EE 8510 Tuesday, March 22, 2005
Advanced Topics in Communications Prof. N. Jindal

Homework Set 6
Due: Tuesday, March 29, 2005

1. *Joint vs. Marginal Distributions for BC.* In class we claimed that the capacity region of the broadcast channel only depends on the marginal distributions $p(y_1|x)$ and $p(y_2|x)$ and not on the joint distribution $p(y_1, y_2|x)$. In this problem we will prove this result.

   We defined the average probability of error as:

   $$P_e^{(n)} = P\{\hat{W}_1 \neq W_1 \text{ or } \hat{W}_2 \neq W_2\}$$

   Let us also define the individual probabilities of error by

   $$P_1^{(n)} = P\{\hat{W}_1 \neq W_1\}, \quad P_2^{(n)} = P\{\hat{W}_2 \neq W_2\}$$

   A rate pair (R_1, R_2) is defined to be achievable if there exist a sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes such that $P_e^{(n)} \to 0$ as $n \to \infty$, and the capacity region is defined as the closure of the set of all achievable rate pairs. In problem 6 of homework 5 we showed that the capacity region of the multiple-access channel is the same if we consider a rate pair to be achievable if $P_1^{(n)}$ and $P_2^{(n)}$ can be driven to zero. The same argument is also true for the broadcast channel. Use this to show that the capacity region of the broadcast channel only depends on the marginal distributions of the channel, and not on the joint.

2. *Common Information.* Consider a two-user broadcast channel where there is only a common message W_0 chosen uniformly from $\{1, \ldots, 2^n R\}$ that is intended for *both* receivers. Let $\hat{W}_1$ be the estimate of W_0 at receiver 1, and $\hat{W}_2$ be the estimate of W_0 at receiver 2. The probability of error is defined as:

   $$P_e^{(n)} = P\{\hat{W}_1 \neq W_0 \text{ or } \hat{W}_2 \neq W_0\}.$$

   Show that the capacity C_0 of this channel is given by:

   $$C_0 = \max_{p(x)} \min\left(I(X; Y_1), I(X; Y_2)\right)$$

   Achievability and converse only need to be outlined.

3. *Alternative Decoder.* Consider the proof of achievability for the degraded broadcast channel. Suppose we modify the decoding strategy for receiver Y_1: declare that $(\hat{w}_1, \hat{\hat{w}}_2)$ is sent if it is the unique pair such that $(x^n(\hat{w}_1, \hat{\hat{w}}_2), y_1^n) \in A_\epsilon^{(n)}$.

   (a) Show that this modified rule gives the same constraints as the one presented in class (where receiver Y_1 declared $(\hat{w}_1, \hat{\hat{w}}_2)$ is sent if it is the unique pair such that $(u^n(\hat{\hat{w}}_2), x^n(\hat{w}_1, \hat{\hat{w}}_2), y_1^n) \in A_\epsilon^{(n)}$).

   (b) What if receiver 1 compiles a list of all pairs (w_1, w_2) such that $(x^n(w_1, w_2), y_1^n) \in A_\epsilon^{(n)}$. If there are no such pairs, he declares an error. If there is one such pair, he declares w_1 was sent. If there are multiple such pairs but the w_1 portion is the same for all such pairs, he declares w_1 was sent. If there are multiple such pairs that do not all agree on w_1, he declares an error. Does this decoder outperform the decoder from part (a)? Does this increase the capacity region?

4. *Inner and outer bounds on capacity of broadcast channel.* Consider a general DM broadcast channel $(\mathcal{X}, p(y_1, y_2|x), \mathcal{Y}_1, \mathcal{Y}_2)$.

   (a) Show that any rate pair (R_1, R_2) such that

   $$R_1 < I(U; Y_1), \quad R_2 < I(V; Y_2)$$

   for some $p(u)p(v)p(x|u, v)p(y_1, y_2|x)$ is achievable. Don't repeat arguments from lecture.

   (b) Show that for any sequence of $(2^{nR_1}, 2^{nR_2}, n)$ codes with $P_e^{(n)} \to 0$

   $$\begin{aligned} R_1 &\le I(U; Y_1) \\ R_2 &\le I(V; Y_2) \\ R_1 + R_2 &\le I(U, V; Y_1, Y_2) \end{aligned}$$

   for some $p(u, v)p(x|u, v)p(y_1, y_2|x)$.

   Note that these two bounds on the capacity region are not in general equal. Marton provided a tighter inner bound, but it is not known whether her bound is tight, i.e. there is no converse and no counterexample to show that it is not tight.

5. *Identical broadcast channels.* Consider a DM broadcast channel where the channels of both receivers are statistically identical, i.e. $p(y_1|x) = p(y_2|x)$ for all $x \in \mathcal{X}$ and $y_1, y_2 \in \mathcal{Y}$ $(= \mathcal{Y}_1 = \mathcal{Y}_2)$, but are conditionally independent, i.e. $p(y_1, y_2|x) = p(y_1|x)p(y_2|x)$.

   (a) What is the capacity region of this broadcast channel?

   (b) What if the channel outputs are always identical, i.e. $\Pr\{Y_1 = Y_2\} = 1$? Does this channel have the same capacity region as in (a)?