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Practice Problems for Final Exam

1. Consider the following three waveforms:

$$s_0(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{else} \end{cases} \qquad s_1(t) = \begin{cases} -\frac{1}{\sqrt{\Delta}} & 0 \le t \le \Delta \\ 0 & \text{else} \end{cases} \qquad s_2(t) = \begin{cases} -\frac{1}{\sqrt{1-\Delta}} & \Delta \le t \le 1 \\ 0 & \text{else} \end{cases}$$

Assume equal message priors, i.e., $\pi(0) = \pi(1) = \pi(2)$, white Gaussian noise with variance $\sigma^2 = N_0/2$, and that Δ is a constant between 0 and 1.

   (a) Find an orthonormal basis for the waveforms and provide the signal space representation in terms of this basis.
   (b) Sketch the ML decision regions for this constellation.
   (c) Is $P_{e|1}$ increasing or decreasing in Δ?
   (d) Derive the union bound for this constellation.

2. Consider the $R = 1/3$ binary convolutional code with memory 2 that outputs the following three coded bits for each information bit:

$$u[k] + u[k-1] + u[k-2]$$
$$u[k] + u[k-1] + u[k-2]$$
$$u[k] + u[k-2]$$

(in octal notation this is the [7,7,5] code)

   (a) Draw the trellis diagram (with outputs labeled for each branch) for this code.
   (b) Compute d_{free} for this code.
   (c) Compare this code's coding gain to the coding gain of the $R = 1/2$, $[7, 5]$ code (i.e., the initial convolutional code we studied in class).

3. Amongst all binary convolutional codes with a particular rate and with a particular number of states (i.e., memory), we are often interested in the code that has the largest d_{free}. For example, the largest d_{free} for codes with $R = 1/2$ and 4 states is 5, which is achieved by the $[7, 5]$ code (non-systematic and non-recursive) we studied in class.

   (a) Prove that for codes with $R = 1/8$ and 4 states, the largest d_{free} is greater than or equal to 20.
   (b) Prove that for codes with $R = 1/8$ and 16 states, the largest d_{free} is greater than or equal to 20.

4. In this problem we will study the BCJR algorithm for the systematic recursive $[7, 5]$ binary convolutional code we studied in class. Assume there are ten information bits followed by two terminating bits (such that the code terminates in state 00), and that $\sqrt{E_s} = 1$ and $\sigma^2 = 1$.

   Consider the received sequence $1.2, 2.3, -3.1, 2.8, \theta, 0$, followed by 18 arbitrary values.

   (a) Determine what happens to the LLR of the third information bit as $\theta \to \infty$.

   (b) If $\theta = 0$, write down the expression for the LLR of the third information bit in terms of the relevant α and β values. (You do not need to solve for the values of α and β). Indicate which parts of the received sequence determine the α and β values in your expression.

5. Consider a system where the impulse response of the TX filter and of the channel are given by:

$$g_{\mathrm{TX}}(t) = \begin{cases} 1 & 0 \le t \le 1 \\ -1 & 1 \le t \le 2 \\ 0 & \text{else} \end{cases} \qquad\qquad g_C(t) = \delta(t) + \delta(t-4)$$

   The symbol period is 2.

   (a) Assuming a sample period of $T = 2$, compute $p(t)$ and $h[n]$.

   (b) If MLSE was to be performed (assuming BPSK), how many states would be required in the Viterbi implementation?

   (c) Assume white Gaussian noise with PSD σ^2 and that BPSK with ± 1 is used. We are interested in the MMSE equalizer (on the matched filter outputs for $T = 2$) of length 5. Compute $\mathbf{U}$, the matrix that maps from bits to the 5 received symbols, and $\mathbf{C}_w$, the noise covariance matrix (for 5 received symbols), and write out the equation for the MMSE received filter.

   (d) Now assume that the following RX filter is used:

$$g_{\mathrm{RX}}(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{else} \end{cases}$$

   (instead of the matched filter), and that it is sampled with a period $T = 1$. Repeat part (c) if you wish to design the MMSE filter of length 6.