Problem 8.3

Take a look at the walkthrough for problem 8.2 for a description of the Poisson process. Now, we will use yet another property of Poisson processes (all of this is in pages 363-364 in the textbook).

If we let $M_{01}$ be the number of occurrences between times $t - \tau_1$ and $t - \tau_0$, so $M_{01} = x(t - \tau_0) - x(t - \tau_1)$, and if $M_{12}$ is the number of occurrences between times $t - \tau_2$ and $t - \tau_1$, so $M_{12} = x(t - \tau_1) - x(t - \tau_2)$, then $M_{01}$ and $M_{12}$ will be independent, because the time intervals $(t - \tau_1, t - \tau_0]$ and $(t - \tau_2, t - \tau_1]$ are non-overlapping. We can extend this result, and conclude that $M_{01}, M_{12}, M_{23}, M_{34}, \ldots$ are all independent, as long as $\tau_0 < \tau_1 < \tau_2 < \tau_3 < \tau_4 < \ldots$. Further, if we set $M = x(t) - x(t - \tau_0)$, and $N = x(t - \tau_0) - x(0)$, then by the same argument, $M$ will be independent of $M_{01}, M_{12}, M_{23}, M_{34}, \ldots$, and $M$ will also be independent of $N$. Since the Poisson process starts at the origin, $x(0) = 0$, so $N = x(t - \tau_0)$.

The trick to figuring out what is independent here is to write everything in terms of increments, i.e., differences between values of $x(t)$ evaluated at various points in time. Then, we know that if the time intervals are non-overlapping, the differences must be independent. In case we have to deal with a lone quantity, like $x(t - \tau_0)$, that is not written in the form of a difference between 2 $x(t)$ values, we can always write it as $x(t - \tau_0) = x(t - \tau_0) - x(0)$, because $x(0) = 0$.

One direct implication of the independence property is the following conditional expectation:

$$E[M|N, M_{01}, M_{12}, M_{23}, M_{34}, \ldots] = E[M]$$

that is, the estimate of $M$ does not depend on $N,M_{01}, M_{12}, M_{23}, M_{34}, \ldots$. We can then $E[M]$ as follows:

$$E[M] = E[x(t) - x(t-\tau_0)]$$

$$= \lambda \tau_0$$

because we know that the quantity $x(t) - x(t-\tau_0)$ is Poisson distributed with parameter $\lambda \tau_0$ (look at the solution to problem 8.2 if you’re confused about this), and Poisson distributed random variables have a mean equal to the parameter $\lambda \tau_0$.

The problem asks us to estimate $x(t)$ given $x(t - \tau_i), i = 0, 1, 2, \ldots, n$. This estimate is:

$$\hat{x}_{MMSE} = E[x(t)|x(t-\tau_0), x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_n)]$$

$$= E[x(t)|x(t-\tau_0), x(t-\tau_1), x(t-\tau_2) - x(t-\tau_1), \ldots, x(t-\tau_n) - x(t-\tau_{n-1})]$$

$$= E[x(t)|x(t-\tau_0), M_{01}, M_{12}, \ldots, M_{n,n-1}]$$

$$= E[M + x(t-\tau_0)|x(t-\tau_0), M_{01}, M_{12}, \ldots, M_{n,n-1}]$$

$$= E[M] + E[x(t-\tau_0)|x(t-\tau_0), M_{01}, M_{12}, \ldots, M_{n,n-1}]$$

$$= \lambda \tau_0 + E[x(t-\tau_0)|x(t-\tau_0), M_{01}, M_{12}, \ldots, M_{n,n-1}]$$

$$= \lambda \tau_0 + x(t - \tau_0)$$
Here is a brief explanation for how we got all of this:

Remember that if we condition on certain quantities, they can be treated as constants for the time being. This means, we can add and subtract constants together, and this doesn’t change anything as far as the conditional information is concerned. For example, if we are told the value of $\mathbf{x}(t - \tau_0), \mathbf{x}(t - \tau_1)$ and $\mathbf{x}(t - \tau_2)$, then it is the same as being told the value of $\mathbf{x}(t - \tau_0), \mathbf{x}(t - \tau_1) - \mathbf{x}(t - \tau_0)$ and $\mathbf{x}(t - \tau_2) - \mathbf{x}(t - \tau_1)$, because we can work out the values of $\mathbf{x}(t - \tau_0), \mathbf{x}(t - \tau_1)$ and $\mathbf{x}(t - \tau_2)$ from these three. This is the idea behind the step in (2).

We get (3) and (4) by simply substituting our definitions of $N$, $M_{ij}$ and $M$, and we get (5) and (6) using (1). Finally, notice that when $\mathbf{x}(t - \tau_0)$ is given, it is effectively a constant, so $E [\mathbf{x}(t - \tau_0)|\mathbf{x}(t - \tau_0), \text{anything else}]$ is nothing but $\mathbf{x}(t - \tau_0)$ itself. We use this to get (7).