High SNR Analysis for MIMO Broadcast Channels: Dirty Paper Coding Versus Linear Precoding

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***Abstract*—In this correspondence, we compare the achievable throughput for the optimal strategy of dirty paper coding (DPC) to that achieved with suboptimal and lower complexity linear precoding techniques (zero-forcing and block diagonalization). Both strategies utilize all available spatial dimensions and therefore have the same multiplexing gain, but an absolute difference in terms of throughput does exist. The sum rate difference between the two strategies is analytically computed at asymptotically high SNR. Furthermore, the difference is not affected by asymmetric channel behavior when each user has a different average SNR. Weighted sum rate maximization is also considered. In the process, it is shown that allocating user powers in direct proportion to user weights asymptotically maximizes weighted sum rate.**

***Index Terms*—Broadcast channels, channel capacity, multiple-input–multiple-output (MIMO) systems.**

I. Introduction

Dirty paper coding (DPC) was proved to achieve the capacity region of the multiple antenna broadcast channel (BC) [1]. However, implementation of DPC requires significant complexity at both transmitter and receiver, and the problem of finding practical dirty paper codes close to the capacity limit is still open [2]. On the other hand, linear precoding is a low complexity but suboptimal transmission technique (with complexity roughly equivalent to point-to-point multiple-input–multiple-output (MIMO)) that is able to transmit the same number of data streams as a DPC-based system. Linear precoding therefore achieves the same multiplexing gain (which characterizes the slope of the capacity versus SNR curve) as DPC, but incurs an absolute rate/power offset relative to DPC.

The contribution of this work is the quantification of this rate/power offset using the affine approximation developed by Shamai and Verdú [3]. At high SNR, the channel capacity $C(P)$ is well approximated by an affine function of SNR (P)

$$C(P) = \mathcal{S}_\infty \left(\log_2 P - \mathcal{L}_\infty\right) + o(1) \tag{1}$$

where $\mathcal{S}_\infty$ represents the multiplexing gain and $\mathcal{L}_\infty$ represents the power offset (in 3 dB units) that are defined as

$$\mathcal{S}_\infty = \lim_{P\to\infty} \frac{C(P)}{\log_2(P)} \tag{2}$$

$$\mathcal{L}_\infty = \lim_{P\to\infty} \left(\log_2(P) - \frac{C(P)}{\mathcal{S}_\infty}\right). \tag{3}$$

The multiplexing gain $\mathcal{S}_\infty$ is equal to the minimum of the number of transmit and receive antennas (for either point-to-point or downlink MIMO channels), and thus is essentially independent of the fading environment and signaling strategy. However, the power offset $\mathcal{L}_\infty$ does depend on the actual fading statistics and the signaling strategy. Reference [4] provides an exact characterizations of these offset terms for point-to-point MIMO channels for the most common fading models, such as independent and identically distributed (i.i.d.) Rayleigh fading, spatially correlated fading, and Ricean (line-of-sight) fading. Indeed, one of the key insights of [4] is the necessity to consider these rate offset terms, because considering only the multiplexing gain can lead to rather erroneous conclusions, e.g., spatial correlation does not affect MIMO systems at high SNR.

In a similar vein, in this correspondence, we utilize the high SNR approximation to quantify the difference between optimal dirty paper coding and simpler linear precoding in an independent and indentically distributed (i.i.d.) Rayleigh-fading environment. By investigating the differential offsets between these two strategies, we are able to quantify the throughput degradation that results from using linear precoding rather than the optimal DPC strategy in spatially white fading.[1] We are also able to derive simple expressions for the average rate offset as a function of only the number of transmit and receive antennas and users for systems in which the aggregate number of receive antennas is no larger than the number of transmit antennas. Note that past work has analyzed the *ratio* between the sum rate capacity and the linear precoding sum rate [5], [6]. However, such analyses essentially capture only multiplexing gain effects and, thus, are limited in scope. By alternatively studying the absolute difference between the different sum rates, we are able to derive more meaningful and accurate conclusions.

In addition to the sum rate, we study weighted sum rate maximization (using DPC and linear precoding) and provide simple expressions for the rate offsets. One of the most interesting results is that weighted sum rate (for either DPC or linear precoding) is maximized at asymptotically high SNR by allocating power directly proportional to user weights. A similar result was recently observed in [7] in the context of parallel single-user channels (e.g., for OFDMA systems). Because the linear precoding strategies we study result in parallel channels, the result of [7] shows that it is asymptotically optimal to allocate power in direct proportion to user weights whenever linear precoding is used. By showing that weighted sum rate maximization when DPC is employed can also be simplified to power allocation over parallel channels, we are able to show that the same strategy is also asymptotically optimal for DPC.

Because of space limitations, the correspondence is limited to a brief presentation of the main technical results and proofs. For additional interpretation and numerical results, see [8] and [9].

II. System Model

We consider a K-user Gaussian MIMO BC in which the transmitter has M antennas and each receiver has N antennas with $M \geq KN$, i.e., the number of transmit antennas is no smaller than the aggregate number of receive antennas. The received signal $\mathbf{y}_k$ of user k is given by

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{x} + \mathbf{n}_k, \quad k = 1, \ldots, K \tag{4}$$

where $\mathbf{H}_k (\in \mathbb{C}^{N\times M})$ is the channel gain matrix for user k, $\mathbf{x}$ is the transmit signal vector having a power constraint $\mathrm{tr}(\mathbb{E}[\mathbf{x}\mathbf{x}^H]) \leq P$, and $\mathbf{n}_k$ $(k = 1, \ldots, K)$ is complex Gaussian noise with unit variance per vector component (i.e., $\mathbb{E}[\mathbf{n}_k\mathbf{n}_k^H] = \mathbf{I}$). We assume that the transmitter has perfect knowledge of all channel matrices and each

[1] Although we do not pursue this avenue in the present publication, it would also be interesting to investigate the DPC-linear precoding offset in other fading environments, e.g., Ricean and spatially correlated fading. However, one must be careful with respect to channel models because some point-to-point MIMO models do not necessarily extend well to the MIMO broadcast channel. For example, in point-to-point channels spatial correlation captures the effect of sparse scattering at the transmitter and/or receiver and is a function of the angle-of-arrival. In a broadcast channel, the angle-of-arrival is typically different for every receiver because they generally are not physically co-located; as a result, using the same correlation matrix for all receivers is not well motivated in this context.

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receiver has perfect knowledge of its own channel matrix. For the sake of notation, the concatenation of the channels is denoted by $\mathbf{H}^H = \left[\mathbf{H}_1^H \mathbf{H}_2^H \cdots \mathbf{H}_K^H\right] (\in \mathbb{C}^{M \times KN})$, which can be decomposed into row vectors as $\mathbf{H}^H = \left[\mathbf{h}_{1,1}^H \mathbf{h}_{1,2}^H \cdots \mathbf{h}_{1,N}^H \mathbf{h}_{2,1}^H \mathbf{h}_{2,2}^H \cdots \mathbf{h}_{2,N}^H \cdots \mathbf{h}_{K,N}^H\right]$, where $\mathbf{h}_{k,n} (\in \mathbb{C}^{1 \times M})$ is the nth row of $\mathbf{H}_k$. We develop rate offset expressions on a per-realization basis, as well as averaged over the standard i.i.d. Rayleigh-fading distribution, where the entries of $\mathbf{H}$ are i.i.d. complex Gaussian with unit variance.

III. High SNR Sum Rate Approximations

In this section, we compute the affine approximations to the dirty paper coding sum rate and the linear precoding sum rate at high SNR. In the following section, these expressions are used to quantify the sum rate degradation incurred by linear precoding relative to DPC.

A. *Dirty Paper Coding*

The sum rate by DPC, which achieves the sum capacity [10], [11], [12], can be written by the duality of the MIMO broadcast channel (BC) and the MIMO multiple-access channel (MAC) [10]

$$\mathcal{C}_{\mathrm{DPC}}(\mathbf{H}, P) = \max_{\sum_k \mathrm{tr}(\mathbf{Q}_k) \le P} \log_2 \left| \mathbf{I} + \sum_{k=1}^{K} \mathbf{H}_k^H \mathbf{Q}_k \mathbf{H}_k \right| \tag{5}$$

where $\mathbf{Q}_k$ represent the $N \times N$ transmit covariance matrices in the dual MAC. No closed-form solution to (5) (which is convex) is known to exist, but it has been shown that $\mathcal{C}_{\mathrm{DPC}}(\mathbf{H}, P)$ converges (absolutely) to the capacity of the point-to-point MIMO channel with transfer matrix $\mathbf{H}$ whenever $M \ge KN$:

Theorem 1 ([13, Th. 3]): When $M \ge KN$ and $\mathbf{H}$ has full row rank

$$\lim_{P \to \infty} \left[\mathcal{C}_{\mathrm{DPC}}(\mathbf{H}, P) - \log_2 \left| \mathbf{I} + \frac{P}{KN} \mathbf{H}^H \mathbf{H} \right| \right] = 0. \tag{6}$$

Using this result we can make a few important observations regarding the optimal covariance matrices at high SNR. Since

$$\log_2 \left| \mathbf{I} + \frac{P}{KN} \sum_{k=1}^{K} \mathbf{H}_k^H \mathbf{H}_k \right| = \log_2 \left| \mathbf{I} + \frac{P}{KN} \mathbf{H}^H \mathbf{H} \right| \tag{7}$$

choosing each of the dual MAC covariance matrices as $\mathbf{Q}_k = \frac{P}{KN}\mathbf{I}$ in (5) achieves sum capacity at asymptotically high SNR. Thus, uniform power allocation across the KN antennas in the dual MAC is asymptotically optimal. As a result, an affine approximation for the sum rate can be found as

$$\mathcal{C}_{\mathrm{DPC}}(\mathbf{H}, P) \cong KN \log_2 P - KN \log_2 KN + \log_2 \left| \mathbf{H}\mathbf{H}^H \right| \tag{8}$$

where $\cong$ refers to equivalence in the limit (i.e., the difference between both sides converges to zero as $P \to \infty$). Since the MIMO BC and the $M \times KN$ point-to-point MIMO channel are equivalent at high SNR (Theorem 1), the high SNR results developed in [4] directly apply to the sum capacity of the MIMO BC channel. It is important to be careful regarding the ordering of the equivalent point-to-point MIMO channel: due to the assumption that $M \ge KN$, the MIMO BC is equivalent to the $M \times KN$ MIMO channel with CSI at the transmitter, which is equivalent to the $KN \times M$ MIMO channel with or without CSI at the transmitter (i.e., open-loop MIMO). When $M > KN$, the level of CSI at the transmitter affects the rate offset of the $M \times KN$ point-to-point MIMO channel. Finally, notice that the high SNR sum rate capacity only depends on the product of K and N and not on their specific values; this is not the case for linear precoding.

B. *Linear Precoding*

Linear precoding is a low-complexity, albeit suboptimal, alternative to DPC. We consider two linear precoding techniques that cancel out the multiuser interference: block diagonalization (BD) and zero-forcing (ZF). Note that eliminating multiuser interference is desirable at high SNR in order to prevent interference-limited behavior.

The sum rate by BD is given by [14], [15]

$$\mathcal{C}_{\mathrm{BD}}(\mathbf{H}, P) = \max_{\mathbf{Q}_k : \sum_{k=1}^{K} \mathrm{tr}\{\mathbf{Q}_k\} \le P} \sum_{k=1}^{K} \log_2 \left| \mathbf{I} + \mathbf{G}_k^H \mathbf{Q}_k \mathbf{G}_k \right| \tag{9}$$

where $\mathbf{G}_k (= \mathbf{H}_k \mathbf{V}_k$, $\mathbf{V}_k$ is the precoding matrix) is the effective channel matrix for user k and the optimal rate is achieved asymptotically by uniform power allocation at high SNR since the channel can be decomposed into parallel channels. Hence, the sum rate is asymptotically given by

$$\mathcal{C}_{\mathrm{BD}}(\mathbf{H}, P) \cong KN \log_2 P - KN \log_2 KN + \log_2 \prod_{k=1}^{K} \left| \mathbf{G}_k \mathbf{G}_k^H \right|. \tag{10}$$

As a special case, the sum rate by ZF is similarly given by

$$\mathcal{C}_{\mathrm{ZF}}(\mathbf{H}, P) \cong KN \log_2 P - KN \log_2 KN + \log_2 \prod_{k=1}^{K} \prod_{n=1}^{N} |g_{k,n}|^2 \tag{11}$$

where $g_{k,n} = \mathbf{h}_{k,n} \mathbf{v}_{k,n}$ and $\mathbf{v}_{k,n}$ denotes the nth column vector of $\mathbf{V}_k$.

C. *Equivalent MIMO Interpretation*

Due to the properties of i.i.d. Rayleigh fading, systems employing linear precoding like ZF or BD are equivalent to parallel point-to-point MIMO channels, as shown in [14]. When ZF is used, the precoding vector for each receive antenna (i.e., each row of the concatenated channel matrix $\mathbf{H}$) must be chosen orthogonal to the other $KN - 1$ rows of $\mathbf{H}$. Due to the isotropic nature of i.i.d. Rayleigh fading, this orthogonality constraint consumes $KN - 1$ degrees of freedom at the transmitter, and reduces the channel from the $1 \times M$ vector $\mathbf{h}_{k,n}$ to a $1 \times (M - KN + 1)$ Gaussian vector. As a result, the effective channels norm $|g_{k,n}|^2$ of each parallel channel is chi-squared with $2(M - KN + 1)$ degrees of freedom (denoted $\chi^2_{2(M-KN+1)}$). Therefore, a ZF-based system with uniform power loading is exactly equivalent (in terms of ergodic throughput) to KN parallel $1 \times (M - KN + 1)$ MIMO channels.

When BD is used, the orthogonality constraint consumes $(K-1)N$ degrees of freedom. This reduces the channel matrix $\mathbf{H}_k$, which is originally $N \times M$, to a $N \times (M - (K-1)N)$ complex Gaussian matrix. As a result, the $N \times N$ matrix $\mathbf{G}_k \mathbf{G}_k^H$ is Wishart with $M - (K-1)N$ degrees of freedom, and therefore a BD-based system is equivalent to K parallel $N \times (M - (K-1)N)$ parallel MIMO channels.

Finally, when DPC is used, the MIMO broadcast channel is equivalent to the $M \times KN$ point-to-point MIMO channel, where $M \ge KN$ and CSIT is again assumed. Note that a MIMO channel of this dimension can be interpreted as a series of parallel channels as well: in this case, the $M \times KN$ channel is equivalent to $1 \times M, 1 \times (M - 1), \ldots, 1 \times (M - KN + 1)$ channels in parallel [16].

For all three cases, the MIMO equivalence is exact when uniform power loading is used. If optimal power allocation is performed (for either ZF, BD, or DPC) the MIMO broadcast systems can achieve a larger ergodic throughput than the MIMO equivalent at finite SNR, but this advantage disappears at asymptotically high SNR because water-filling provides a vanishing benefit.

IV. High SNR Offset Calculations

We define the rate loss as the asymptotic (in SNR) difference between the sum rate capacity (DPC) and the BD sum rate

$$\beta_{\mathrm{DPC-BD}}(\mathbf{H}) \triangleq \lim_{P\to\infty} \left[\mathcal{C}_{\mathrm{DPC}}(\mathbf{H},P) - \mathcal{C}_{\mathrm{BD}}(\mathbf{H},P)\right]. \quad (12)$$

Since each of the capacity curves has a slope of $\frac{KN}{3}$ in units of bps/Hz/dB, this rate offset (i.e., the vertical offset between capacity versus SNR curves) can be immediately translated into a power offset (i.e., a horizontal offset): $\Delta_{\mathrm{DPC-BD}}(\mathbf{H}) = \frac{3}{KN}\beta_{\mathrm{DPC-BD}}(\mathbf{H})$ dB. Because $\Delta_{\mathrm{DPC-BD}}$ is in dB units, we clearly have $\Delta_{\mathrm{DPC-BD}}(\mathbf{H}) = 3\left(\mathcal{L}_\infty^{\mathrm{BD}}(\mathbf{H}) - \mathcal{L}_\infty^{\mathrm{DPC}}(\mathbf{H})\right)$. In terms of the high SNR approximation, the rate offset is precisely the difference between the $\mathcal{L}_\infty$ terms for DPC and linear precoding multiplied by $\mathcal{S}_\infty$.

From the affine approximation to DPC and BD sum rate found in (8) and (10), the rate loss incurred by BD is

$$\beta_{\mathrm{DPC-BD}}(\mathbf{H}) = \log_2 \frac{|\mathbf{H}^H\mathbf{H}|}{\prod_{k=1}^K |\mathbf{G}_k\mathbf{G}_k^H|}. \quad (13)$$

By averaging across the fading distribution, we can calculate the average rate offset: $\bar{\beta}_{\mathrm{DPC-BD}} \triangleq \mathbb{E}_{\mathbf{H}}\left[\beta_{\mathrm{DPC-BD}}(\mathbf{H})\right]$, which allows a comparison of ergodic throughput. Likewise, the average power offset (denoted as $\bar{\Delta}_{\mathrm{DPC-BD}}$) can be calculated in the same fashion. Since the matrices $\mathbf{H}^H\mathbf{H}$ and $\mathbf{G}_k\mathbf{G}_k^H$ are Wishart under i.i.d. Rayleigh fading, we can get a simple closed form expression for the rate offset[2]:

Theorem 2: The expected loss in Rayleigh fading due to block diagonalization is given by

$$\bar{\beta}_{\mathrm{DPC-BD}}(M,K,N) = (\log_2 e)\sum_{k=0}^{K-1}\sum_{n=0}^{N-1}\sum_{i=kN+1}^{(K-1)N}\frac{1}{M-n-i} \text{ (bps/Hz)}. \quad (14)$$

Proof: From (8) and (10), $\bar{\beta}_{\mathrm{DPC-BD}}$ is given by

$$\bar{\beta}_{\mathrm{DPC-BD}} = \mathbb{E}[\log_2 \left|\mathbf{H}^H\mathbf{H}\right|] - K\,\mathbb{E}\left[\log_2\left|\mathbf{G}\mathbf{G}^H\right|\right]$$

where $\mathbf{G}\mathbf{G}^H$ is $N\times N$ Wishart wth $M-(K-1)N$ degrees of freedom. Using the property of Wishart matrix [17], we have

$$\frac{\bar{\beta}_{\mathrm{DPC-BD}}}{\log_2 e} = \sum_{l=0}^{KN-1}\psi(M-l) - K\sum_{n=0}^{N-1}\psi(M-(K-1)N-n)$$

and the result follows from an expansion of the digamma function and some algebraic manipulations. □

We can easily determine the zero forcing offset by noting that it depends only on the product KN

$$\begin{aligned}\bar{\beta}_{\mathrm{DPC-ZF}}(M,K,N) &= \bar{\beta}_{\mathrm{DPC-ZF}}(M,KN,1)\\ &= \bar{\beta}_{\mathrm{DPC-BD}}(M,KN,1).\end{aligned} \quad (15)$$

[2]It is straightforward to use this result to determine the rate offset of BD in terms of the rate offset $\mathcal{L}_\infty^{\mathrm{MIMO}}(KN,M)$ of a $KN\times M$ MIMO channel in i.i.d. Rayleigh fading given in Proposition 1 of [4]: $\mathcal{L}_\infty^{\mathrm{BD}}(M,K,N) = \mathcal{L}_\infty^{\mathrm{MIMO}}(KN,M) + \frac{1}{KN}\bar{\beta}_{\mathrm{DPC-BD}}(M,K,N)$.

In Sections IV-A–C we gain some insight into these results by first further studying zero forcing and then considering block diagonalized systems.

A. Zero Forcing

To understand the rate penalty associated with zero forcing, we study the behavior of the offset as system size increases. The first case of interest is when $M = KN$, i.e., the total number of receive antennas is equal to the number of transmit antennas. In Appendix A we show that the offset in this scenario can be well approximated as

$$\bar{\beta}_{\mathrm{DPC-ZF}}(M,M,1) \approx M\log_2 M \text{ (bps/Hz)} \quad (16)$$

in the sense that the ratio of both sides converges to one as M grows large. In this scenario, the ZF sum rate is associated with the capacity of M parallel 1×1 (SISO) channels while the DPC sum rate is associated with an $M\times M$ MIMO channel. This corresponds to a power offset of $3\log_2 M$ (dB), which is very significant when M is large. Numerical results show that the approximation $3\log_2 M$ (dB) overstates the power penalty by 1 to 1.5 dB for reasonable values of $M(<20)$, but it does capture the growth rate correctly. Such a large penalty is not surprising, since the use of zero-forcing requires inverting the $M\times M$ matrix $\mathbf{H}$, which is poorly conditioned with high probability when M is large.

The behavior of zero-forcing is quite different if the number of receivers is strictly smaller than M. If system size increases such that $M,K\to\infty$ with $M=\alpha KN$ for some $\alpha>1$, the power offset converges to a constant (see proof in Appendix B)

$$\bar{\Delta}_{\mathrm{DPC-ZF}}(\alpha) = -3\left(\log_2 e + \alpha\log_2\left(1-\frac{1}{\alpha}\right)\right) \text{ (dB)}. \quad (17)$$

Thus, for large systems, ZF is a viable low-complexity alternative to DPC if the number of transmit antennas can be made suitably large. A similar conclusion was drawn in [18] where the ratio of the rates achievable with ZF relative to the sum capacity is studied. Note that using ZF on the MIMO downlink channel is identical to using a decorrelating receiver on the multiple antenna uplink channel or in a randomly spread CDMA system; see [3, eq. (152)] for the asymptotic performance of the decorrelating CDMA receiver.

B. Block Diagonalization

To gain some insight into Theorem 2, we first note a simple property of the rate offset. If the number of transmit antennas M is kept fixed but N is increased and K is decreased such that KN is constant, i.e., the number of antennas per receiver is increased but the aggregate number of receive antennas is kept constant, then the rate offset decreases. Indeed, this observation can be reached by simply considering the equivalent MIMO channels discussed in Section III-C.

It is also very useful to analyze the offset between BD (K receivers with N antennas each) and ZF (equivalent to KN receivers with 1 antenna each). Some simple manipulations of the earlier results yield the following theorem:

Theorem 3: If $M=\alpha KN$ with $N>1$ and $\alpha\geq 1$, the expected throughput gain of BD relative to ZF is

$$\begin{aligned}\bar{\beta}_{\mathrm{BD-ZF}} &\triangleq \bar{\beta}_{\mathrm{DPC-ZF}}(M,NK) - \bar{\beta}_{\mathrm{DPC-BD}}(M,N,K)\\ &= (\log_2 e)K\sum_{j=1}^{N-1}\frac{(N-j)}{(\alpha-1)KN+j} \text{ (bps/Hz)}.\end{aligned} \quad (18)$$

Furthermore, when $M = KN$ the power offset between BD and ZF is a function only of N (i.e., is independent of the particular values of M and K)

$$\bar{\Delta}_{\mathrm{BD-ZF}}(N) = \frac{3(\log_2 e)}{N} \sum_{j=1}^{N-1} \frac{N-j}{j} (\mathrm{dB}). \tag{19}$$

For example, consider two system configurations: (i) $\frac{M}{2}$ receivers each have two antennas, and (ii) M receivers each have one antenna. Equation (19) indicates that the power advantage of using BD in the $N = 2$ system is $\bar{\Delta}_{\mathrm{BD-ZF}}(2) = 2.1640\ (\mathrm{dB})$ relative to performing ZF that is independent of M.

C. Unequal Average SNR's

Near-far effects in a wireless broadcast channel environment can lead to asymmetric channel gains; i.e., the channel gain of user k is now $\mathbf{H}_k = \sqrt{\gamma_k}\tilde{\mathbf{H}}_k$, where γ_k denotes the average SNR of user k. The elements of $\tilde{\mathbf{H}}_k$ have Gaussian distribution with mean zero and unit variance by assumption. It is important to note that the uniform power allocation is still asymptotically optimal even when users' SNR are asymmetric since $M \geq KN$. Thus, we can derive that all the sum rates by DPC, ZF, and BD are shifted equally by $N \sum_{k=1}^{K} \log_2 \gamma_k$. As a result, the DPC-ZF and DPC-BD offsets are unaffected.

V. Weighted Sum Rate Analysis

In this section we generalize the rate offset analysis to weighted sum rate. The sum rate offset quantifies the difference between the sum rate points of both regions; the weighted sum rate offset is intended to describe the offset for the other portions of the rate region.

We first show that allocating power in proportion to user weights is asymptotically optimal for either DPC or BD, and then use this result to compute the associated rate offsets.

A. Asymptotically Optimal Power Allocation

Without loss of generality, we assume user weights are given in descending order: $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_K \geq 0$ with $\sum_{k=1}^{K} \mu_k = 1$. The maximum weighted sum rate problem (DPC) can be written in terms of the dual MAC as:

$$\mathcal{C}_{\mathrm{DPC}}(\boldsymbol{\mu}, \mathbf{H}, P) = \max_{\sum_{k=1}^{K} \mathrm{tr}(\mathbf{Q}_k) \leq P} \sum_{k=1}^{K} \mu_k \log_2 \frac{|\mathbf{A}^{(k)}|}{|\mathbf{A}^{(k-1)}|} \tag{20}$$

where $\mathbf{A}^{(k)} = \mathbf{I} + \sum_{j=1}^{k} \mathbf{H}_j^H \mathbf{Q}_j \mathbf{H}_j$ for $k \geq 1$ and $\mathbf{A}^{(0)} = \mathbf{I}$. Notice that the uplink (or the dual MAC) decoding is done in order of increasing weight, i.e., user K does not get the benefit of any interference cancellation while user 1's signal benefits from full interference cancellation and is thus detected in the presence of only noise. From the construction of $\mathbf{A}^{(k)}$, we have

$$\frac{|\mathbf{A}^{(k)}|}{|\mathbf{A}^{(k-1)}|} = \left| \mathbf{I} + \mathbf{Q}_k \mathbf{H}_k (\mathbf{A}^{(k-1)})^{-1} \mathbf{H}_k^H \right|.$$

The following lemma shows that if we limit ourselves to linear power allocation policies of the form $\mathrm{tr}(\mathbf{Q}_k) = \alpha_k P$, then the objective function in (20) can be decoupled at high SNR.

Lemma 1: Let $\mathbf{F}_k$ ($k = 1, \ldots, K$) be the projection of $\mathbf{H}_k$ onto the nullspace of $\{\mathbf{H}_j\}_{j=1}^{k-1}$. If $M \geq KN$, then

$$\lim_{P \to \infty} \left[\mathbf{H}_k (\mathbf{A}^{(k-1)})^{-1} \mathbf{H}_k^H - \mathbf{F}_k \mathbf{F}_k^H \right] = 0, \quad k = 1, \ldots, K. \tag{21}$$

Proof: See Appendix C. □

Once the weighted sum rate maximization has been decoupled into the problem of maximizing weighted sum rate over parallel single-user channels, we can use the result of [7] to show that the optimal power allocation is of the form $P_k^* = \mu_k P + O(1)$.

Theorem 4: When $M \geq KN$, allocating power according to

$$\mathbf{Q}_k = \frac{\mu_k P}{N} \mathbf{I}, \quad k = 1, \ldots, K. \tag{22}$$

asymptotically achieves the optimal solution to (20) at high SNR.

Proof: See Appendix D. □

Theorem 4 generalizes the fact that uniform power allocation achieves the maximum sum rate asymptotically at high SNR. That is, for the sum rate problem the weights are the same (i.e., $\mu_1 = \cdots = \mu_K = 1/K$), thus the uniform power policy is asymptotically optimal.

Meanwhile, the weighted sum rate of BD is given by

$$\mathcal{C}_{\mathrm{BD}}(\boldsymbol{\mu}, \mathbf{H}, P) = \max_{\mathbf{Q}_k : \sum_{k=1}^{K} \mathrm{tr}(\mathbf{Q}_k) \leq P} \sum_{k=1}^{K} \mu_k \log_2 \left| \mathbf{I} + \mathbf{Q}_k \mathbf{G}_k \mathbf{G}_k^H \right| \tag{23}$$

where $\mathbf{G}_k$ is the projection of $\mathbf{H}_k$ onto the null space of $\{\mathbf{H}_1, \ldots, \mathbf{H}_{k-1}, \mathbf{H}_{k+1}, \ldots, \mathbf{H}_K\}$. (cf. $\mathbf{F}_k$ in Lemma 1 is the projection of $\mathbf{H}_k$ onto the null space of $\{\mathbf{H}_1, \ldots, \mathbf{H}_{k-1}\}$.) Likewise, the optimization (23) is the same as the optimization (29) and (30) except that $\mathbf{F}_k$ is replaced by $\mathbf{G}_k$ which does not contribute to the asymptotic solution. Thus, the power allocation policy in (22) is also the asymptotic solution to (23). By the same token, the weighted sum rate optimization by ZF can be easily solved at high SNR.

B. Rate Loss

Using the asymptotically optimal power allocation policy of (22), the difference between the weighted sum rates by DPC and BD can be found as

$$\beta_{\mathrm{DPC-BD}}(\boldsymbol{\mu}, \mathbf{H}) = \sum_{k=1}^{K} \mu_k \log_2 \frac{|\mathbf{F}_k \mathbf{F}_k^H|}{|\mathbf{G}_k \mathbf{G}_k^H|} \tag{24}$$

per realization. In Rayleigh fading, the distribution of $\mathbf{F}_k \mathbf{F}_k^H$ is Wishart with $M - (k-1)N$ degrees of freedom while the distribution of $\mathbf{G}_k \mathbf{G}_k^H$ is Wishart with $M - (K-1)N$ degrees of freedom. Thus, we can compute the expected loss using the property of Wishart matrix

$$\bar{\beta}_{\mathrm{DPC-BD}}(\boldsymbol{\mu}, M, K, N) = (\log_2 e) \sum_{k=1}^{K} \mu_k \left(\sum_{n=0}^{N-1} \sum_{j=M-(K-1)N-n}^{M-(k-1)N-n-1} \frac{1}{j} \right). \tag{25}$$

As done for the sum rate analysis, we can also calculate the rate/power offsets between BD and ZF with simple algebraic manipulations.

Besides that the sum rate problem is a special case of weighted sum rate problems, the sum rate has another property in terms of the rate offset. The expected rate offset is minimized at the sum rate; i.e., when $\mu_1 = \cdots = \mu_k = \frac{1}{K}$. If we let $\zeta_k = \sum_{n=0}^{N-1} \sum_{j=M-(K-1)N-n}^{M-(k-1)N-n-1} \frac{1}{j}$, then then $\zeta_1 > \zeta_2 > \cdots > \zeta_K$ and $\bar{\beta}_{\mathrm{DPC-BD}} = (\log_2 e) \sum_{k=1}^{K} \mu_k \zeta_k$. Since $\{\mu_k\}$ has constraints of $\mu_1 \geq \cdots \geq \mu_K$, $\sum_{k=1}^{K} \mu_1 = 1$, and $\mu_k \geq 0 (1 \leq k \leq K)$, $\bar{\beta}_{\mathrm{DPC-BD}}$ achieves minimum at $\mu_1 = \cdots = \mu_k = \frac{1}{K}$ for a given $\{\zeta_k\}$.

VI. Conclusion

We have investigated the difference between the throughputs achieved by DPC relative to those achieved with linear precoding strategies. When the aggregate number of receive antennas is equal or slightly less than the number of transmit antennas, linear precoding incurs a rather significant penalty relative to DPC, but this penalty is much smaller when the number of transmit antennas is large relative to the number of receive antennas. Additionally, one interesting finding is that allocating power directly proportional to user weights is asymptotically optimal for DPC at high SNR. This simple yet asymptotically optimal power policy may prove to be useful in other setting such as opportunistic scheduling.

Our analysis is limited to channels for more transmit antennas than aggregate receive antennas. If not, no MIMO equivalent channel exists for either DPC or linear precoding. The sum capacity (DPC) is smaller than the capacity of the $M \times KN$ (forward) cooperative channel (in which CSIT is not required at high SNR), but is larger than the capacity of the reverse $KN \times M$ cooperative channel without CSIT. Additionally, ZF and BD are not feasible. Thus, some form of selection (of users and possibly of the number of data streams per receiver) must be performed. As a result of these complications, it does not appear that the high SNR framework will yield closed-form solutions for either DPC or linear precoding when $M < KN$.

Appendix A
Derivation of Equation (16)

Let S_M be the expected rate loss with M antennas; i.e., $S_M = \bar{\beta}_{\mathrm{DPC-ZF}}(M, M, 1)$. Then, we have

$$S_{M+1} - S_M = \sum_{i=1}^{M} \frac{1}{i} \leq 1 + \log_e M, \ \text{for } M \geq 1 \qquad (26)$$

since $\log_e M = \int_1^M \frac{1}{x} dx \geq \sum_{i=2}^{M} \frac{1}{i}$. If we let $f(M) \triangleq M \log_e M$, then $f(M+1) \geq f(M) + 1 + \log_e M$. Since $S_{M+1} - S_M \leq 1 + \log_e M$ and $f(1) = S_1 = 0$, $S_M \leq M \log_e M$ for all $M \geq 1$.

Now we show that S_M converges to $M \log_e M$. We do this by showing that $S_M \geq \theta M \log_e M$ for any $0 < \theta < 1$ for all M larger than some M_0. First notice that $S_{M+1} - S_M = \sum_{i=1}^{M} \frac{1}{i} \geq \log_e M$. Let $g(M) \triangleq \theta M \log M$ for some $0 < \theta < 1$. Then, $g(M+1) \leq g(M) + g'(M+1) = g(M) + \theta + \theta \log_e(M+1)$. Therefore we have

$$\begin{aligned} g(M+1) - S_{M+1} \leq (f(M) - S_M) \\ + \theta + \theta \log_e(M+1) - \log_e M. \end{aligned} \qquad (27)$$

Notice that the term $\theta + \theta \log_e(M+1) - \log_e M$ is a monotonically decreasing function that goes to $-\infty$. Thus, any positive gap between $g(M)$ and S_M must close and go to $-\infty$, i.e., $S_M \geq g(M)$ for sufficiently large M. As a consequence of this, $\lim_{M\to\infty} \frac{S_M}{\theta M \log_e M} \geq 1$, or $\lim_{M\to\infty} \frac{S_M}{M \log_e M} \geq \theta$ for any $\theta < 1$. Since $\frac{S_M}{M \log_e M}$ is bounded above by 1, it must converge; i.e.

$$\lim_{M\to\infty} \frac{S_M}{M \log_e M} = 1 \qquad (28)$$

as desired.

Appendix B
Derivation of Equation (17)

From Theorem 2, if $M = \alpha KN$, the expected power offset, which is now a function of α and KN, can be expressed as

$$\begin{aligned} \bar{\Delta}_{\mathrm{DPC-ZF}}(\alpha, KN) &= \frac{3 \log_2 e}{KN} \sum_{j=1}^{KN-1} \frac{j}{M-j} \\ &= 3 \log_2 e \sum_{j=1}^{KN-1} \frac{\frac{j}{KN}}{\alpha - \frac{j}{KN}} \frac{1}{KN}. \end{aligned}$$

Let $f(x) = \frac{x}{\alpha - x}$ $(\alpha > 1)$, for $x \in [0, 1]$. Then, by the property of integration, we can find the limit of $\bar{\Delta}_{\mathrm{DPC-ZF}}$ as $KN \to \infty$

$$\begin{aligned} \bar{\Delta}_{\mathrm{DPC-ZF}}(\alpha) &= \lim_{KN\to\infty} \bar{\Delta}_{\mathrm{DPC-ZF}}(\alpha, KN) \\ &= 3 \log_2 e \int_0^1 f(x) dx. \end{aligned}$$

Thus, we have the result.

Appendix C
Proof of Lemma 1

If we let the eigenvector matrix and eigenvalues of $\sum_{j=1}^{k-1} \mathbf{H}_j^H \mathbf{Q}_j \mathbf{H}_j$ be $\mathbf{U}$ and $\lambda_1, \ldots, \lambda_{k-1}$ with $\lambda_j > 0$, then

$$(\mathbf{A}^{(k-1)})^{-1/2} = \mathbf{U} \mathbf{\Lambda} \mathbf{U}^H$$

where

$$\mathbf{\Lambda} = \mathrm{diag}\left(\frac{1}{\sqrt{1+\lambda_1}}, \ldots, \frac{1}{\sqrt{1+\lambda_{k-1}}}, 1, \ldots, 1 \right).$$

As P goes to infinity, λ's tend to infinity. Thus, the first $k-1$ eigenvalues of $\mathbf{\Lambda}$ converge to 0. The eigenvectors corresponding to the unit eigenvalues span the nullspace $\{\mathbf{H}_j\}_{j=1}^{k-1}$; i.e.

$$\lim_{P\to\infty} \left[\mathbf{H}_k (\mathbf{A}^{(k-1)})^{-1/2} - \mathbf{F}_k \right] = 0.$$

This completes the proof.

Appendix D
Proof of Theorem 4

By Lemma 1, the optimization (20) can be decomposed into the two optimizations at high SNR

$$\mathcal{C}_{\mathrm{DPC}}(\boldsymbol{\mu}, \mathbf{H}, P) \cong \max_{\sum_{k=1}^{K} P_k \leq P} \sum_{k=1}^{K} \mu_k \xi_k(P_k) \qquad (29)$$

where

$$\xi_k(P_k) = \max_{\mathrm{tr}(\mathbf{Q}_k) = P_k} \log_2 \left| \mathbf{I} + \mathbf{Q}_k \mathbf{F}_k \mathbf{F}_k^H \right|. \qquad (30)$$

By applying the high SNR affine approximation to (30) and applying [3, Th. 3], we have the result.

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On the Multivariate Conditional Probability Density of a Vector Perturbed by Gaussian Noise

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***Abstract*—This correspondence examines the joint conditional probability density function (pdf) of the main variables (envelope, phase, and their n-order time derivatives) of a time-varying random signal in the presence of additive Gaussian noise. The main variables are conditioned with respect to the given variables, which are the amplitude, phase, and their derivatives of the signal alone. We prove a theorem stating that some of the conditional pdfs of the main variables do not depend on some of the given variables. This theorem, together with Bayes's theorem, can substantially simplify the derivations of conditional pdfs and give alternative forms of them. Both theorems can also help in finding reasonable approximations, as we demonstrate for the phase and first time derivative of the envelope.**

***Index Terms*—Joint probability density function, phase, random signal, time derivative of the envelope, time-varying vector.**

I. INTRODUCTION

Statistical properties of an information bearing narrowband signal perturbed by noise have been studied for decades beginning with the early works of Rice. In spite of this, as will become clear in the sequel, one of the important problems still remains unsolved.

Most generally, a signal is represented in the form of

$$\begin{aligned} s(t) &= \sqrt{2S(t)}\cos[\omega_0 t + \vartheta(t)] \\ &= U_c(t)\cos\omega_0 t - U_s(t)\sin\omega_0 t \end{aligned} \tag{1}$$

where $U_c = U\cos\vartheta$, $U_s = U\sin\vartheta$, $2S(t)$ is an instantaneous power, and ω_0 is an angular carrier frequency. Here $U(t) = \sqrt{2S(t)}$ and $\vartheta(t)$ are the signal amplitude and phase, respectively.

A common case is that, at a receiver, (1) is perturbed by narrowband Gaussian noise, whose model is

$$\begin{aligned} \xi(t) &= A(t)\cos[\omega_0 t + \phi(t)] \\ &= A_c(t)\cos\omega_0 t - A_s(t)\sin\omega_0 t \end{aligned} \tag{2}$$

where $A_c = A\cos\phi$ and $A_s = A\sin\phi$ are orthogonal, low-pass, stationary, and zero-mean Gaussian processes with equal variances $\sigma^2 = \sigma_c^2 = \sigma_s^2$. Also, it is supposed that $\xi(t)$ is continuous and multiply differentiable. Both $s(t)$ and $\xi(t)$ are mixed at the receiver additively, so that the signal becomes noisy

$$v(t) = V(t)\cos[\omega_0 t + \varphi(t)] \tag{3a}$$

$$= V_c(t)\cos\omega_0 t - V_s(t)\sin\omega_0 t \tag{3b}$$

$$= (U_c + A_c)\cos\omega_0 t - (U_s + A_s)\sin\omega_0 t \tag{3c}$$

where $V_c = V\cos\varphi$, $V_s = V\sin\varphi$, $V(t)$ is the envelope, and $\varphi(t)$ is the phase.

Equating the amplitudes of the harmonic functions in (3b) and (3c) produces the Gaussian variables

$$A_c(t) = V(t)\cos\varphi(t) - U(t)\cos\vartheta(t) \tag{4}$$

$$A_s(t) = V(t)\sin\varphi(t) - U(t)\sin\vartheta(t) \tag{5}$$

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