Lecture 2

Saturday, January 21, 2012
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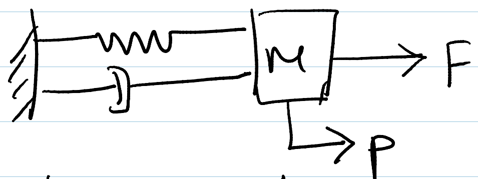
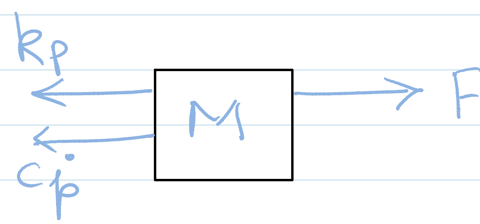
Last time:

Consider a spring mass damper system



- The position of the mass is p
- A force F is applied to the mass M

Free body diagram



$$M\ddot{p} = F - kp - c\dot{p}$$

and thus

$$M\ddot{p} + c\dot{p} + kp = F$$

- Converting into State Space

Let

$$x_1 \doteq p$$

$$x_2 \doteq \dot{p}$$

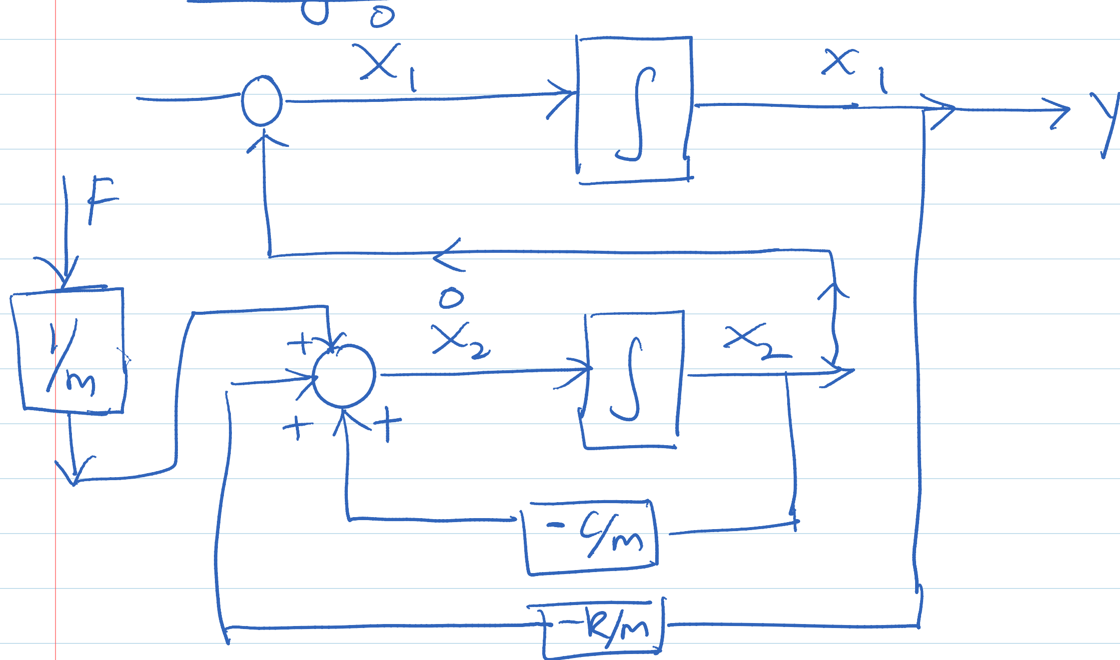
Thus,

$$\dot{x}_1 = \dot{p} = x_2$$
$$\dot{x}_2 = \ddot{p} = -\frac{k}{m}p - \frac{c}{m}\dot{p} + F$$

$$= -\frac{k}{m}x_1 - \frac{c}{m}x_2 + F$$

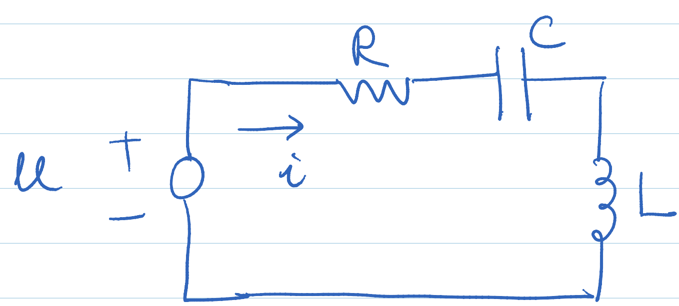
$$\therefore \quad \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F$$

Analog Circuit:



Example 2:

A RLC Circuit



V_R = Voltage drop across resistor $= iR$

V_C = Voltage drop across capacitor

$$- \quad i = C\frac{dV_C}{dt}$$

V_L = voltage drop across inductor

$$= L\frac{di}{dt}$$

Applying Kirchoffs voltage law

$$V_R + V_C + V_L = u \quad \text{---} \quad (1)$$

$$V_R = iR; \quad C\frac{dV_C}{dt} = i; \quad V_L = L\frac{di}{dt}$$

Lets choose as states

$$x_1 = V_C$$
$$x_2 = i$$

Then we get

$$\dot{x}_1 = \frac{dV_C}{dt} = \frac{i}{C} = \frac{x_2}{C}$$

$$\dot{x}_2 = \frac{di}{dt}$$

From (1) we have

$$iR + V_C + L\frac{di}{dt} = u$$

$$\Rightarrow \frac{di}{dt} = \frac{u}{L} - \frac{V_C}{L} - \frac{iR}{L}$$

$$= \frac{u}{L} - \frac{x_1}{L} - \frac{x_2 R}{L}$$

$\therefore$

$$\mathring{x} = \begin{bmatrix} \mathring{x}_1 \\ \mathring{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$; y is the voltage across the resistor

If we had defined

$$x_1 = C V_c$$
$$x_2 = i$$

Then we obtain

$$\mathring{x}_1 = C \frac{dV_c}{dt} = i = x_2$$

$$\mathring{x}_2 = \frac{u}{L} - \frac{V_c}{L} - \frac{V_R}{L}$$
$$= \frac{u}{L} - \frac{x_1}{LC} - \frac{R x_2}{L}$$

And therefore it follows that

$$\begin{pmatrix} \mathring{x}_1 \\ \mathring{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Compare this to the state-space of Spring-mass-damper System

$$\begin{pmatrix} \mathring{x}_1 \\ \mathring{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F$$

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} F$$

Lets identify

mechanical stiffness $k \equiv \frac{1}{C}$ where C is the capacitance

mechanical damping $c \equiv R$; R is the resistance

mechanical mass $m \equiv L$; the inductance

This provides a mapping between Electrical and mechanical systems.

<u>Example</u>: <u>Another choice of state variables</u>.

Note that we have

$$V_R + V_L + V_C = u \quad --- (1)$$

where $V_R = iR$

$$V_L = L\frac{di}{dt}$$

$$C\frac{dV_C}{dt} = i$$

We can differentiate (1) to obtain

$$R\frac{di}{dt} + L\frac{d^2i}{dt^2} + \frac{i}{C} = \frac{du}{dt}$$

Looks like a choice for state variables

is

$$x_1 \doteq i; \quad x_2 = \frac{di}{dt}$$

where we obtain

$\mathring{x}_1 = x_2$ and

$$\mathring{x}_2 = \frac{1}{L}\frac{du}{dt} - \frac{i}{LC} - \frac{R}{L}\frac{di}{dt}$$

$$= \frac{1}{L}\frac{du}{dt} - \frac{x_1}{LC} - \frac{R}{L}x_2$$

Thus,

$$\mathring{x} = \begin{bmatrix} \mathring{x}_1 \\ \mathring{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} \frac{du}{dt}$$

$$y = [1 \ 0]x$$

Here we have $\frac{du}{dt}$; not u

The above is not in the form

$$\mathring{x} = Ax + Bu$$

Consider the same state space equations with input u and output $\tilde{y}$

$$\mathring{x} = Ax + Bu$$

$$\tilde{y} = Cx = x_1$$

where $x_2 = \mathring{x}_1$

If the input is u for the above system the output is $\tilde{y} = x_1$

→ As the system is linear, if the input is changed to $\frac{du}{dt}$ the output will be $\frac{d\tilde{y}}{dt} = \frac{d}{dt}x_1 = \dot{x}_1 = x_2$

Thus,

$$\dot{x} = Ax + Bu$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2$$