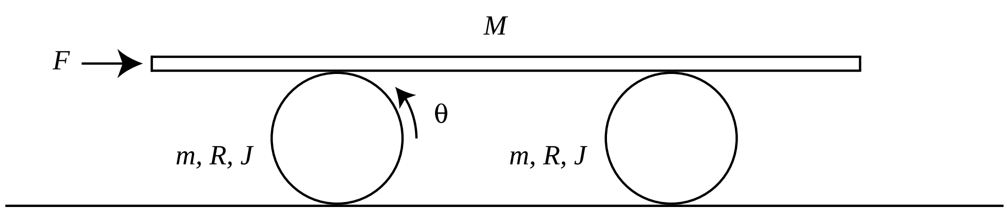
Homework on Lagrange's Method

1. In simpler times, the world agreed that Pluto was a planet and Charon was its moon. In fact, Pluto and Charon differ in mass by less than one order of magnitude, and orbit one another as a planar binary system. Let m_p denote the mass of Pluto and (x_p, y_p) its position with respect to a point fixed in the ecliptic plane; let m_c denote the mass of Charon and (x_c, y_c) its position with respect to the same fixed point. The gravitational potential energy in this system is given by

$$P.E. = -\frac{Gm_p m_c}{r},$$

where G is a constant and r denotes the distance between Pluto and Charon. Express the kinetic energy, potential energy, and Lagrangian for this system in terms of the generalized coordinates x_p, y_p, x_c, and y_c and their derivatives with respect to time. Derive the two equations of motion for Charon as the Euler-Lagrange equations corresponding to the generalized coordinates x_c and y_c.

2. Consider the system shown: The board with mass M rolls without slipping atop two identical wheels,



each with mass m, radius R, and moment of inertia J. The wheels, in turn, roll on the ground without slipping. The force F is applied horizontally to the board.

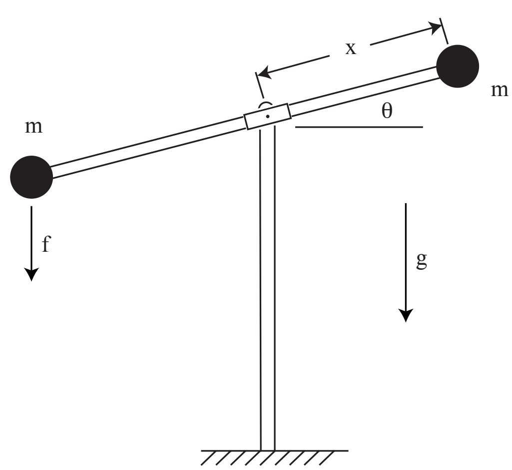
   (a) Express the total kinetic energy T in this system in terms of $\dot{\theta}$, M, m, R, and J.

   (b) The equations of motion for this system can be written in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = Q_\theta.$$

   Find Q_θ in terms of F and R and hence find the equations of motion.

3. Consider the system shown:

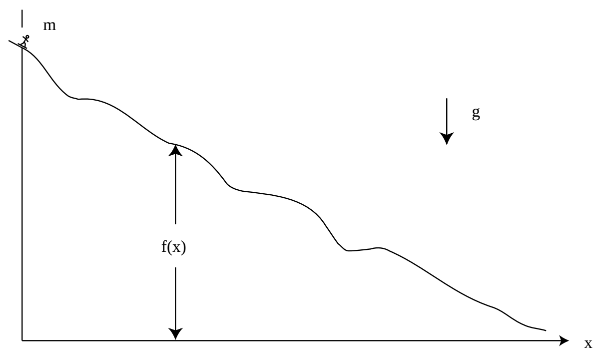


The rod joining the two masses slides frictionlessly in the collar, which pivots frictionlessly; an internal linear spring of stiffness k N/m provides a restoring force to center the rod with respect to the collar. The rod joining the masses is two meters long and x and θ are expressed in meters and radians/s.

(a) Find the differential equations that describe the motion of the above system.

(b) Optional bonus question: Use MATLAB with `diffeq.m` and `run_diffeq.m` to simulate the dynamics of the system starting centered from rest with $f(t) = \sin t$, assuming that $g = 9.81$ m/s^2, $k = 2$ N/m, and $m = 1$ kg. What is the maximum extension of the spring?

4. Gandalf, whose mass is m, decides to try his hand at snowboarding. He stands atop one of the many tall mountains near Isengard, the altitude of its slope varying with horizontal distance from his starting point as $f(x)$. It is an icy and thus frictionless morning.

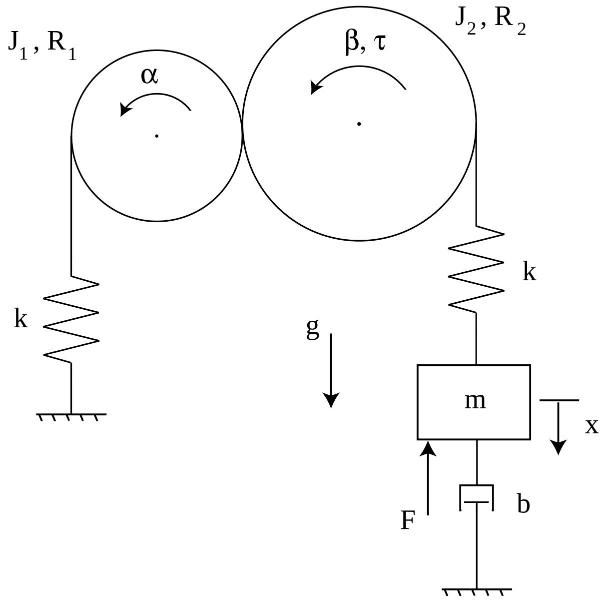


(a) Derive the differential equation in $x(t)$ governing Gandalf's date with destiny in terms of the functions $f(x)$, $\frac{\partial f(x)}{\partial x}$, and $\frac{\partial^2 f(x)}{\partial x^2}$. Assume that Gandalf never leaves the ground.

(b) Lying atop his cot in the hospital tent, Gandalf wonders if he would have fared better with some fresh powder on the slope. Imagine a thin layer of snow, parameterized by the linear drag coefficient b, to have impeded his motion. Find the corresponding generalized force in the x direction.

(c) Optional bonus question: Use MATLAB with `diffeq.m` and `run_diffeq.m` to simulate Gandalf's snowboard run for a mountain with shape $f(x) = 0.0005(x - 1000)^2$, where both x and f are measured in meters. Assume that Gandalf starts at rest from $x = 0$ and take $m = 80$ kg, $g = 9.81$ m/s^2, and $b = 0.1$ Ns/m. How long does it take for Gandalf to reach the bottom (at $x = 1000$ m)? What is Gandalf's maximum speed?

5. Express the total kinetic energy and total potential energy in the system shown...



in terms of angles α, its derivative $\dot{\alpha}$ and position x and $\dot{x}$. Obtain the dynamics using Lagrange's method.