Sum Power Iterative Water-filling for Multi-Antenna Gaussian Broadcast Channels

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Abstract

In this paper we consider the problem of maximizing sum rate on a multiple-antenna downlink in which the base station and receivers have multiple-antennas. The optimum scheme for this system was recently found to be "dirty paper coding". Obtaining the optimal transmission policies of the users when employing this dirty paper coding scheme is a computationally complex non-convex problem. We use a "duality" to transform this problem into a convex multiple access problem, and then obtain a simple and fast iterative algorithm that gives us the optimum transmission policies.

1 Introduction

There has been a great interest in characterizing and computing the capacity region of downlink channels in recent years. An achievable region was found by [6], and this achievable region was shown to be sum rate optimal in [6, 2, 9, 10].

Unfortunately, the characterization of the region in [6] leads to a non-convex non-linear optimization problem that is difficult to solve, and hence obtaining the optimal rates and transmission policies of each user is computationally complex. Note that, in the single antenna case, although the problem is still non-convex, it simplifies to only the best user transmitting at any time instant. Such a policy is, however, not the optimal policy in the multiple antenna case. A duality technique presented in [8, 2] transforms the non-convex downlink problem into a convex *sum power* uplink (MAC) problem, which is much easier to solve. In this sum power uplink or sum power MAC problem, the users in the system have a joint power constraint instead of the individual constraints in the conventional MAC. As in the case of the conventional MAC, there exist standard interior point convex optimization algorithms [11] that solve the sum power MAC problem. A new interior point based method has also been found in [12].

However, employing a interior point convex optimization algorithm to tackle as well structured a problem as sum capacity is inefficient. In this paper, we exploit the structure in this sum capacity problem to obtain a simple iterative algorithm for calculating sum capacity. This algorithm is inspired by and is very similar to an iterative algorithm for the conventional individual power constraint MAC problem by Yu and Cioffi [1]. Although a rigorous proof of the optimality of the algorithm for the general case is unknown, its working is highly intuitive and is found to converge in all simulation results so far. Here, we first provide an argument that shows that the algorithm either converges or oscillates between two points.

This paper is structured as follows. In the next section, we present the system model. In Section 3, we present some background on dirty paper coding and duality. In Section 4, we study the Kuhn-Tucker conditions of the problem and present the algorithm. Finally, we present an analysis of the properties of this algorithm in Section 5 and conclude with Section 6.

2 System Model

The downlink channel model considered is shown in Figure 1. Note that an uplink model is depicted alongside it. This is the *dual* uplink, where the users in the system have a sum power constraint. The significance of this dual uplink model is explained in Section 3

The downlink and uplink channel models are as below:

$$y_i = H_i x + n_i, \quad \forall\ i \quad \text{Downlink channel model} \tag{1}$$

$$v = \sum_{i=1}^{M} H_i^{\dagger} x_i + w. \quad \text{Dual uplink channel model} \tag{2}$$

where we assume $H_1, H_2, \ldots, H_K$ to be the channel matrices of users 1 through K respectively on the downlink, and a transmit power constraint of P.

In the next section we provide some background on

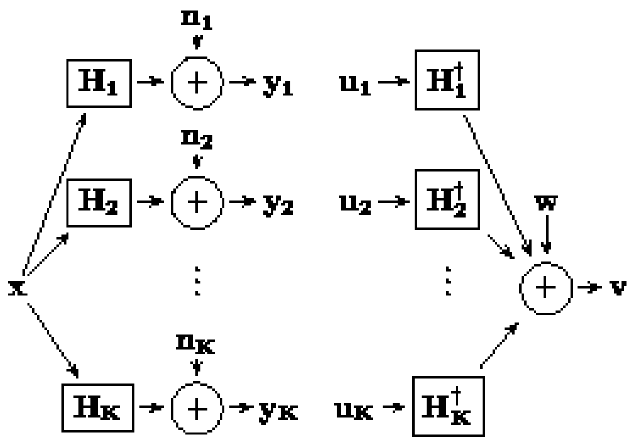

Figure 1: System models of the BC MIMO(left) and the MAC MIMO (right) channels

two important concepts that lead to the algorithm - dirty paper coding and duality.

3 Background

3.1 Dirty Paper Coding

Caire and Shamai [6] developed an achievable set of rates for the MIMO broadcast channel based on the "dirty paper coding" result of Costa [5], and hence this region is termed the dirty paper region. This coding strategy allows a channel with interference known at the transmitter to achieve the same data rate as if the interference did not exist. This translates to the following coding strategy: The transmitter first picks a codeword for receiver 1. The transmitter then chooses a codeword for receiver 2 with full (non-causal) knowledge of the codeword intended for receiver 1. Therefore receiver 2 does not see the codeword intended for receiver 1 as interference. Similarly, the codeword for receiver 3 is chosen such that receiver 3 does not see the signals intended for receivers 1 and 2 as interference. This process continues for all K receivers. Since the ordering of the users clearly matters in such a procedure, the following is an achievable set of rates

$$R_{\pi(i)} = \frac{1}{2}\log\frac{|I + H_{\pi(i)}(\sum_{j\geq i}\Sigma_{\pi(j)})H_{\pi(i)}^\dagger|}{|I + H_{\pi(i)}(\sum_{j>i}\Sigma_{\pi(j)})H_{\pi(i)}^\dagger|} i = 1, \ldots, K. \tag{3}$$

The dirty-paper region $\mathcal{C}_{\text{dirtypaper}}(P, H)$ is defined as the union of all such rate vectors over all covariance matrices $\Sigma_1, \ldots, \Sigma_K$ such that $\text{Tr}(\Sigma_1 + \ldots \Sigma_K) = \text{Tr}(\Sigma_x) \leq P$ and over all decoding order permutations $(\pi(1), \ldots, \pi(K))$. The transmitted signal is $x = x_1 + \ldots + x_K$ and the input covariance matrices are of the form $\Sigma_i = E[x_i x_i{}^\dagger]$.

3.2 Duality

Next, we introduce the concept of duality with the following theorem:

Theorem 1 **([2])** *The dirty paper region of a MIMO BC channel with power constraint P is equal to the the capacity region of the dual MIMO MAC with sum power constraint P.*

$$\mathcal{C}_{\text{dirtypaper}}(P, H) = \mathcal{C}_{\text{union}}(P, H^\dagger).$$

Since the dual MIMO MAC with sum power P is, in fact, a convex problem, while the original dirty paper problem is not, this duality result is of great use, both computationally and analytically.

Using this duality, the sum rate of the downlink has been shown to be achievable by dirty-paper coding [2]. The sum rate maximization problem is:

$$\max_{\Sigma_1\geq 0,\ \sum_{i=1}^M \text{Tr}(\Sigma_i)\leq \text{P}} \log|I + H_1\Sigma_1 H_1^\dagger| + \log\frac{|I + H_2(\Sigma_1+\Sigma_2)H_2^\dagger|}{|I + H_2\Sigma_1 H_2^\dagger|} + \cdots + \log\frac{|I + H_M(\Sigma_1+\cdots+\Sigma_M)H_M^\dagger|}{|I + H_M(\Sigma_1+\cdots+\Sigma_{M-1})H_M^\dagger|}. \tag{4}$$

As noted for the general dirty paper region, this problem is not convex. By using duality, however, we get the following equivalent uplink sum rate maximization problem:

$$\max_{S_i\geq 0,\ \sum_{i=1}^M \text{Tr}(\text{S}_i)\leq \text{P}} \log|I + \sum_{i=1}^M H_i^\dagger S_i H_i|. \tag{5}$$

This problem is convex and can be solved using convex maximization techniques which are polynomial in complexity [11].

3.3 Iterative Waterfilling by Yu and Cioffi

The iterative waterfilling algorithm for the conventional MIMO MAC problem, with *individual* power constraints on each user was obtained by Yu and Cioffi in [1]. This algorithm can also be applied to the sum power MIMO MAC problem, but is however, inefficient, since it requires a search for covariances over all power splits amongst the K users in the system. This iterative waterfilling algorithm, however, forms the basis on which we develop the sum-power iterative waterfilling algorithm in this paper.

4 The Algorithm

We propose a specialized algorithm which is found to converge quickly to the optimal covariance matrices. This algorithm is based on the same mathematical quantity as the most common algorithms in fading and multi-antenna theory - the Karush Kuhn Tucker (KKT) conditions. The KKT conditions were used in [13] to obtain the time-waterfilling power distribution for single antenna point to point fading channels. The space waterfilling results in [3] using singular value decomposition in multi-antenna systems, and power allocation results in [14] for the single antenna MAC can also be shown to be connected to KKT conditions.

The KKT conditions have also been used to obtain iterative algorithms. For the MIMO MAC, an iterative algorithm was found by [1] that performs significantly better than convex optimization software employed to solve the problem. The algorithm we present here for the MIMO BC is inspired by the MAC algorithm in [1]. In our algorithm, we first solve the dual sum power MAC problem, and then use the duality transformations in [2] to obtain the downlink covariances. Before analyzing the dual sum power MAC problem, let us review the KKT conditions of a multi-antenna point to point system. This problem can be written mathematically as

$$\max_{\{S:Tr(S)\leq P\}} \log|I+(H^{eff})^{\dagger}SH^{eff}| \tag{6}$$

where H^{eff} is the channel of this user. The KKT conditions for this problem are given by

$$\lambda I = H^{eff}(I+(H^{eff})^{\dagger}SH^{eff})^{-1}(H^{eff})^{\dagger}+\Psi$$

along with complementary slackness conditions, where Ψ is a slackness variable. Note that these KKT conditions have a deep connection with the celebrated space-waterfilling algorithm [3] to obtain the optimum covariance S, with the inverse of the Lagrangian constant $1/\lambda$ corresponding to the waterlevel. Next, let us focus on the dual sum power MAC problem. It can be written mathematically as

$$\max_{S_i} \log|I+\sum_i H_i^{\dagger}S_iH_i|$$

such that

$$\sum Tr(S_i)\leq P$$
$$S_i \geq 0.$$

We can obtain the Lagrangian for the above problem, and differentiating it with respect to S_i, the KKT conditions are found to be:

$$\lambda I = H_i Z_i^{-1/2}(I+Z_i^{-1/2}H_i^{\dagger}S_iH_iZ_i^{-1/2})^{-1}Z_i^{-1/2}H_i^{\dagger}+\Psi_i$$

where $Z_i = (I+\sum_{j\neq i}^{M} H_j^{\dagger}S_jH_j)$.

Note that these KKT conditions are very similar to the KKT condition of the point to point channel above. In fact, for each user i, if $H^{eff} = H_j(I+\sum_{i\neq j}^{M} H_i^{\dagger}S_iH_i)^{-1/2}$, then the KKT conditions are identical. This observation was made in [1] to obtain the iterative waterfilling algorithm for the MAC channel with separate power constraints, and hence different water levels $1/\lambda_i$. In our case, we further find that the water level $1/\lambda$ is the same for all i. This inspires the following sum power iterative waterfilling algorithm:

1. Initialize covariance matrices to zero: $S_i(0) = 0\ \forall\ i$.

2. For iteration l : Generate effective channels $H_j^{eff} = H_j(I+\sum_{i\neq j}^{M} H_i^{\dagger}S_i(l-1)H_i)^{-1/2}$.

3. Treating these effective channels as parallel, non-interfering channels, obtain the new covariance matrices by waterfilling with total power P.

   $$\{S_i(l)\}_{i=1}^{M} = \text{argmax}\sum_{i=1}^{M}\log|I+(H_i^{eff})^{\dagger}Q_iH_i^{eff}|$$

   over the set

   $$Q_i\geq 0, \sum_{i=1}^{M}\text{Tr}(Q_i)\leq P$$

   This maximization is equivalent to waterfilling the block diagonal channel with diagonals equal to H_j^{eff}.

4. Return to Step 2 until desired accuracy is reached.

Any set of covariance matrices that are a fixed point of this algorithm can be shown to satisfy the KKT conditions of (5). It is then easy to transform these uplink covariance matrices to downlink covariance matrices using the transformations specified in [2].

This algorithm is different from that in [1] in that it performs a *joint* waterfilling on all the users in the system instead of a user-by-user waterfilling. Note that, to perform user-by-user waterfilling, individual power constraints on the users are essential. Thus, the joint waterfilling algorithm is designed for the case when there is a joint power constraint on all the users.

5 Analysis of the Algorithm

For this analysis, we assume a simple system with two users. We use the subscript l to denote the iteration number. Let us consider instead, the following optimization problem:

$$\max \log |I + H_1^\dagger S_1(l) H_1 + H_2^\dagger S_2(l-1) H_2| + \log |I + H_1^\dagger S_1(l-1) H_1 + H_2^\dagger S_2(l) H_2| \quad (7)$$

such that

$$Tr(S_1(l) + S_2(l)) = P$$
$$Tr(S_1(l-1) + S_2(l-1)) = P$$

Although each iteration of the sum power iterative algorithm may not increase the objective value of (5), we show that it increases the objective value of the optimization problem (7) above. Note that the optimization problem in (7) can be rewritten as:

$$\begin{aligned} \max_{S_1(l), S_2(l), S_1(l-1), S_2(l-1)} & \log |I + H_1^\dagger S_1(l) H_1| \\ & + \log |I + (I + H_1^\dagger S_1 H_1)^{-1} H_2^\dagger S_2(l-1) H_2| \quad (8) \\ & + \log |I + (I + H_2^\dagger S_2(l) H_2)^{-1} H_1^\dagger S_1(l-1) H_1| \\ & + \log |I + H_2^\dagger S_2(l) H_2|. \end{aligned}$$

After the $l+1$th iteration in the algorithm, we obtain new estimates $S_1(l+1)$ and $S_2(l+1)$. Substituting them into the expression (8) above, we get

$$\begin{aligned} \max_{S_1(l), S_2(l), S_1(l+1), S_2(l+1)} & \log |I + H_1^\dagger S_1(l) H_1| \\ & + \log |I + (I + H1^\dagger S_1 H_1)^{-1} H_2^\dagger S_2(l+1) H_2| \quad (9) \\ & + \log |I + (I + H_2^\dagger S_2(l) H_2)^{-1} H_1^\dagger S_1(l+1) H_1| \\ & + \log |I + H_2^\dagger S_2(l) H_2| \end{aligned}$$

Note that, by the nature of the algorithm, the expression in (9) is greater than the expression in (8). Moreover (9) can be rewritten as

$$\max \log |I + H_1^\dagger S_1(l+1) H_1 + H_2^\dagger S_2(l) H_2| + \log |I + H_1^\dagger S_1(l) H_1 + H_2^\dagger S_2(l+1) H_2|$$

such that

$$Tr(S_1(l) + S_2(l)) = P$$
$$Tr(S_1(l+1) + S_2(l+1)) = P$$

Thus, the objective function in (7) is always increasing with the iteration l in the algorithm. Since the expression in (7) is jointly convex in $S_1(l), S_2(l), S_1(l-1), S_2(l-1)$, and the function is bounded, the increments in the function must converge to zero.

Note that, after the increments in the function reduce to zero, the uniqueness of the waterfilling algorithm [3] guarantees that $S_i(l-1) = S_i(l+1)$ for $i = 1, 2$. Thus, the algorithm either oscillates between two sets of values for S_i but or it converges to a fixed point. If S_i converges to a fixed point, then we are done. If it does not, we must modify the algorithm to guarantee convergence. We know, from optimization theory, that such a fixed point for the KKT conditions exist and that this fixed point is the optimum point. We must initialize the algorithm with matrices S_i that are in the neighborhood of this optimal fixed point. This can be achieved, for example, using greedy algorithm techniques [15].

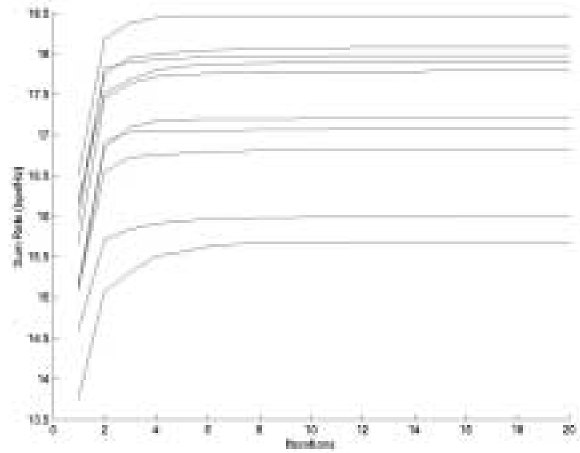

Figure 2: Convergence to sum rate of the iterative algorithm for different channel realizations

This concludes our analysis of this algorithm. A plot showing the convergence properties of this algorithm for different random channel realizations in a two transmit antenna, two receivers with two antennas each is shown in Figure 2. Note that the algorithm converges within 7 iterations in all these cases.

6 Conclusion

We provide an iterative algorithm to efficiently compute the optimal transmit policies corresponding to the sum capacity of broadcast (downlink) systems. This algorithm is based on the Kuhn-Tucker conditions of the dual sum power multiple access channel.

References

[1] W. Yu, W. Rhee, S. Boyd, J. Cioffi, "Iterative Water-filling for Vector Multiple Access Channels", pp. 322, Proc. IEEE Int. Symp. Inf. Theory, (ISIT), Washington DC, June 24-29, 2001.

[2] S. Vishwanath, N. Jindal, and A. Goldsmith, "On the Capacity of Multiple Input Multiple Output

Broadcast Channels", Proc. IEEE Int. Conf. Commun. (ICC), New York, April 2002.

[3] I. E. Telatar, "Capacity of Multi-antenna Gaussian Channels", AT&T Labs Tech. Report, 1995.

[4] G.J. Foschini and M.J. Gans, " On Limits of Wireless Communications in a Fading Environment when Using Multiple Antennas", Wireless Personal Communications, vol 6.,311-335, 1998.

[5] M. Costa, "Writing on Dirty Paper", IEEE. Trans. Inform. Theory, v. 29, pp 439-441, May 1983.

[6] G. Caire and S. Shamai, "On the Achievable Throughput of a Multi-antenna Gaussian Broadcast Channel", submitted to IEEE Trans. on Inform. Theory, July 2001. See also Proc. IEEE Int. Symp. Inf. Theory, ISIT 2001, Washington D.C., June 2001.

[7] T. Cover and J. Thomas, "Elements of Information Theory", J. Wiley, NY.

[8] N. Jindal, S. Vishwanath and A. Goldsmith, "On the Duality of MAC and BC", submitted to IEEE Trans. Inf. Theory. See also Proc. IEEE Int. Symp. Inf. Theory, June 2002, pp. 500.

[9] W. Yu and J. Cioffi, "Sum Capacity of Vector Gaussian Broadcast Channels", submitted to IEEE Trans. inf. Theory. See also http://www.comm.utoronto.ca/ weiyu/publications.html.

[10] P. Viswanath and D. N. Tse, " "Sum Capacity of Vector Gaussian Broadcast Channels", submitted to IEEE Trans. inf. Theory. See also http://degas.eecs.berkeley.edu/ dtse/pub.html.

[11] S. Boyd and L. Vandenberghe, "Introduction to convex optimization with engineering applications," Course Reader, 2001, http://www.stanford.edu/class/ee364.

[12] H. C. Huang, S. Venkatesan and H. Viswanathan, "Downlink Capacity Evaluation of Cellular Networks with Known Interference Cancellation", Proc. DIMACS Workshop on Wireless Commun., Rutgers univ., NJ Oct. 7-9 2002.

[13] A. Goldsmith and P.Varaiya, "Capacity of Fading Channels with Channel Side Information", IEEE Trans. Inf. Theory, Vol. 43, No. 6, pp. 1986-1992, Nov. 1997.

[14] D. N. Tse and S. Hanly, "Multiacess Fading Channels I: Polymatroid Structure, Optimal Resource Allocation and Throughput Capacities", IEEE Trans. Inf. Theory, Vol. 44, Issue 7, pp. 2796-2815, Nov. 1998.

[15] Syed A. Jafar and Andrea J. Goldsmith, "Optimal Power Allocation for Multiuser Multicellular Multiple Antenna Systems", Proc. IEEE Int. Symp. Inf. Theory 2002.