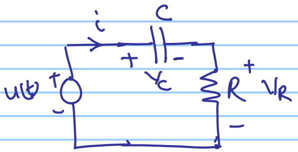
Note Title 9/23/2009

Systems

- A system processes a "time" signal to yield another "time" signal as output

Example:

Consider



$$i = C\frac{dv_c}{dt} \quad ; \quad v_R = iR$$

Also, $$v_c + v_R - u = 0$$

$$\Rightarrow \quad v_c + iR - u = 0$$

$$\Rightarrow \quad CR\frac{dv_c}{dt} + v_c = u(t)$$

$$\Rightarrow \quad \frac{dv_c}{dt} + \frac{1}{RC}v_c = \frac{u(t)}{RC}$$

Assume initial capacitor voltage is zero

i.e. $v_c(0) = 0$

Therefore the physics is described by

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{u(t)}{V_c} ; \quad V_c(0) = 0$$

where $u(t)$ is the input to the "System".

(*) Suppose current through the resistor is the signal of interest. Let the output be denoted by y. Thus

$$y(t) = i(t) = C \frac{dV_c(t)}{dt}$$

(*) Thus the input $u(t)$ gives the output according to the following rules

input

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c = \frac{u(t)}{RC} ; \quad V_c(0) = 0$$

output $\longrightarrow y(t) = \dfrac{C\,dV_c(t)}{dt}$

(*) Note that if the above rules are followed there is no "ambiguity" of what the signal

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$y(t)$ is going to be given $u(t)$ is specified.

The Impulse Response and Convolution:

⊛ Note that the impulse function has the property that

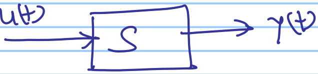
$$\int_{-\infty}^{\infty} f(\tau)\, \delta(t-\tau)\, d\tau = f(t)$$

which is called the "Sampling" property

Linearity of a System:

Suppose S is a system with input $u(t)$ and output $y(t)$ denoted by

$$u(t) = (Sy)(t).$$



S is said to linear if

$$S(u_1+u_2) = (Su_1) + (Su_2)$$

and

$$S(\alpha u) = \alpha\, S(u)$$ where α is a constant

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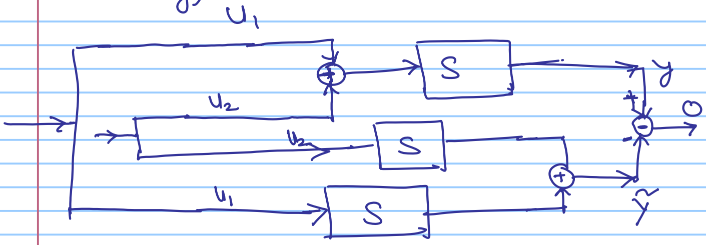
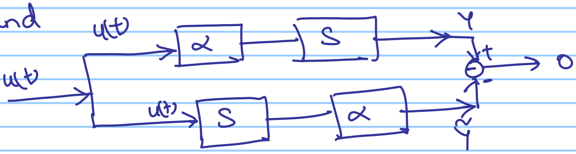
Pictorially,



$y = \tilde{y}$

and



$y = \tilde{y}$

Example:

System: $\left[\dfrac{dV_c}{dt} + \dfrac{1}{RC} V_c = \dfrac{u(t)}{RC} \; ; \; V_c(0) = 0 \right.$

and $\left. y(t) = C \dfrac{dV_c}{dt} \right]$.

- Suppose system is provided with input $u_1(t)$ and the corresponding output is $y_1(t)$ then

$$\frac{dV_{c,1}}{dt} + \frac{1}{RC} V_{c,1} = \frac{u_1(t)}{RC} \; ; \; V_{c,1}(0) = 0$$

$$y_1(t) = C \frac{dV_{c,1}}{dt}$$

- Suppose system is provided with input $u_2(t)$ and the corresponding output is $y_2(t)$ then

$$\frac{dV_{c,2}}{dt} + \frac{1}{RC} V_{c,2} = \frac{u_2(t)}{RC} \; ; \; V_{c,2}(0) = 0$$

$$y_2(t) = C \frac{dV_{c,2}}{dt}$$

→ Suppose the input is $u = (\alpha_1 u_1 + \alpha_2 u_2)$. Then the output $y(t)$ is given by

$$\left.\begin{array}{l}\frac{dV_c}{dt} + \frac{1}{RC} V_c = \alpha_1 u_1 + \alpha_2 u_2 \quad ; V_c(0) = 0 \\ \text{and } y(t) = C \frac{dV_c}{dt}\end{array}\right\} \text{---} \ (*)$$

we claim that $V_c(t) = \alpha_1 V_{c,1} + \alpha_2 V_{c,2}$

satisfies $(*)$. lets check this.

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c =$$

$$= \frac{d(\alpha_1 V_{c,1} + \alpha_2 V_{c,2})}{dt} + \frac{1}{RC}\left[V_{c,1} + V_{c,2}\right]$$

$$= \frac{d}{dt}(\alpha_1 V_{c,1}) + \frac{d}{dt}\alpha_2(V_{c,2}) + \frac{V_{c,1}}{RC} + \frac{V_{c,2}}{RC}$$

$$= \underbrace{\alpha_1 \frac{dV_{c,1}}{dt} + \frac{V_{c,1}}{RC}}_{\alpha_1 u_1} + \underbrace{\alpha_2 \frac{dV_{c,2}}{dt} + \frac{V_{c,2}}{RC}}_{\alpha_2 u_2}$$

$$= \alpha_1 u_1 + \alpha_2 u_2.$$

Also, $V_c(0) = (\alpha_1 V_{c,1} + \alpha_2 V_{c,2})(0)$

$$= \alpha_1 V_{c,1}(0) + \alpha_2 V_{c,2}(0)$$

$$= 0 + 0 = 0$$

$\therefore$ With $V_c = \alpha_1 V_{c,1} + \alpha_2 V_{c,2}$

$V_c(0) = 0$ and

$$\frac{dV_c}{dt} + \frac{1}{RC} V_c(t) = \alpha_1 u_1 + \alpha_2 u_2$$

There the output of the system with input $\alpha_1 u_1 + \alpha_2 u_2$ is

$$y(t) = C\frac{dV_c}{dt} = C\frac{d}{dt}(\alpha_1 V_{c,1} + \alpha_2 V_{c,2})$$

$$= \alpha_1 C\frac{d}{dt} V_{c,1} + \alpha_2 \frac{d}{dt} V_{c,2}$$

$$= \alpha_1 y_1 + \alpha_2 y_2$$

$$\therefore \underbrace{S(\alpha_1 u_1 + \alpha_2 u_2)}_{y(t)} = \alpha_1 \underbrace{(S u_1)}_{y_1} + \alpha_2 \underbrace{(S u_2)}_{y_2}.$$

This proves the linearity of the system S described by ordinary differential equations.

Time-Invariance:

Definition: The Shift operator:

$$(Sh_\tau u)(t) \triangleq u(t-\tau)$$

$$u(t) \longrightarrow \boxed{Sh_\tau} \longrightarrow u(t-\tau).$$

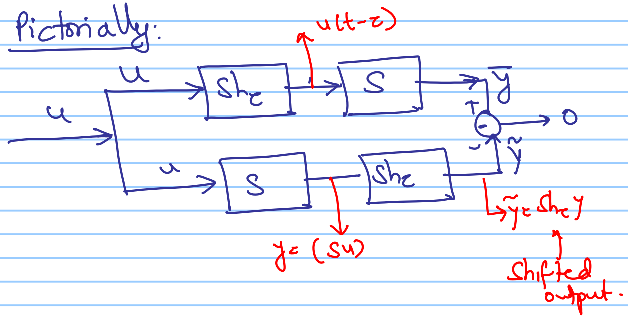
(*) Verify that the Shift operator is Linear

Let S be a system. If

$$S \cdot Sh_\tau = Sh_\tau \cdot S \qquad \text{then}$$

S is said to be time-invariant.

Pictorially:



Example

Back to the RC Circuit

- Suppose the input is $u(t)$ and the corresponding output is $y(t)$. Then let $v_c(t)$ be such that

$$\frac{dv_c}{dt} + \frac{1}{RC} v_c = \frac{u}{RC} \quad ; v_c(0) = 0$$

$$y(t) = C\frac{dv_c}{dt}.$$

- Suppose the input is $(Sh_\tau u)(t) = u(t-\tau)$. τ is a constant.

Consider $\tilde{v}_c(t) = v_c(t-\tau)$.

Note that

$$\frac{d\tilde{v}_c(t)}{dt} = \frac{dv_c(t-\tau)}{dt}$$

$$= \frac{dv_c(\sigma)}{d\sigma} \quad ; \text{ let } \sigma \text{ be } \sigma = t-\tau.$$

$$\therefore \frac{d\tilde{v}_c(t)}{dt} + \frac{1}{RC}\tilde{v}_c(t) = \frac{dv_c(\sigma)}{d\sigma} + \frac{1}{RC} v_c(t-\tau)$$

$$= \dot{v}_c(\sigma) + \frac{1}{RC} v_c(\sigma)$$

$$= u(\sigma)$$

$$= u(t-\tau)$$

$$\therefore \frac{d\tilde{v}_c(t)}{dt} + \frac{1}{RC}\tilde{v}_c(t) = u(t-\tau);$$

Also, $\tilde{y}(t) = C\frac{d\tilde{v}_c(t)}{dt}$

$$= C\frac{dv_c(t-\tau)}{dt}$$

$$= C\frac{dv_c(\sigma)}{d\sigma}$$

$$= y(\sigma)$$

$$= y(t-\tau).$$

Therefore

$$u(t) \xrightarrow{S} y(t)$$

$$u(t-\tau) \xrightarrow{S} y(t-\tau).$$

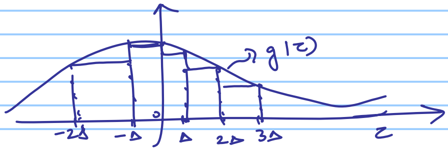
(Shift the input by τ) results in an output that is shifted by τ.

Linearity and time invariance play a very important role in describing a number of systems.

Essentially for linear and time invariant systems determining the output due to one special input suffices to determine the output of any other general input. The special input is the "impulse" input

The Impulse Function:

$$f(t) = \int_{-\infty}^{\infty} g(\tau)\, d\tau$$



$$\int_{-\infty}^{\infty} g(\tau)\, d\tau = g(0)\cdot\Delta + g(-\Delta)\Delta + g(\Delta)\Delta + g(-2\Delta)\Delta + g(2\Delta)\Delta + \cdots$$

$$\cong \sum_{n=-\infty}^{\infty} g(n\Delta)\,\Delta.$$

with $g(\tau) = f(\tau)\,\delta(t-\tau)$

$$\int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau) \cong \sum_{n=-\infty}^{\infty} f(n\Delta)\,\delta(t-n\Delta)\,\Delta$$

Thus any function 'u' can be written as a summation of scaled and shifted impulses $\delta(t-n\Delta)$ i.e.

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\,\delta(t-\tau)\,d\tau = \sum_{n=-\infty}^{\infty} u(n\Delta)\,\delta(t-n\Delta)\,\Delta$$

Impulse Response of a System.

Suppose the output of a linear time invariant system S for an impulse input at time $t=0$ is $h(t)$

$$\delta(t) \longrightarrow \boxed{\ S\ } \longrightarrow h(t)$$

Then consider an arbitrary input $u(t)$

$$u(t) \longrightarrow \boxed{\ S\ } \longrightarrow y(t)$$

Note that

$$u(t) = \sum_{n=-\infty}^{\infty} u(n\Delta)\,\delta(t-n\Delta)\,\Delta.$$

$$\therefore\ (Su)(t) = S\left[\sum_{n=-\infty}^{\infty} u(n\Delta)\,\Delta\,\delta(t-n\Delta)\right]$$

Linearity $\longleftarrow$ $$= \sum_{n=-\infty}^{\infty}\left[u(n\Delta)\,\Delta\, S\,\delta(t-n\Delta)\right]$$

Linearity $\longleftarrow$ $$= \sum_{n=-\infty}^{\infty}\left[u(n\Delta)\,\Delta\, S\cdot Sh_{n\Delta}\,\delta(t)\right]$$

time-invariance $\longleftarrow$ $$= \sum_{n=-\infty}^{\infty}\left[u(n\Delta)\,\Delta\, Sh_{n\Delta}(S\delta)(t)\right]$$

Impulse response is $h(t)$

$$= \sum_{n=-\infty}^{\infty} u(n\Delta)\,\Delta\, S h_{n\Delta}^{h(t)}$$

$$= \sum_{n=-\infty}^{\infty} u(n\Delta)\,\Delta\, h(t-n\Delta).$$

$$\cong \int_{-\infty}^{\infty} h(t-\tau)\,u(\tau)\,d\tau.$$

$\therefore$ The output due to an arbitrary input $u(t)$ is

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\,u(\tau)\,d\tau$$

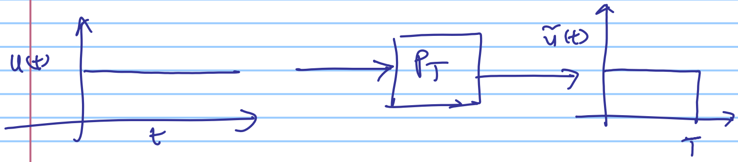
where $h(t) = (S\delta)(t)$ is the impulse response of the time-invariant system S.

Causality:

① **Definition:** (Truncation operator)

P_T is a system that has the following property on signals

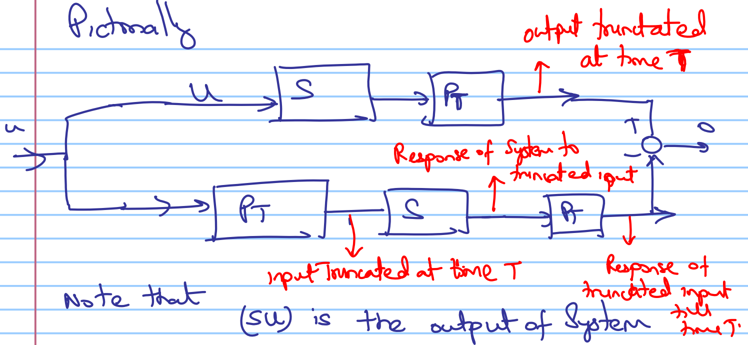
$$\tilde{u}(t) \stackrel{\circ}{=} (P_T u)(t) = u(t) \quad \text{if } t \le T$$
$$= 0 \quad \text{if } t > T$$



① **Definition:** A system S is said to be causal if the future of an input does not affect the present or the past of the output. Mathematically this notion is captured by

$$P_T(Su) = P_T(S(P_T u))$$
for all T

Pictorially



Note that

(Su) is the output of System S with input u

(a) $P_T(Su)$ is the output of System till time $t \le T$ and is zero otherwise

(b) $P_T S P_T u$ is the output of the System till time T when the input is truncated at time T.

As (a) and (b) are the same the following can be concluded:

(*) The output of the System till time T is completely determined by the input till time T

→ i.e. the future has no effect on the past ⇒ Causality!

Example of a Causal System:

$$u(t) \longrightarrow \boxed{S} \longrightarrow y(t)$$

Suppose System S is described by

$$y(t) = (Su)(t) = \int_0^t u(\tau)\,d\tau.$$

Then

$$(P_T(Su))(t) = P_T\,y(t) = \int_0^t u(\tau)\,d\tau \quad \text{if } t \le T$$
$$= 0 \quad \text{otherwise.}$$

$$(P_T S P_T u)(t) = P_T\left[\int_0^t (P_T u)(\tau)\,d\tau\right]$$
$$= \int_0^t (P_T u)(\tau)\,d\tau \quad \text{if } t \le T$$
$$= 0 \quad \text{otherwise}$$

$$\therefore\ (P_T S P_T u)(t) = \int_0^t u(\tau)\,d\tau \quad \text{if } t \le T$$
$$= 0 \quad \text{otherwise}$$

$$(P_T S u)(t) = (P_T S P_T u)(t),$$

$\therefore$ System is Causal.

——— x ———

We know that for a linear time-invariant system

$$y(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau.$$

where $h(t)$ is the response of the system to an impulse at time $t=0$.

Note also that a linear system has 0 as the output when the input is 0:

$$\left[\quad S(\alpha u) = \alpha S u \right.$$

$$\left. \therefore S(0 \cdot u) = 0 \cdot S u = 0 . \right]$$

The impulse at time $t=0$ is zero before time $t=0$.

Therefore if the system is causal then

$$h(t) = 0 \quad \text{if } t < 0.$$

$$\therefore \quad y(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

$$\boxed{y(t) = \int_{-\infty}^{t} u(\tau)\, h(t-\tau)\, d\tau}$$

If we take the $t=0$ as the time origin then

$$\boxed{y(t) = \int_{0}^{t} u(\tau)\, h(t-\tau)\, d\tau.}$$

Laplace Transform

Consider a signal $r(t)$ that is defined for $0 \leq t < \infty$.

The one-sided Laplace Transform of $r(t)$ is defined by

$$R(s) = \int_0^\infty r(t)\, e^{-st}\, dt$$

for every $s \in \mathbb{C}$

* $\mathbb{C}$ denotes the complex plane.

Convolution in Time Domain Equivalent to multiplication in Laplace domain:

Suppose

$$y(t) = \int_0^\infty u(\tau)\, h(t-\tau)\, d\tau.$$

$$=: (u*h)(t) \quad \text{where } h(t)=0 \text{ if } t<0$$

Then

$$Y(s) = \int_0^\infty y(t)\, e^{-st}\, dt$$

$$= \int_0^\infty \int_0^\infty u(\tau)\, h(t-\tau)\, e^{-st}\, d\tau\, dt$$

$$= \int_0^\infty \int_0^\infty u(\tau)\, h(t-\tau)\, e^{-st}\, dt\, d\tau$$

$$= \int_0^\infty u(\tau) \left(\int_0^\infty h(t-\tau)\, e^{-st}\, dt \right) d\tau.$$

$t-\tau = \tau'$

$$= \int_0^\infty u(\tau) \int_0^\infty h(\tau')\, e^{-(\tau'+\tau)s}\, d\tau'\, d\tau$$

$$= \int_0^\infty u(\tau) \int_{-\tau}^\infty h(\tau')\, e^{-\tau' s}\, e^{-\tau s}\, d\tau'\, d\tau$$

$$= \left(\int_0^\infty u(\tau) e^{-s\tau} d\tau \right) \left(\int_{-\tau}^\infty h(\tau') e^{-\tau' s} d\tau' \right)$$

$$= \left(\int_0^\infty u(\tau) e^{-s\tau} d\tau\right) \left(\int_0^\infty h(\tau') e^{-\tau' s} d\tau'\right)$$

$$= U(s) \quad H(s).$$

where

$$U(s) = \mathcal{L}\, u(t) = \int_0^\infty u(\tau) e^{-s\tau} d\tau$$

$$H(s) = \mathcal{L}\, h(t) = \int_0^\infty h(\tau) e^{-s\tau} d\tau$$

$$\therefore \quad (h * u)(t) \rightleftarrows H(s)\, U(s).$$

Laplace transforms:

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt.$$

⊙ $\mathcal{L} f(t) = F(s)$

Example: $f(t) = e^t$

$$F(s) = \int_0^\infty e^t e^{-st}\, dt$$

$$= \int_0^\infty e^{(1-s)t} dt$$

$$= \frac{1}{1-s} e^{(1-s)t} \Big|_0^\infty$$

$$= \lim_{t \to \infty} \left[\frac{1}{1-s} \cancel{e^{(1-s)t}} \right] - \frac{1}{1-s} \cdot 1.$$

$$\boxed{F(s) = \frac{1}{s-1}}$$

with the condition that

$$e^{(1-s)t} \to 0 \quad \text{as} \quad t \to \infty$$

$$e^{Re(1-s)t + j\, Img(1-s)t} \to 0 \quad \text{as} \quad t \to \infty$$

This is true if $\underline{\underline{Re(s) > 1}}$.

⊛ However the formula $\frac{1}{s-1}$ is well defined for all $s \in \mathbb{C}$; except $s = 1$.

Example: (unit Step)

$$f(t) = 1 \quad \text{if } t \geq 0$$
$$= 0 \quad \text{otherwise}$$

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$
$$= \int_0^\infty e^{-st}\, dt$$
$$= -\frac{1}{s} e^{-st} \Big|_0^\infty$$
$$= -1\left[\lim_{t \to \infty} \frac{e^{-st}}{s} - \frac{1}{s}\right].$$
$$= \frac{1}{s} \quad \text{if} \quad \lim_{t \to \infty} e^{-st} = 0.$$

$Re(s) > 0$

① The formula $1/s$ holds for all values of $s \in \mathbb{C}$ except $s = 0$.

Laplace transform of a Sinusoid:

$$f(t) = \cos \omega t$$

$$= \frac{1}{2} e^{j\omega t} + \frac{1}{2} e^{-j\omega t}$$

$$F(s) = \frac{1}{2} \int_0^{\infty} e^{-st} \left[e^{j\omega t} + e^{-j\omega t} \right] dt$$

$$= \frac{1}{2} \left[\int_0^{\infty} e^{(-s+j\omega)t} dt + \int_0^{\infty} e^{(-s-j\omega)t} dt \right]$$

$$= \frac{1}{2} \frac{1}{s-j\omega} + \frac{1}{2} \frac{1}{s+j\omega} = \frac{s}{s^2+\omega^2}$$

if $Re(s) > 0$

(a) The formula $\frac{s}{s^2+\omega^2}$ is well defined for all $s \neq \pm j\omega$ in the complex plane.

power of t : t^n ; $n \geq 1$.

Integration by parts

$$\int_a^b u v' \, dt = uv \Big|_a^b - \int_a^b u' v \, dt$$

$$F(s) = \int_0^\infty e^{-st} t^n \, dt$$

$$= t^n \frac{e^{-st}}{-s} \Big|_0^\infty - \int_0^\infty n t^{n-1} \frac{e^{-st}}{-s} \, dt$$

$$= + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} \, dt$$

$$= \frac{n}{s} \mathcal{L} \, t^{n-1}$$

$$F(s) = \frac{n!}{s^{n+1}}$$

* holds true if $Re(s) > 0$
* formula holds true if $s \neq 0$.

Impulse:

$$F(s) = \int_0^\infty \delta(t)\, e^{-st}\, dt$$

$$= 1.$$

Properties of Laplace Transforms:

Linearity:

$$\mathcal{L}(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \mathcal{L} f_1 + \alpha_2 \mathcal{L} f_2$$

Check this!

One to One property:

$$\mathcal{L}(f_1) = \mathcal{L} f_2$$

then $f_1 = f_2$.

Example: $F(s) = \dfrac{3s-5}{s-1}$

$$= \frac{3(s-1)+3-5}{s-1}$$

$$= 3 - \frac{2}{s-1}$$

$$f(t) = \mathcal{L}^{-1} F(s) = \mathcal{L}^{-1} 3 - 2 \mathcal{L} \frac{1}{s-1}$$

$$= 3 \mathcal{L}^{-1} 1 - 2 \mathcal{L} \frac{1}{s-1}$$

$$= 3\,\delta(t) - 2e^{t}$$

Inverse Laplace transform:

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s)\, e^{st}\, ds$$

where σ is large enough that $F(s)$ is defined for $\mathrm{Re}(s) \geq \sigma$.

→ We will "never" use this formula.

Exponential Scaling:

Let f be a signal and α be a real constant

Let

$$\mathcal{L} f(t) = F(s)$$

then

$$\mathcal{L}\, e^{at} f(t) = F(s-a)$$

Proof:

$$\mathcal{L}\, e^{at} f(t) = \int_0^\infty e^{at} f(t)\, e^{-st}\, dt$$

$$= \int_0^\infty f(t)\, e^{-(s-a)}\, dt$$

$$= F(s-a)$$

Example:

We know that $\mathcal{L} \cos t = \dfrac{s}{s^2+\omega^2}$ $\triangle$

$\therefore$

$$\mathcal{L}\, e^{-t} \cos t = \frac{s+1}{(s+1)^2+\omega^2}.$$

Time delay:

f be a signal and $T > 0$; Let

$$g(t) = \begin{cases} 0 & 0 \le t \le T \\ f(t-T) & \text{if } t \geq T \end{cases}$$

$$\mathcal{L}\, g(t) = \int_0^\infty e^{-st} g(t)\, dt$$

$$= \int_T^\infty e^{-st} f(t-T)\, dt$$

$$= \int_0^\infty e^{-s(\tau+T)} f(\tau)\, d\tau \qquad t-T=\tau$$

$$= e^{-sT} \int_0^\infty e^{-s\tau} f(\tau)\, d\tau$$

$$= e^{-sT} F(s).$$

where $F(s) = \mathcal{L}\, f(t)$.

Derivative:

Let $g(t) = f'(t) = \frac{d}{dt} f(t)$.

$$\mathcal{L}\, g(t) = \int_0^\infty f'(t)\, e^{-st}\, dt$$

$$= e^{-st} f(t) \Big|_0^\infty - \int_0^\infty f(t)(-s)\, e^{-st}\, dt$$

$$= \lim_{t\to\infty} e^{-st} f(t) - f(0) + s\int_0^\infty f(t) e^{-st}\, dt$$

$$= s\,F(s) - f(0).$$

Note that

$$\mathcal{L}\frac{d^2 f}{dt^2} = s\,\mathcal{L}\frac{df}{dt} - \frac{df(0)}{dt}$$

$$= s\,[s\,F(s) - f(0)] - f'(0)$$

$$= s^2 F(s) - s f(0) - f'(0)$$

Examples:

1. Let $f(t) = e^t$; $F(s) = \frac{1}{s-1}$

$f'(t) = e^t$

$\mathcal{L} f'(t) = sF(s) - f(0)$

$= \frac{s}{s-1} - 1$

$= \frac{s-(s-1)}{s-1} = \frac{1}{s-1} = F(s).$

2. $\sin\omega t = -\frac{1}{\omega}\frac{d}{dt}\cos\omega t$

We know $\mathcal{L}\cos\omega t = \frac{s}{s^2+\omega^2}$

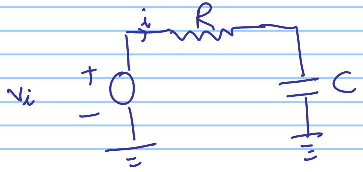
$\Rightarrow \mathcal{L}\sin\omega t = -\frac{1}{\omega}\mathcal{L}\frac{d}{dt}(\cos\omega t)$

$= -\frac{1}{\omega}\left[s\frac{s}{s^2+\omega^2} - 1\right]$

$= -\frac{1}{\omega}\left[\frac{s^2 - s^2 - \omega^2}{s^2+\omega^2}\right]$

$= \frac{\omega}{s^2+\omega^2}$

Example: Step Response of a RC Circuit:



Let $R = 1$; $C = 1$.

Then
$$\frac{dv_c}{dt} + v_c = v_i$$

Taking Laplace transforms we get

$$sV_c(s) + V_c(s) - V_c(0) = V_i(s).$$

$$\Rightarrow (s+1)\, V_c(s) = V_c(0) + V_i(s)$$

$$\Rightarrow \quad V_c(s) = \frac{V_c(0)}{s+1} + \frac{V_i(s)}{s+1}$$

If $v_i(t)$ is a step then $V_i(s) = \mathcal{L}\, v_i(t) = 1/s$

$$\therefore \quad V_c(s) = \frac{V_c(0)}{s+1} + \frac{1}{s(s+1)}$$

$\therefore$ $$V_c(s) = V_c(0)\frac{1}{s+1} + \frac{1}{s(s+1)}$$

$$= V_c(0)\frac{1}{s+1} + \frac{1}{s} - \frac{1}{s+1}$$

So

$$v_c(t) = V_c(0)\mathcal{L}^{-1}\frac{1}{s+1} + \mathcal{L}^{-1}\frac{1}{s} - \mathcal{L}^{-1}\frac{1}{s+1}$$

$$= V_c(0)e^{-t} + 1 - e^{-t}$$

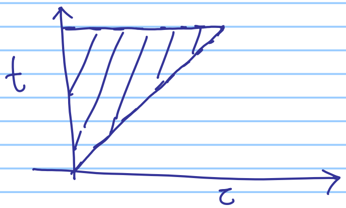
Same as we obtained solving a differential equation directly.

Laplace of a integral:

Let
$$y(t) = \int_0^t f(\tau)\, d\tau$$

Let
$$\mathcal{L} f(t) = F(s).$$

then
$$\mathcal{L} y(t) = \int_0^\infty y(t) e^{-st} dt$$
$$= \int_0^\infty \left(\int_0^t f(\tau)\, d\tau\right) e^{-st} dt$$
$$= \int_{t=0}^\infty \int_{\tau=0}^t f(\tau)\, e^{-st}\, d\tau\, dt$$



Switching integration order

$$\mathcal{L}\{y(t)\} = \int_{\tau=0}^{\infty}\int_{t=\tau}^{\infty} f(\tau)\, e^{-st}\, dt\, d\tau$$

$$= \int_{\tau=0}^{\infty} f(\tau) \left[\left(\int_{t=\tau}^{\infty} e^{-st}\, dt \right) \right] d\tau$$

$$= \int_{\tau=0}^{\infty} f(\tau) \left[-\frac{1}{s} e^{-st} \Big|_{t=\tau}^{\infty} \right] d\tau$$

$$= \int_{\tau=0}^{\infty} f(\tau) \left(-\frac{1}{s}\right)\left(0 - e^{-s\tau}\right) \Big]\, d\tau$$

$$= \frac{1}{s} \int_{\tau=0}^{\infty} f(\tau)\, e^{-s\tau}\, d\tau$$

$$= \frac{1}{s} F(s)$$

Multiplication by t.

Suppose $\mathcal{L} f(t) = F(s)$

then

$$\mathcal{L}\, t f(t) = -\frac{dF(s)}{ds}.$$

Indeed

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt$$

and therefore

$$\frac{dF(s)}{ds} = \frac{d}{ds} \int_0^\infty f(t) e^{-st}\, dt$$

$$= \int_0^\infty f(t)(-t) e^{-st}\, dt$$

$$\Rightarrow \frac{dF(s)}{ds} = -\int_0^\infty t f(t) e^{-st}\, dt$$

$$\Rightarrow \mathcal{L}\, t f(t) = -\frac{dF(s)}{ds}$$

Examples :

(1) $f(t) = e^{-t}$

then $F(s) = \dfrac{1}{s+1}$.

$$\Rightarrow \quad \mathcal{L}\, t f(t) = -\frac{d}{ds}\left(\frac{1}{s+1}\right)$$

$$= \frac{1}{(s+1)^2}$$

(x) Let $f(t) = te^{-t}$ and $F(s) = 1/(s+1)^2$

then $\mathcal{L}\, t^2 e^{-t} = \mathcal{L}\, t f(t) = -\dfrac{d}{ds}\left(\dfrac{1}{s+1}\right)^2 = \dfrac{2}{(s+1)^3}$

(x) In general $\mathcal{L}\, t^n e^{-t} = \dfrac{(n-1)!}{(s+1)^{n+1}}$

The Final Value theorem:

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

Proof:

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

and we have

$$\int_0^\infty f'(t)e^{-st}dt = e^{-st}f(t)\Big|_0^\infty - \int f(t)(-s)e^{-st}dt$$

$$= sF(s) - f(0)$$

Now $\displaystyle\lim_{s\to 0}\int^\infty f'(t)e^{-st}dt = \lim_{s\to 0}[sF(s) - f(0)]$

$$\Rightarrow \int_0^\infty \left(\lim_{s\to 0} f'(t)e^{-st}\right)dt = \lim_{s\to 0} sF(s) - f(0)$$

$$\Rightarrow \Rightarrow \int_0^\infty f'(t)dt = \lim_{s\to 0} sF(s) - f(0)$$

$$\Rightarrow \lim_{t\to\infty} f(t) - f(0) = \lim_{s\to 0} sF(s) - f(0)$$

$$\Rightarrow \boxed{\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)}$$

Initial Value theorem:

Let $\mathcal{L}f(t) = F(s)$

then $f(0) = \lim_{s \to \infty} s\,X(s)$

Proof:

Note that

$$\int_0^\infty f'(t)\,e^{-st}\,dt = sF(s) - f(0)$$

$$\Rightarrow \lim_{s \to \infty} \int_0^\infty f'(t)\,e^{-st}\,dt = \lim_{s \to \infty} sF(s) - f(0)$$

$$\Rightarrow \int_0^\infty f'(t) \lim_{s \to \infty} e^{-st}\,dt = \lim_{s \to \infty} sF(s) - f(0)$$

$$\Rightarrow 0 = \lim_{s \to \infty} sF(s) - f(0)$$

$$\Rightarrow \boxed{f(0) = \lim_{s \to \infty} sF(s)}$$