Note Title 9/13/2008

Partial Fraction Expansions

⊛ Simple Roots

Consider

$$G(s) = \frac{3s+5}{2s^2-5s-3}.$$

Consider the denominator $2s^2-5s-3$. The roots are given by

$as^2+bs+c=0$ has roots $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$s_1 = \frac{+5+\sqrt{25-4(2)(-3)}}{2(2)} = \frac{5+\sqrt{25+24}}{4} = \frac{5+7}{4} = 3$$

$$s_2 = \frac{+5-\sqrt{25-4(2)(-3)}}{4} = \frac{5-\sqrt{49}}{4} = \frac{5-7}{4} = \frac{-2}{4} = -1/2$$

$$\therefore 2s^2-5s-3 = 2(s-s_1)(s-s_2)$$
$$= 2(s-3)(s+\tfrac{1}{2})$$
$$= (s-3)(2s+1)$$

$$\therefore \frac{3s+5}{2s^2-5s-3} = \frac{3s+5}{2(s-3)(s+1/2)}$$

Let

$$G(s) = \frac{3s+5}{2(s-3)(s+1/2)} = \frac{A}{s-3} + \frac{B}{s+\frac{1}{2}}$$

Note that

$$G(s)(s-3) = A + \frac{B}{(s+\frac{1}{2})}(s-3)$$

$$\Rightarrow \quad \frac{3s+5}{2(s+\frac{1}{2})} = A + \frac{B(s-3)}{s+1/2}$$

$$\Rightarrow \quad \left.\frac{3s+5}{2(s+\frac{1}{2})}\right|_{s=3} = A$$

$$\Rightarrow A = \frac{3\cdot 3+5}{2(3\cdot 5)} = \frac{9+5}{7} = \frac{14}{7} = 2$$

<u>Similarly</u>

$$G(s)(s+1/2)\Big|_{s=-1/2} = B$$

$$\Rightarrow \quad \left.\frac{3s+5}{2(s-3)}\right|_{s=-1/2} = B$$

$$\Rightarrow B = \frac{3(-\frac{1}{2})+5}{2(-\frac{1}{2}-3)}$$

$$\Rightarrow \quad B = \frac{-1\cdot 5+5}{2(-3\cdot 5)} = \frac{+3\cdot 5}{2(-3\cdot 5)} = \frac{-1}{2}$$

$$\therefore \quad G(s) = \frac{2}{s-3} - \frac{1}{2(s+1/2)}$$

$$= \frac{2}{s-3} - \frac{1}{2(s+1/2)}$$

Note that $e^{3t} \rightleftarrows \frac{2}{s-3}$

$$e^{-\frac{1}{2}t} \rightleftarrows \frac{1}{s+1/2}$$

$$\therefore \quad \mathcal{L}\left(\frac{2}{s-3} - \frac{1}{2(s+1/2)}\right) = 2e^{3t} - \frac{1}{2}e^{-\frac{1}{2}t}$$

In general if

$$G(s) = \frac{a_m s^m + b_{m-1} s^{m-1} + \dots + a_0}{(s-s_1)(s-s_2)\dots(s-s_n)} \quad ; n \geq m$$

$s_i \neq s_j$ if $i \neq j$

then

$$G(s) = \frac{A_1}{(s-s_1)} + \frac{A_2}{(s-s_2)} + \dots + \frac{A_n}{(s-s_n)}$$

where $A_i = G(s)(s-s_i)\Big|_{s=s_i}$

and

$$g(t) = \mathcal{L}^{-1} G(s) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \dots + A e^{s_n t}$$

clearly if for any i; $Re(s_i) > 0$ then

$$g(t) \longrightarrow \infty \quad \text{as } t \to \infty$$

and if $Re(s_i) < 0 \quad \forall i = 1 \dots n$ then

$g(t) \longrightarrow 0$ as $t \to \infty$

Example: (Repeated roots)

$$G(S) = \frac{3s+1}{(S-1)^2(s+2)}$$

$$= \frac{A_1}{(S+2)} + \frac{A_2}{(S-1)} + \frac{A_3}{(S-1)^2}$$

Note that

$$G(S)(S+2)\Big|_{S=-2} = A_1 + \frac{A_2(S+2)}{(S-1)}\Big|_{S=-2} + \frac{A_3(S+2)}{(S-1)^2}\Big|_{S=-2}$$

$$\Rightarrow \frac{3s+1}{(S-1)^2}\Big|_{s=-2} = A_1$$

$$\therefore A_1 = \frac{-6+1}{3^2} = -\frac{5}{9}.$$

Note that

$$G(S)(S-1)^2\Big|_{S=1} = \frac{A_1(S-1)^2}{(S+2)}\Big|_{S=1} + A_2(S-1)\Big|_{S=1} + A_3$$

$$\Rightarrow \frac{3s+1}{S+2}\Big|_{S=1} = A_3$$

$$\Rightarrow A_3 = \frac{4}{3}$$

Note that

$$\frac{d}{ds}\left(G(S)(S-1)^2\right) = \frac{d}{ds}\left[\frac{A_1(S-1)^2}{s+2}\right] + \frac{d A_2(S-1)}{ds} + \frac{d A_3}{ds}$$

$$= \left[\frac{2(S-1)A_1}{S+2} - \frac{A_1(S-1)^2}{(S+2)^2}\right] + A_2$$

$$\Rightarrow \quad \frac{d}{ds}\left[G(s)(s-1)^2\right]\Big|_{s=1} = A_2$$

$$\Rightarrow \frac{d}{ds}\left[\frac{3s+1}{s+2}\right]\Big|_{s=1} = A_2$$

$$\Rightarrow \left[\frac{-(3s+1)}{(s+2)^2} + \frac{3}{s+2}\right]\Big|_{s=1} = A_2$$

$$\Rightarrow \quad A_2 = -\frac{4}{9} + \frac{3}{3}$$

$$= \frac{-4+9}{9} = +5/9.$$

$$\therefore \quad G(s) = \frac{-5/9}{s+2} + \frac{5/9}{(s-1)} + \frac{4}{3(s-1)^2}$$

$$g(t) = \mathcal{L}^{-1}\frac{-5/9}{s+2} + \mathcal{L}^{-1}\frac{5/9}{(s-1)} + \mathcal{L}^{-1}\frac{4/3}{(s-1)^2}$$

$$= \frac{-5}{9}e^{-2t} + \frac{5}{9}e^{+t} + \frac{4}{3}\mathcal{L}^{-1}\left(1/(s-1)^2\right)$$

$$= \frac{-5}{9}e^{-2t} + \frac{5}{9}e^{+t} + \frac{4}{3}te^{t}$$

Note that $\mathcal{L}e^{t} = \frac{1}{(s-1)}$

$\therefore \mathcal{L}te^{t} = -\frac{d}{ds}\left(\frac{1}{s-1}\right)$

$= \frac{1}{(s-1)^2}$

Example: $G(s) = \dfrac{1}{s^2+s+1}$

Then the roots of s^2+s+1 are given by

$$s_1 = \frac{-1+\sqrt{1-4}}{2} \; ; \; s_2 = \frac{-1-\sqrt{1-4}}{2}$$

$$= \frac{-1+\sqrt{-3}}{2} \; ; \; s_2 = \frac{-1-\sqrt{-3}}{2}$$

$$= \frac{-1+\sqrt{3}\,j}{2} \; ; \; s_2 = \frac{-1-\sqrt{3}\,j}{2} \; ; \; \alpha = -\frac{1}{2} \; ;$$

$$\beta = +\frac{\sqrt{3}}{2}$$

$$s_1 = \alpha + j\beta \; ; \quad s_2 = \alpha - j\beta.$$

$$\frac{1}{s^2+s+1} = \frac{A_1}{(s-s_1)} + \frac{A_2}{(s-s_2)}$$

$$G(s)\,(s-s_1)\Big|_{s=s_1} = A_1$$

$$s^2+s+1 = (s-s_1)(s-s_2)$$

$$\Rightarrow \quad G(s)(s-s_1)\Big|_{s=s_1} = \frac{1}{(s-s_1)(s-s_2)}(s-s_1)\Big|_{s=s_1}$$

$$= \frac{1}{(s-s_2)}\Big|_{s=s_1}$$

$$= \frac{1}{\alpha+j\beta-\alpha+j\beta} = \frac{1}{2j\beta} = -\frac{1}{2\beta}j$$

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$$G(s)(s-s_2)\Big|_{s=s_2} = \frac{1}{s-s_1}\Big|_{s=s_2}$$

$$= \frac{1}{\alpha - j\beta - \alpha - j\beta} = -\frac{1}{2\beta j}$$

$$= \frac{1}{2\beta} j$$

$\therefore$

$$G(s) = \frac{-\frac{1}{2\beta} j}{(s-s_1)} \quad \frac{+\frac{1}{2\beta} j}{(s-s_2)}$$

$$\Rightarrow g(t) = -\frac{1}{2\beta} j \, \mathcal{L}^{-1} \frac{1}{s-s_1} + \frac{1}{2\beta} j \, \mathcal{L}^{-1} \frac{1}{(s-s_2)}$$

$$= -\frac{1}{2\beta} j \, e^{s_1 t} + \frac{1}{2\beta} j \, e^{s_2 t}$$

$$= \frac{j}{2\beta} \left[-e^{s_1 t} + e^{s_2 t} \right]$$

$$= \frac{j}{2\beta} \left[-e^{(\alpha + j\beta)t} + e^{+(\alpha - j\beta)t} \right]$$

$$= \frac{j}{2\beta} e^{\alpha t} \left[-e^{j\beta t} + e^{-j\beta t} \right]$$

$$= -\frac{e^{\alpha t}}{2\beta j} \left[-e^{j\beta t} + e^{-j\beta t} \right]$$

$$= \frac{+e^{\alpha t}}{2\beta j} \left[e^{j\beta t} - e^{-j\beta t} \right]$$

$$= \frac{e^{\alpha t}}{\beta} \cdot \left[\frac{e^{j\beta t} - e^{-j\beta t}}{2j}\right]$$

$$= \frac{e^{\alpha t}}{\beta} \sin \beta t \;; \alpha = -\frac{1}{2} \;; \beta = \frac{\sqrt{3}}{2}$$

<u>Another method</u>:

$$\frac{1}{s^2+s+1} = \frac{1}{s^2 + 2 \cdot \frac{1}{2} \cdot s + \frac{1}{4} - \frac{1}{4} + 1}$$

$$= \frac{1}{(s+\frac{1}{2})^2 + 3/4} = \frac{1}{(s+1)^2 + (\frac{\sqrt{3}}{2})^2}$$

$$= \frac{1}{(s+\alpha)^2 + \beta^2} = \frac{1}{\beta} \frac{\beta}{(s+\alpha)^2 + \beta^2}$$

$$= \frac{1}{\beta} e^{\alpha t} \sin \beta t.$$

Note that any polynomial with real coefficients, if it has any complex root, then its complex conjugate will also be a root. Thus, if $a+jb$ (where a and b are real numbers) is a root the $a-jb$ is also a root.

<u>Example</u>: Consider

$$G(s) = \frac{5s}{s^3 - s^2 - s - 2}$$

step 1 is to factorize the denominator

$$s^3 - s^2 - s - 2 = (s^2+s+1)(s-2)$$

$$\therefore G(s) = \frac{5s}{(s^2+s+1)(s-2)}$$

$$= \frac{As+b}{(s^2+s+1)} + \frac{C}{s-2}$$

$$\Rightarrow G(s)(s-2)\Big|_{s=2} = C$$

$$\Rightarrow \frac{5s}{s^2+s+1}\Big|_{s=2} = C$$

$$\Rightarrow c = \frac{10}{4+2+1} = \frac{10}{7}$$

Note also that

$$G(s)(s^2+s+1)\Big|_{s=0} = b + \frac{c(s^2+s+1)}{s-2}\Big|_{s=0}$$

$$\Rightarrow \frac{5s}{s-2}\Big|_{s=0} = b + \frac{c}{-2}$$

$$\therefore 0 = b - 5/7 \Rightarrow b = 5/7$$

Note that

$$G(s)(s^2+s+1)(s-2)\Big|_{s=1} = (As+b)(s-2)\Big|_{s=1} + c(s^2+s+1)\Big|_{s=1}$$

$$\Rightarrow 5 = (A+b)(-1) + c(1+1+1)$$

$$\Rightarrow 5 = -(A+b) + 3c$$

$$\Rightarrow A+b = 3c-5$$

$$\Rightarrow A = 3c - 5 - b$$

$$= 3\cdot\frac{10}{7} - 5 - 5/7$$

$$= \frac{30-40}{7} = -10/7$$

$$\therefore G(s) = \frac{5s}{s^3-s^2-s-2} = \frac{-\frac{10}{7}s + \frac{5}{7}}{s^2+s+1} + \frac{10/7}{s-2}$$

Consider

$$G_1(s) = \frac{-\frac{10}{7}s + \frac{5}{7}}{s^2+s+1}$$

$$= \frac{-\frac{10}{7}s + \frac{5}{7}}{s^2+2\cdot\frac{1}{2}\cdot s+\frac{1}{4}-\frac{1}{4}+1}$$

$$= \frac{-\frac{10}{7}s + \frac{5}{7}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \frac{-\frac{10}{7}s + \frac{5}{7}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{-\frac{10}{7}\left(s+\frac{1}{2}\right)+\frac{5}{7}+\frac{5}{7}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{-\frac{10}{7}\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{\frac{10}{7}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{10}{7}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{10}{7}\frac{2}{\sqrt{3}}\frac{\sqrt{3}/2}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$g(t) = \mathcal{L}^{-1} G(S) = \mathcal{L}^{-1}\left(-\frac{10}{7}\right) \frac{S+\frac{1}{2}}{(S+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$+ \frac{20}{7\sqrt{3}} \mathcal{L}^{-1} \frac{\sqrt{3}/2}{(S+\frac{1}{2})^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{10}{7} \quad e^{-\frac{1}{2}t} \cos\frac{\sqrt{3}}{2}t$$

$$+ \frac{20}{7\sqrt{3}} \quad e^{-1/2\, t} \sin\frac{\sqrt{3}}{2}t$$

Definition

Suppose

$$G(s) = \frac{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}$$

where $m \geq n$ then

$G(s)$ is said to be a proper transfer function

An Important Conclusion is that if the denominator of a transfer function that is proper has any root with a strictly positive real part then

$$g(t) = \mathcal{L}^{-1} G(s)$$

will be such that

$$g(t) \to \infty \text{ as } t \to \infty$$

Furthermore if all the roots have strictly negative real parts then

$$g(t) \to 0 \text{ as } t \to \infty.$$