore, choose the zero at $1/T$ to be around 1 decade below the new corner frequency ω_c

Step 4: Determine β and the pole location...

- ⋆ We now examine $|G_1(j\omega_c)|$ to find out how much it is greater than 0 dB. This is equal to $20 \log \beta$ i.e.

$$0\ (dB) - |G_1(j\omega_c)|\ (dB) = -20 \log \beta$$

$$|G G_c(j\omega_c)| = 1$$

$$\Rightarrow \left| G(j\omega_c) k \frac{Tj\omega_c + 1}{\beta T j\omega_c + 1} \right| = 1$$

$$\Rightarrow |kG(j\omega_c)| \left| \frac{Tj\omega_c + 1}{\beta T j\omega_c + 1} \right| = 1$$

$$\Rightarrow 20\lg|G_1(j\omega_c)| + 20\lg\left|\frac{jT\omega_c + 1}{j\beta T\omega_c + 1}\right| = 0$$

Analysis

Friday, November 20, 2009
5:35 PM

$$\frac{1}{T} \lll \omega_c$$

$$\Rightarrow \frac{1}{\beta T} \ll \omega_c$$

$$\therefore \quad \frac{|jT\omega_c + 1|}{|j\beta T\omega_c + 1|}$$

$$= \frac{T}{\beta T} \frac{|j\omega_c + 1/T|}{|j\omega_c + 1/\beta T|}$$

$$\approx \frac{1}{\beta}$$

$$\therefore 20\lg |G_1(j\omega_c)| + 20\lg \frac{1}{\beta} = 0$$

$$\Rightarrow \quad 20\lg \beta = 20\lg |G_1(j\omega_c)|$$

—— x ——

Example

Friday, November 20, 2009
5:35 PM

Example

- For a given plant $G(s) = \frac{1}{s(s+1)(0.5s+1)}$ design a ~~lead~~ lag controller so that the resulting unity feedback closed loop system has $GM > 10$, $PM > 40^o$ and $K_v = 5$.

Design Steps:

Step 1: Choose k to satisfy static error constants (K_v)

⋆

$$K_v = 5 \Rightarrow \lim_{s\to 0} sG_c(s)G(s) = 0$$

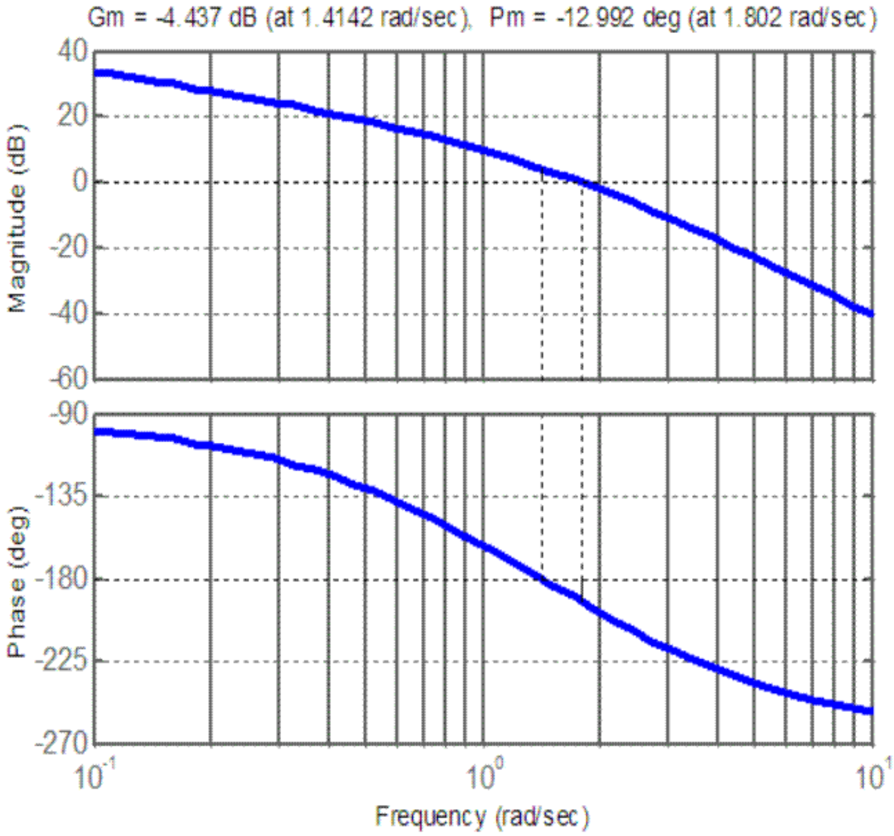
$$\Rightarrow \lim_{s\to 0} s\left(k\frac{Ts+1}{\beta Ts+1}\right)\left(\frac{1}{s(s+1)(0.5s+1)}\right) = 5 \Rightarrow k = 5$$

$$\Rightarrow k = 5$$

Step 2

Friday, November 20, 2009
5:36 PM

Step 2: Draw Bode diagram of $G_1(s) = kG(s)$



- Required PM $= 40^o + 10^o = 50^o$
- ω_c is that frequency where $\angle(G_1(j\omega_c) = PM - 180 = -130^o$. Therefore $\omega_c = 0.5\ rad/s$ (from the bode plot)

Steps 3 and 4

Friday, November 20, 2009
5:38 PM

Step 3: Choose the corner frequency of the zero

- ⋆ Choose the zero at $1/T$ to be around 1 decade below the new corner frequency ω_c; i.e. $\frac{1}{T} = 0.05$ which implies $T = 20$.

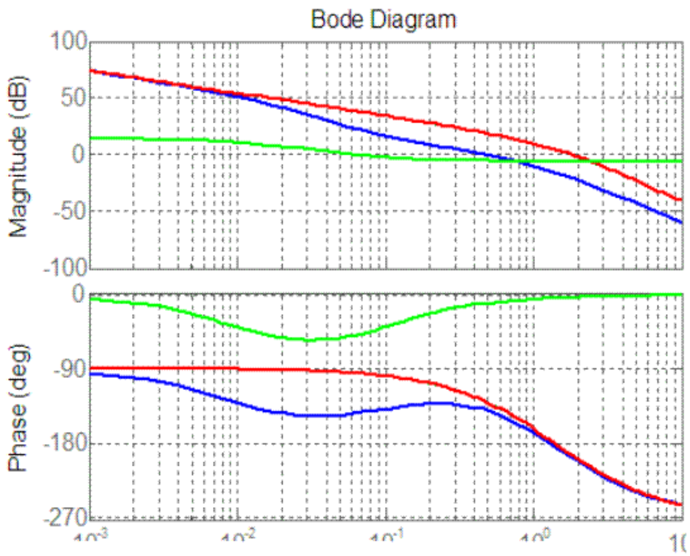
Step 4: Determine β

- ⋆ $|G_1(j\omega)| = 20\ dB$ at $\omega = \omega_c = 0.5\ rad/s$
  Therefore $20 \log \beta = 20 \Rightarrow \beta = 10$

$$G_c(s) = \frac{5(20s+1)}{200s+1}$$

Results

Friday, November 20, 2009
5:40 PM



- Frequency Response
- Gain Margin = 14.3dB
- Phase Margin = 42 deg
- Specifications met.
- Green = $G_c(s)$
- Red = $G_1(s)$
- Blue = $G_cG_1(s)$