Problem 8.2

If $x(t)$ is a Poisson process (the textbook uses the notation $N(t)$) with parameter $\lambda$, then the number of arrivals between times $t$ and $t + \tau$ is Poisson distributed with parameter $\lambda\tau$ (assuming $\tau > 0$).

This means that $x(t)$ is equal to the number of times some event has occurred (this can be customers arriving at a lunch counter, or earthquakes occurring) up till time $t$. For example, if $x(1) = 2$, then 2 customers arrived before time 2, and if $x(2) = 2$, then we know that no new customers arrived between times 1 and 2. Clearly, $x(t)$ will be either constant or increasing in $t$.

Let $M$ be the number of occurrences between times $t$ and $t + \tau$, i.e., $M = x(t + \tau) - x(t)$. Then $M$ is distributed as follows:

$$\text{Prob}(M = k) = \frac{(\lambda\tau)^k}{k!} e^{-\lambda\tau}, \text{ for } k = 0, 1, 2, 3, \ldots$$

Now we’ll get a little comfortable with summing these probabilities up. What is the probability that $M$ is a non-negative integer? This is $\text{Prob}(M = 0) + \text{Prob}(M = 1) + \text{Prob}(M = 2) + \ldots$, which is $\sum_{i=0}^{\infty} \text{Prob}(M = i)$. Remember that the Taylor’s series expansion for $e^x$ is $\sum_{i=0}^{\infty} \frac{x^i}{i!}$, we’ll use this to find the value of $\sum_{i=0}^{\infty} \text{Prob}(M = i)$:

$$\text{Prob}(M = \text{any non-zero integer}) = \sum_{i=0}^{\infty} \text{Prob}(M = i)$$

$$= \sum_{i=0}^{\infty} \left( \frac{(\lambda\tau)^i}{i!} e^{-\lambda\tau} \right)$$

$$= e^{-\lambda\tau} \left( \sum_{i=0}^{\infty} \frac{(\lambda\tau)^i}{i!} \right)$$

$$= e^{-\lambda\tau} e^{\lambda\tau}$$

$$= 1$$

This should come as no surprise, since we know that $M$ is 0 or 1 or 2 or 3, etc., with probability 1 (this is just the sum over the PDF, which is 1).

Now we’ll compute the probability that $M$ is even:

$$\text{Prob}(M = \text{any even integer}) = \sum_{i=0,2,4,\ldots} \text{Prob}(M = i)$$

$$= \sum_{i=0,2,4,\ldots} \left( \frac{(\lambda\tau)^i}{i!} e^{-\lambda\tau} \right)$$

$$= \frac{e^{-\lambda\tau}}{2} \left( \sum_{i=0,2,4,\ldots} \frac{(\lambda\tau)^i}{i!} + \sum_{i=0,2,4,\ldots} \frac{(\lambda\tau)^i}{i!} \right)$$

$$= \frac{e^{-\lambda\tau}}{2} \left( \sum_{i=0,2,4,\ldots} \frac{(\lambda\tau)^i}{i!} + \sum_{i=1,3,5,\ldots} \frac{(\lambda\tau)^i}{i!} - \sum_{i=1,3,5,\ldots} \frac{(\lambda\tau)^i}{i!} + \sum_{i=0,2,4,\ldots} \frac{(\lambda\tau)^i}{i!} \right)$$
\[
\begin{align*}
&= \frac{e^{-\lambda \tau}}{2} \left( \sum_{i=0}^{\infty} \frac{(\lambda \tau)^i}{i!} - \sum_{i=1,3,5,...} (\lambda \tau)^i i! + \sum_{i=2,4,...} (\lambda \tau)^i i! \right) \\
&= \frac{e^{-\lambda \tau}}{2} \left( \sum_{i=0}^{\infty} \frac{(\lambda \tau)^i}{i!} + \sum_{i=0}^{\infty} (-1)^i \frac{(\lambda \tau)^i}{i!} \right) \\
&= \frac{e^{-\lambda \tau}}{2} \left( \sum_{i=0}^{\infty} \frac{(\lambda \tau)^i}{i!} + \sum_{i=0}^{\infty} \frac{(-\lambda \tau)^i}{i!} \right) \\
&= \frac{e^{-\lambda \tau}}{2} \left( e^{\lambda \tau} + e^{-\lambda \tau} \right) \\
&= \frac{1}{2} \left( 1 + e^{-2\lambda \tau} \right)
\end{align*}
\]

You can similarly find the probability that \( M \) is odd.