Week 2 :

Note Title 9/11/2009

Solution of a first order scalar differential equation :

$$\frac{dx}{dt} = ax + bu;$$ where a is a constant

$x(t)$ is a scalar signal.

Then

$$e^{-at}\frac{dx}{dt} = e^{-at}ax + e^{-at}bu$$

$$\Rightarrow \quad e^{-at}\frac{dx}{dt} - e^{-at}ax = e^{-at}bu$$

Let

$$z = e^{-at}x$$

then
$$\frac{dz}{dt} = \frac{d}{dt}\left(e^{-at}x(t)\right)$$
$$= \left(\frac{d}{dt}e^{-at}\right)x(t) + e^{-at}\frac{dx(t)}{dt}.$$
$$= -e^{-at}a x(t) + e^{-at}\frac{dx(t)}{dt}$$
$$= e^{-at}bu$$

Aside:
Note
$$\frac{d}{dt}\left[\alpha(t)\beta(t)\right]$$
$$= \left(\frac{d\alpha(t)}{dt}\right)\beta(t) + \alpha(t)\frac{d\beta(t)}{dt}.$$

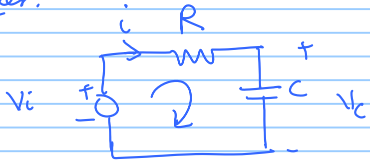
Thus,
$$\frac{dz}{dt} = e^{-at}bu$$
Integrating both sides we obtain
$$\int_{t_0}^{t}\frac{dz}{d\tau}d\tau = \int_{t_0}^{t}e^{-a\tau}bu(\tau)\,d\tau$$
$$\Rightarrow \quad z(t) - z(t_0) = \int_{t_0}^{t}e^{-a\tau}bu(\tau)\,d\tau$$
$$\Rightarrow \quad e^{-at}x(t) - e^{-at_0}x(t_0) = \int_{t_0}^{t}e^{-a\tau}bu(\tau)\,d\tau$$
$$\Rightarrow \quad x(t) - e^{a(t-t_0)}x(t_0) = e^{at}\int_{t_0}^{t}e^{-a\tau}bu(\tau)\,d\tau$$

$$\Rightarrow \quad x(t) = e^{a(t-t_0)} x(t_0) + e^{at} \int_{t_0}^{t} e^{-a\tau} b u(\tau) d\tau$$

$$\Rightarrow \quad \boxed{x(t) = e^{a(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{a(t-\tau)} b u(\tau) d\tau}$$

Example:

Consider:



apply KVL

$$-V_i + iR + V_c = 0$$

$$i = C \frac{dV_c}{dt}$$

$$\Rightarrow \quad RC \frac{dV_c}{dt} + V_c = V_i$$

$$\Rightarrow \quad \frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{V_i}{RC}$$

Let

$x = V_c$

$$\frac{dx}{dt} = -\frac{1}{RC}x + \frac{V_i}{RC} \; ; \quad a = -1/RC$$

Suppose initial charge on the capacitor is $V_c(0)$ -

Then we obtain

$$x(t) = e^{-\frac{1}{RC}(t-0)} V_c(0) + \frac{1}{RC}\int_0^t e^{-1/RC(t-\tau)} V_i(\tau)\, d\tau$$

so $$V_c(t) = e^{-\frac{1}{RC}t} V_c(0) + \frac{1}{RC}\int_0^t e^{-1/RC(t-\tau)} V_i(\tau)\, d\tau.$$

Suppose $V_i(\tau)$ is a step i.e.

$$V_i(\tau) = 1 \quad \text{if } \tau \geq 0$$
$$= 0 \quad \text{otherwise}$$

Then

$$V_c(t) = e^{-\frac{1}{RC}t} V_c(0) + \frac{1}{RC}\int_0^t e^{-1/RC(t-\tau)}\, d\tau$$

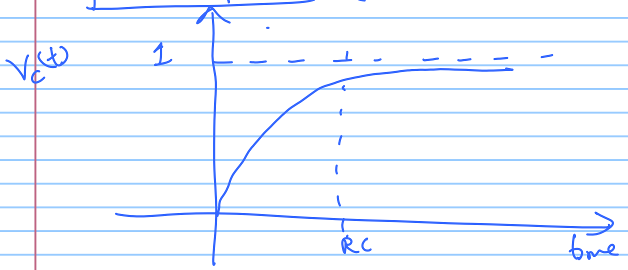
$$= e^{-\frac{1}{RC}t} V_c(0) + \frac{1}{RC} e^{-\frac{1}{RC}t} \int_0^t e^{\frac{1}{RC}\tau}\, d\tau$$

$$= e^{-\frac{1}{RC}t} V_c(0) + \frac{1}{RC} e^{-\frac{1}{RC}t} RC \left[e^{\frac{1}{RC}\tau} \right]_0^t$$

$$V_c(t) = e^{-\frac{1}{RC}t} V_c(0) + e^{-\frac{1}{RC}t}\left[e^{\frac{1}{RC}t} - 1\right]$$

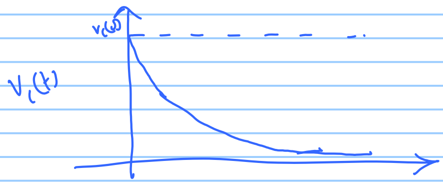
$$= \underbrace{\left(1 - e^{-1/RC\, t}\right)}_{\text{forced response}} + \underbrace{e^{-\frac{1}{RC}t} V_c(0)}_{\text{initial condition response.}}$$

<u>forced response</u> (assume $V_c(0) = 0$)



Time constant governed by RC

<u>Initial condition response:</u> (assume $V_i = 0$)



Impulse Response of a first order System

$\dot{x} = ax + bu$;

Suppose $u = \delta(t)$; $t \geq 0$
$= 0$ otherwise

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)} b u(\tau)\, d\tau$$

$$= e^{at}x(0) + \int_0^t e^{a(t-\tau)} b\, \delta(\tau)\, d\tau$$

Aside:

The impulse function is defined by its sampling property:

$$\int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\, d\tau = f(t).$$

$$= e^{at}x(0) + \int_0^t e^{a(t-\tau)} b\, \delta(\tau)\, d\tau$$

$$= e^{at}x(0) + \int_{-\infty}^{\infty} e^{a(t-\tau)} b\, \delta(\tau)\, d\tau$$

$$= e^{at}x(0) + e^{a(t-\tau)} b \Big|_{\tau=0}$$

$$= e^{at}x(0) + b e^{at}$$

Impulse response obtained by setting $x(0) = 0$ is given by

$$h(t) = b e^{at}.$$

→ Note that if $a > 0$ then $h(t) \to \infty$ as $t \to \infty$
if $a < 0$ then $h(t) \to 0$ as $t \to \infty$

Sinusoidal Response of a first order System:

Consider an input

$$u(t) = \sin\omega t \quad \text{if } t \geq 0$$
$$= 0 \quad \text{if } t < 0.$$

Then

$$x(t) = e^{at} x(0) + \int_0^t e^{a(t-\tau)} b\, u(\tau)\, d\tau$$

$$= e^{at} x(0) + e^{at} \int_0^t e^{-a\tau} b \sin(\omega\tau)\, d\tau$$

$$= e^{at} x(0) + e^{at} \int_0^t e^{-a\tau} b \left[\frac{e^{j\omega\tau} - e^{-j\omega\tau}}{2j}\right] d\tau$$

$$= e^{at} x(0) + e^{at} \int_0^t b \left[\frac{e^{(j\omega-a)\tau} - e^{-(a+j\omega)\tau}}{2j}\right] d\tau$$

$$= e^{at} x(0) + e^{at} b \left[\frac{1}{(j\omega-a)} e^{(j\omega-a)\tau} - \frac{1}{(j\omega+a)} e^{(j\omega+a)\tau}\right]_{\tau=0}^{t}$$

$$= e^{at} x(0) + e^{at} b \left[e^{-a\tau} \frac{(-a\sin\omega\tau - \omega\cos\omega\tau)}{\omega^2+a^2}\right]_{\tau=0}^{t}$$

$$= e^{at}x(0) - be^{at}\left[e^{-at}\frac{[a\sin\omega\tau + \omega\cos\omega\tau]}{a^2+\omega^2} - \frac{\omega}{\omega^2+a^2}\right]$$

$$= e^{at}x(0) - b\left[\frac{a}{a^2+\omega^2}\sin\omega t + \frac{\omega}{a^2+\omega^2}\cos\omega t\right] + be^{at}\frac{\omega}{\omega^2+a^2}$$

$$= e^{at}x(0) - \frac{b}{\sqrt{a^2+\omega^2}}\left[\sin\omega t\cos\theta + \cos\omega t\sin\theta\right] + be^{at}\frac{\omega}{\omega^2+a^2}$$

$$= e^{at}x(0) - \frac{b}{\sqrt{a^2+\omega^2}}\left[\sin(\omega t+\theta)\right] + be^{at}\frac{\omega}{\omega^2+a^2}$$

where $\cos\theta = \dfrac{a}{\sqrt{a^2+\omega^2}}$

$$= e^{at}\left[x(0) + \frac{b\omega}{\omega^2+a^2}\right] - \frac{b}{\sqrt{a^2+\omega^2}}\sin(\omega t+\theta)$$

If $a<0$ then

$$e^{at}\left[x(0) + \frac{b\omega}{\omega^2+a^2}\right] \longrightarrow 0 \text{ as } t\to\infty$$

and the steady state response is

$$- \frac{b \sin(\omega t + \theta)}{\sqrt{a^2+\omega^2}}.$$

→ Features: input is a sinusoid at frequency ω

Steady state output is also a sinusoid with frequency ω.

gain in amplitude = $\frac{b}{\sqrt{a^2+\omega^2}}$ is

phase = $\theta + \pi$; where $\theta = \cos^{-1}\left(\frac{a}{\sqrt{a^2+\omega^2}}\right)$

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State Space description

Consider Example 1 of a Spring-mass-damper System described by

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

and the RLC circuit of example 4

$$LC\frac{d^2V_c}{dt^2} + RC\frac{dV_c}{dt} + V_c = V_i$$

Both are Second degree ordinary differential equation.

A n^{th} degree ordinary scalar differential equation has the form

$$a_n\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = f(t) \quad \text{---(ODE)}$$

where f is the external input and $a_i;\ i = 0 \ldots n$ are constants.

Fact:

An nth order ordinary differential equation admits a unique solution if $f(t);\ t \geq 0$ is specified and

$$y(0),\ \left.\frac{dy}{dt}\right|_{t=0},\ \left.\frac{d^2y}{dt^2}\right|_{t=0},\ \dots,\ \left.\frac{d^{n-1}y}{dt^{n-1}}\right|_{t=0}$$

are all specified.

Consider a scalar different equation

$$\frac{dx}{dt} = ax \quad \text{----} \quad \text{[note the external forcing is } f(t)=0]$$

$$\Rightarrow \frac{dx}{x} = a\,dt$$

$$\Rightarrow \int_{x_0}^{x} \frac{dx}{x} = \int_0^t a\,dt$$

$$\Rightarrow \ln x \Big|_{x(0)}^{x(t)} = a(t-0)$$

$$\Rightarrow \ln x - \ln x(0) = at$$

$$\Rightarrow \ln \frac{x}{x(0)} = at$$

$$\Rightarrow x(t) = x(0)\, e^{at}$$ (the solution is completly determined if initial condition $x(0)$ is specified).

Converting a n^{th} degree Scalar differential equation to a first order vector differential equation

Consider the n^{th} degree ordinary differential equation

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_0 y = f(t)$$

where $y(t)$ is a scalar function of time; then define the following variables

$$x_1 \triangleq y$$

$$x_2 \triangleq \frac{dy}{dt}$$

$$x_3 \triangleq \frac{d^2 y}{dt^2}$$

$$\vdots$$

$$x_n \triangleq \frac{d^{n-1} y}{dt^{n-1}}$$

Then it follows that

$$\frac{dx_1}{dt} = \frac{dy}{dt} = x_2 = 0x_1 + 1x_2 + 0x_3 + \cdots + 0x_n$$

$$= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\frac{dx_2}{dt} = \frac{d^2y}{dt^2} = x_3 = 0x_1 + 0x_2 + 1x_3 + 0 \cdots + 0$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

$\vdots$

$$\frac{dx_{n-1}}{dt} = \frac{d}{dt}\left(\frac{d^{n-2}y}{dt^{n-2}}\right) = \frac{d^{n-1}y}{dt^{n-1}} = x_n$$

$$= 0x_1 + 0x_2 \cdots + 1x_n$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

and finally

$$\frac{dx_n}{dt} = \frac{d^ny}{dt^n} = f(t) - \frac{1}{a_n}\left[a_0 y + a_1\frac{dy}{dt} + a_2\frac{d^2y}{dt^2} + \cdots + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}}\right]$$

$$= -\frac{a_0}{a_n}x_1 - \frac{a_1}{a_n}x_2 - \cdots - \frac{a_{n-1}}{a_n}x_n + f(t)$$

$$= \begin{bmatrix} -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & -\frac{a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + f(t).$$

We will further use the notation

$$x \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

and

$$\frac{dx}{dt} \triangleq \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}$$

$$\therefore \quad \frac{dx}{dt} \triangleq \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & \cdots & \cdots & -\frac{a_{n-1}}{a_n} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{B} f(t)$$

$\therefore \quad \dot{x} = Ax + Bf.$

where x is a $(n \times 1)$ column vector.

<u>State-Space for Spd System:</u>

Consider the Spring-mass-damper System

$$m\ddot{z} + c\dot{z} + kz = f(t)$$

Let $x_1 = z$

$x_2 = \dot{z}$

Then, $\dot{x}_1 = x_2 = \dot{z} = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0f$

$$\dot{x}_2 = \ddot{z} = \frac{f(t) - kz - c\dot{z}}{m}$$

$$= -\frac{k}{m}x_1 - \frac{c}{m}x_2 + \frac{f(t)}{m}.$$

$$= \left(-\frac{k}{m} \quad -\frac{c}{m}\right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [1/m] f(t)$$

$$\therefore \quad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}_{B} f(t)$$

$$\therefore \quad \mathring{x} = Ax + Bf$$

with $\quad A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}$

$\qquad B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}.$

State Space for RLC Circuit:

$$LC\frac{d^2 v_c}{dt^2} + RC\frac{dV_c}{dt} + V_c = V_i$$

Let

$x_1 \triangleq V_c$

$x_2 \triangleq \mathring{V}_c$

Then
$$\mathring{x}_1 = \frac{dV_c}{dt} = x_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

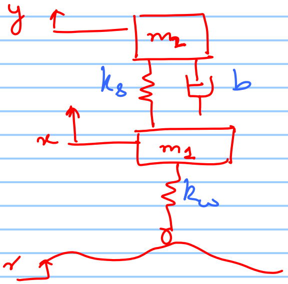
$$\mathring{x}_2 = \frac{d^2 V_c}{dt^2} = V_i - \frac{1}{LC}\left[V_c + RC\frac{dV_c}{dt}\right]$$

$$= V_i - \frac{1}{LC}V_c - \frac{R}{L}\mathring{V}_c$$

$$= -\frac{1}{LC}x_1 - \frac{R}{L}x_2 + 1 \cdot V_i$$

$\therefore$ $$\dot{x} = \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ 1\end{bmatrix} v_i$$

State Space representation of a Quarter Car model:



The equation of motion are

$$m_1 \frac{d^2x}{dt^2} + b\dot{x} - b\dot{y} + k_s x - k_s y + k_w x = k_w r$$

$$m_2 \frac{d^2y}{dt^2} + k_s(y-x) + b(\dot{y} - \dot{x}) = 0.$$

The initial conditions needed for a unique solution are

$$x(0);\ \dot{x}(0);\ y(0);\ \dot{y}(0)$$

Let

$$z_1 \triangleq x$$
$$z_2 \triangleq \dot{x}$$
$$z_3 \triangleq y$$
$$z_4 \triangleq \dot{y}$$

$$\Rightarrow \quad \dot{z}_1 = \dot{x} = z_2$$

$$\dot{z}_2 = \frac{\ddot{x}}{m_1} = \frac{k_w r}{m_1} - \frac{(k_s + k_w)}{m_1} x - \frac{b}{m_1}\dot{x} + \frac{k_s}{m_1} y + \frac{b}{m_1}\dot{y}$$

$$= -\frac{(k_s + k_w)}{m_1} z_1 - \frac{b}{m_1} z_2 + \frac{k_s}{m_1} z_3 + \frac{b}{m_1} z_4 + \frac{k_w r}{m_1}$$

$$\dot{z}_3 = \dot{y} = z_4$$

$$\dot{z}_4 = \ddot{y} = -\frac{k_s}{m_2}(y - x) - \frac{b}{m_2}(\dot{y} - \dot{x})$$

$$= +\frac{k}{m_2} z_1 + \frac{b}{m_2} z_2 - \frac{k_s}{m_2} z_3 - \frac{b}{m_2} z_4$$

$$\therefore\ \dot{z}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

$$\dot{z}_2 = \begin{bmatrix} \frac{-(k_s+k_w)}{m_1} & -\frac{b}{m_1} & \frac{k_s}{m_1} & \frac{b}{m_1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} + \begin{bmatrix} \frac{k}{m_1} \end{bmatrix} r$$

$$\dot{z}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

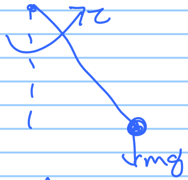
$$\dot{z}_4 = \begin{bmatrix} \frac{k}{m_2} & \frac{b}{m_2} & -\frac{k_s}{m_2} & -\frac{b}{m_2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

$$\therefore\ \dot{z} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{(k_s+k_w)}{m_1} & -\frac{b}{m_1} & \frac{k_s}{m_1} & \frac{b}{m_1} \\ 0 & 0 & 1 & 0 \\ \frac{k}{m_2} & \frac{b}{m_2} & -\frac{k_s}{m_2} & -\frac{b}{m_2} \end{bmatrix}}_{A} z + \underbrace{\begin{bmatrix} 0 \\ \frac{k}{m_1} \\ 0 \\ 0 \end{bmatrix}}_{B} r$$

$$\therefore \boxed{\dot{z} = Az + B\gamma}$$

with initial condition $z(0)$.

State Space of a pendulum:



The equation of motion is

$$\ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{\tau}{ml^2}.$$

Let

$$x_1 = \theta$$

$$x_2 = \dot{\theta}$$

then

$$\dot{x}_1 = \dot{\theta} = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 + \frac{\tau}{ml^2}$$

$\therefore \quad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{\ell} \sin x_1 + \tau / M\ell^2 \end{bmatrix}$

$$=: f(x, \tau).$$

Equilibrium points:

Consider a physical system that admits a state-space representation given by

$$\dot{x} = f(x,u) \; ; \quad (*)$$

$\bar{x}$ is a equilibrium point of $(*)$ if

$$f(\bar{x},0)=0$$

Example:

Consider the system

$$\dot{x} = Ax + Bu.$$

What are the equilibrium points of the system?

Note that

$$f(x,u) = Ax+Bu$$

$$\therefore \quad f(x,0) = Ax$$

$\therefore$ $\bar{x}$ is a equilibrium point if

$$f(\bar{x},0) = 0$$

$$\Rightarrow \quad Ax = 0$$

If A is an invertible matrix (it need not be)

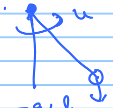
then

$$A^{-1}A\bar{x} = A^{-1}0 = 0$$

$\Rightarrow$ $\bar{x} = 0$ is the only equilibrium point.

Example

Consider the pendulum:



$$\ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{u}{ml^2}$$

$$\dot{x} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 + \frac{u}{ml^2} \end{bmatrix}$$

The equilibrium points are obtained by setting

$f(x, 0) = 0$ i.e.

Setting

$$\begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix} = 0$$

$\Rightarrow x_2 = 0, \ \sin x_1 = 0$

$\Rightarrow x_2 = 0, \ x_1 = n\pi$

The Equilibrium points are $\begin{bmatrix} 0 \\ n\pi \end{bmatrix}$, $n = 0, 1, \ldots$

The distinct equilibrium points are

$\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\bar{\bar{x}} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$

Back to the Scalar Case:

$\dot{x} = ax + bu$

$e^{-at}\dot{x} = e^{-at}ax + e^{-at}bu.$

$\Rightarrow \quad e^{-at}\dot{x} - e^{-at}ax = e^{-at}bu$

$z(t) \triangleq e^{-at}x(t)$

$\Rightarrow \dfrac{dz}{dt} = e^{-at}\dfrac{dx(t)}{dt} - e^{-at}ax$

$= e^{-at}\, b\, u(t)$

$\Rightarrow \quad z(t) - z(t_0) = \displaystyle\int_{t_0}^{t} e^{-a\tau}\, b\, u(\tau)\, d\tau$

$\Rightarrow \quad e^{-at}x(t) - e^{-at_0}x(t_0) = \displaystyle\int_{t_0}^{t} e^{-a\tau}\, b u(\tau)\, d\tau$

$\Rightarrow \quad x(t) = e^{a(t-t_0)}x(t_0) + \displaystyle\int_{t_0}^{t} e^{a(t-\tau)}\, b u(\tau)\, d\tau.$

What is e^{at}??

It is simply a symbol for the series

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$$

Note that

$$\frac{d}{dt}(e^{at}) = 0 + a + \frac{2a^2t}{2!} + \frac{3a^3t^2}{3!} + \cdots$$

$$\Rightarrow \frac{de^{at}}{dt} = a + a^2t + \frac{a^3t^2}{2!} + \cdots$$

$$= \left(1 + at + \frac{a^2t^2}{2!} + \cdots\right) a$$

$$= e^{at} a.$$

Also,

$$e^{-at} = 1 - at + \frac{a^2t^2}{2!} + \cdots$$

$$\Rightarrow \frac{de^{-at}}{dt} = -a + \frac{2a^2t}{2!} - \frac{3a^3t^2}{3!} + \cdots$$

$$= -a + a^2t - \frac{3a^3t^2}{3!} + \cdots$$

$$= \left[1 - at + \frac{a^2t^2}{2!} + \cdots\right](-a)$$

$$= -e^{-at} a.$$

Suppose we define for a square matrix

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$$

Then

$$\frac{d}{dt} e^{At} = 0 + A + \frac{2A^2t}{2!} + \frac{3A^3t^2}{3!} \cdots$$

$$= \left[I + At + \frac{A^2t^2}{2!} + \cdots\right](A).$$

Similarly

$$\dot{e}^{-At} = -e^{-At}A.$$

Then

$$\frac{dx}{dt} = Ax + Bu$$

$$\Rightarrow \quad e^{-At}\frac{dx}{dt} = e^{-At}Ax + e^{-At}Bu.$$

$$\Rightarrow \quad e^{-At}\frac{dx}{dt} - e^{-At}Ax = e^{-At}Bu$$

Let $\quad z(t) = e^{-At}x$

$$\Rightarrow \quad \frac{dz}{dt} = e^{-At}\left(\frac{dx}{dt}\right) + \frac{d}{dt}(e^{-At}).$$

$$= e^{-At}\frac{dx}{dt} - e^{-At}Ax$$

$$\Rightarrow \quad \frac{dz}{dt} = e^{-At}Bu$$

$$\Rightarrow \quad z(t) = \int_{t_0}^{t} e^{-A\tau}Bu(\tau)\,d\tau + z(t_0)$$

$$\Rightarrow \quad e^{-At}x(t) = e^{-At_0}x(t_0) + \int_{t_0}^{t} e^{-A\tau}Bu(\tau)\,d\tau$$

$$\Rightarrow \quad \boxed{x(t) = e^{A(t-t_0)} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau.}$$

Computing e^{At} for Simple Cases

Suppose the state-space description is given by

$$\dot{x}_1 = -1x_1 + u$$

$$\dot{x}_2 = -2x_2 + u$$

$$y = x_1 + x_2.$$

Therefore

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -t & 0 \\ 0 & -2t \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} t^2 & 0 \\ 0 & 4t^2 \end{bmatrix}$$

$$+ \frac{1}{3!}\begin{bmatrix} -t^3 & 0 \\ 0 & -8t^3 \end{bmatrix} + \cdots$$

$$= \begin{bmatrix} 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots & 0 \\ 0 & 1 - (2t) + \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + \cdots \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}$$

Suppose $u(t) = 1$ if $t \geq 0$

$= 0$ otherwise

and suppose $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Then

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-z)} B\, u(z)\, dz$$

$$= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-z)} & 0 \\ 0 & e^{-2(t-z)} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} dz$$

$$= \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-(t-z)} \\ e^{-2(t-z)} \end{bmatrix} dz$$

$$= \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix} + \begin{bmatrix} \int_0^t e^{-t} e^{z}\, dz \\ \int_0^t e^{-2t} e^{2z}\, dz \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} \\ 0 \end{bmatrix} + \begin{bmatrix} e^{-t} [e^{t} - 1] \\ e^{-2t} \frac{1}{2} [e^{2t} - 1] \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} + (1 - e^{-t}) \\ \frac{1}{2}\left[1 - e^{-2t}\right] \end{bmatrix}$$

$$\therefore \quad x_1(t) = 1$$

$$x_2(t) = \frac{1 - e^{-2t}}{2}.$$

—x—

Note that if $\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

then

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

Suppose

$A = P\Lambda P^{-1}$ where Λ is diagonal; then

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$$

However;

$$A^2 = AA = (P\Lambda P^{-1})P\Lambda P^{-1} =$$

$$= P\Lambda(P^{-1}P)\Lambda P^{-1}$$

$$= P\Lambda^2 P^{-1}$$

$$A^3 = A^2 \cdot A = (P\Lambda^2 P^{-1})(P\Lambda P^{-1})$$

$$= P\Lambda^3 P^{-1}$$

$$\therefore \quad A^m = P\Lambda^m P^{-1}$$

$$\therefore \quad e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$$

$$= PIP^{-1} + (P\Lambda P^{-1})t + \frac{(P\Lambda^2 P^{-1})t^2}{2!} + \cdots$$

$$= P\left[I + \Lambda t + \Lambda^2 t^2 + \cdots\right] P^{-1}$$

$$= P e^{\Lambda t} P^{-1}$$

$$= P \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} P^{-1} \quad \text{--- --- ---} \ (*)$$

Thus if $A = P \Lambda P^{-1}$ with Λ a diagonal matrix

e^{At} can be computed in a closed form as given by $(*)$ above