Affine Geometry, Curve Flows, and Invariant Numerical Approximations

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Abstract. A new geometric approach to the affine geometry of curves in the plane and affine-invariant curve shortening is presented. We describe methods of approximating the affine curvature with discrete finite difference approximations, based on a general theory of approximating differential invariants of Lie group actions by joint invariants. Applications to computer vision are indicated.

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1. Introduction.

This paper is concerned with a modern presentation of the basic theory of affine geometry in the plane and related questions of invariant approximations of affine differential invariants. Although affine geometry does not have as long or distinguished a history as either Euclidean or projective geometry, its recent importance in the rapidly developing area of computer vision warrants a modern reassessment of the basics. Affine geometry received its first comprehensive treatment in the seminal work of Blaschke, [**6**], who was inspired by Klein's general Erlanger Programm, that provided the foundational link between groups and geometry, and Einstein's theory of relativity.(The latter motivation, though, is, to a modern thinker, more mysterious.) Affine geometry is based on the affine, or unimodular affine group. In the plane, affine geometry is the "geometry of area", just as Euclidean geometry is the geometry of distance. Besides the basic work of Blaschke, we refer the reader to [**22**], and the more modern texts [**14**], [**27**] for a more comprehensive treatment of the subject.

Even though our primary focus is mathematical, a key motivation for pursuing this line of research comes from certain practical issues from computer vision. Indeed, certain visually-based symmetry groups and their associated differential invariants have, in recent years, assumed great significance in the study of computer vision and image processing. One such problem is that of finding and recognizing a planar object (which may be occluded), whose shape has been transformed by a geometric viewing transformation (that is, an element of the projective group acting on the plane). This common type of shape recognition task naturally brings in the use of invariants under various groups of viewing transformations. Research in model based shape analysis and recognition has already resulted in many useful products, such as optical character recognizers, handwriting recognition systems for computers, and printed-circuit board visual inspection systems. Space limitations preclude us from discussing direct applications of our results to computer vision, which shall be dealt with in subsequent papers.

In the practical application of invariant theory to computer vision, a robust and efficient numerical computation is crucial. We are interested in numerical approximations to differential invariants which are themselves invariant under the transformation group in question. This will enable us to compute the "differential invariant signatures" for plane curves in a manner which will be unaffected by group transformations. The ideal approximation will be geometric, in the sense that it can be computed by specifying a finite number of points, and hence its invariance means that it must be re-expressed in terms of the joint invariants of the group in question. Thus our general question is how to systematically utilize joint invariants to approximate differential invariants. One motivation comes from the results of M. Green [**11**], generalized in [**17**], that relates the number of differential invariants of curves to the number of joint invariants of the group action; the numerological implications of Green's results are thus to be given an analytical justification.

The construction of efficient and practical numerical approximations to differential invariants is a nontrivial problem in that the more important differential invariants, such as the affine and projective curvatures, depend on high order derivatives of the parametrizing functions of the curve. The theory of "noise resistant" differential invariants developed by Weiss, [**29**], provides one approach to this problem. Weiss replaces the higher order differential invariants by lower order derivatives, but, in our view, this is only a partial resolution of the difficulty. In our approach, a fully noise resistant finite difference approximation to the affine (and Euclidean) curvatures are proposed. Another approach to invariant numerical schemes for solving partial differential equations having a prescribed symmetry group appears in the work of Dorodnitsyn, [**7**], [**8**].

Our approach to approximating differential invariants and invariant differential equations is governed by the following philosophy. Consider a group G acting on a space E. We are particularly interested in how the geometry, in the sense of Klein, induced by the transformation group G applies to (smooth) curves[†] , $\subset E$. A differential invariant I of G is a real-valued function, depending on the curve and its derivatives, which is unaffected by the action of G. The simplest example is the Euclidean curvature of a plane curve, which is invariant under the Euclidean group consisting of translations and rotations. The theory of differential invariants dates back to the original work of S. Lie, [**16**]; see [**17**] for further historical remarks and a modern exposition. In order to construct a numerical approximation to the differential invariant I, we use a finite difference approach and use a mesh or discrete sequence of points $P_i \in$, , $i = 0, 1, 2, \ldots$, to approximate the curve, and use appropriate combinations of the coordinates of the mesh points in our approximation scheme. The approximation will be invariant under the underlying group G, and hence its numerical values will not depend on the group transformations, provided it depends on the joint invariants of the mesh points. In general, a joint invariant of a group action on E is a real valued function $J(P_1, \ldots, P_n)$ depending on several points $P_i \in E$ which is unaffected by the simultaneous action of G on the points, so $J(g \cdot P_1, \ldots g \cdot P_n) = J(P_1, \ldots, P_n)$. Again, the simplest example is provided by the Euclidean distance $\mathbf{d}_e(P, Q)$ between points in the plane, which depends on two points. Thus, *any G-invariant numerical approximation to a differential invariant must be governed by a function of the joint invariants of G.* For instance, any Euclidean invariant approximation to the curvature of a plane curve must be based on the distances between the mesh points. Such a formula is known — see Theorem 3.2 below. In this paper, we illustrate this general method by deriving a fully affine invariant finite difference approximation to the affine curvature of a plane curve. The resulting Taylor series expansion leads us to a general conjecture on the approximation of group-invariant curvatures for arbitrary regular transformation groups in the plane. We will also indicate some methods for determining similar approximations to higher order invariants.

Motivated by such questions, in this paper we will give a detailed discussion of equiaffine geometry, which includes new geometric approaches to the equiaffine normal and curvature. We discuss finite difference approximations of Euclidean and affine differential invariants. Finally, we provide some new, remarkable solutions to the affine curvature flow. Even though, this paper is essentially devoted to the derivation of a number of new

[†] More generally, we can develop the same theory for surfaces or arbitrary submanifolds of the space E. In this work, just for simplicity, we restrict our attention to curves.

results in the theory of affine invariants, we will also provide a number of background results to make this work accessible to the largest possible audience and mathematicians and researchers in computer vision, so that the paper will also have a tutorial flavor.

2. Some Fundamental Concepts.

When one treats Euclidean or affine geometry from the analytic standpoint, one must deal with two distinct spaces: the *space of points* (the Euclidean space proper), denoted E, and the finite-dimensional real[†] *vector space* TE consisting of translations (or displacements) of E. Within the space of points, there is one main operation — subtraction: Given two points $P, Q \in E$, the object $v = Q - P$ is the unique displacement vector in TE mapping E onto itself that takes the point P to the point Q. The group of transformations of E that preserve this structure is known as the *affine group*, denoted by $\mathrm{A}(n)$ or $\mathrm{A}(n, \quad)$, where n is the dimension of E. An element of $\mathrm{A}(n)$ consists of a linear transformation $A \in \mathrm{GL}(n)$, which operates on TE, coupled with a displacement vector $b \in TE$; the full action on the point space takes the form $P \mapsto AP + b$. Note that this induces the purely linear action $v \mapsto Av$ on the displacement vector space, and thus underlies the necessity of distinguishing between E and TE.

An affine coordinate system on E is prescribed by an *affinely independent set* of points $(P_0, P_1, \ldots, P_n)$ in E, meaning that the displacment vectors $e_i = P_i - P_0$ form a basis of E. A displacement vector $v = \sum_k y^k e_k \in TE$ is identified with the coordinate n-tuple $(y^1, \ldots, y^n)$, while we associate points $P \in E$ with their relative displacement vectors $v_P = P - P_0 = \sum_i x^i e_i$. In this way, we identify the affine group $\mathrm{A}(n) \simeq \mathrm{GL}(n) \qquad {}^n$ with the semidirect product of the general linear group with the displacement or translation subgroup.

If TE has dimension n, then the space $\bigwedge^n TE$ of volume forms on E is a one-dimensional vector space. The affine transformations act on $\bigwedge^n TE$ according to the determinantal representation $(A, b) \mapsto \det A$. Given two sets of points $(P_0, P_1, \ldots, P_n)$, $(Q_0, Q_1, \ldots, Q_n)$, not necessarily distinct, such that $(P_0, P_1, \ldots, P_n)$ is an affinely independent set, there is a unique affine endomorphism of E that maps P_i onto Q_i for each $1 \leq i \leq n$. Its homogeneous linear part, i.e., the linear endomorphism of TE taking each $v_i = P_i - P_0$ to $w_i = Q_i - Q_0$, has a determinant that, if nonzero, expresses the ratio of (oriented) volumes of the n-parallelotope determined by the w's to that determined by the v's, or, equivalently, the ratio of volumes of the n-simplex spanned by the Q's to that spanned by the P's. Thus the full affine group $\mathrm{A}(n)$ preserves the *ratios* between volumes of subsets of E, or of TE, while volumes themselves are *relative invariants* of the group.

An *orientation* on TE is prescribed by the choice of one of the two connected components of $\bigwedge^n TE \setminus \{0\}$; the orientation-preserving affine transformations are those having positive determinant. The notion of volume on E is fixed by specifying what constitutes a "unit volume", which is represented by a fixed form $\Omega_0 = e_1 \wedge e_2 \wedge \cdots \wedge e_n \in \bigwedge^n TE$, where $\{e_1, \ldots, e_n\}$ form a basis of TE, and the volume of the n-paralleltope spanned by the e_i's is normalized to be 1. In this case, the the oriented volume of the parallelotope

[†] One can, of course, develop much of the general theory over the complex numbers or other fields. Again, for simplicity, we restrict our attention to real geometry throughout.

determined by the displacement vectors $v_i = \sum_k y_i^k e_k$, $i = 1, \ldots, n$, is calculated by the fundamental determinantal bracket expression

$$[v_1, \ldots, v_n] = \det \begin{vmatrix} y_1^1 & y_1^2 & \cdots & y_1^n \\ y_2^1 & y_2^2 & \cdots & y_2^n \\ \vdots & \vdots & \ddots & \vdots \\ y_n^1 & y_n^2 & \cdots & y_n^n \end{vmatrix}. \tag{2.1}$$

Similarly, the volume of the n-simplex having vertices $P_0, P_1, \ldots, P_n$ in E, with P_i having coordinates $(x_i^1, \ldots, x_i^n)$ with respect to some affine coordinate system is given by

$$\Delta(P_0, P_1, \ldots, P_n) = \frac{1}{2^n}[P_0, P_1, \ldots, P_n], \tag{2.2}$$

where

$$\begin{aligned} [P_0, P_1, \ldots, P_n] &= [P_1 - P_0, P_2 - P_0, \ldots, P_n - P_0] \; = \\ &= \det \begin{vmatrix} x_1^1 - x_0^1 & x_1^2 - x_0^2 & \cdots & x_1^n - x_0^n \\ x_2^1 - x_0^1 & x_2^2 - x_0^2 & \cdots & x_2^n - x_0^n \\ \vdots & \vdots & \ddots & \vdots \\ x_n^1 - x_0^1 & x_n^2 - x_0^2 & \cdots & x_n^n - x_0^n \end{vmatrix} = \det \begin{vmatrix} x_0^1 & x_0^2 & \cdots & x_0^n & 1 \\ x_1^1 & x_1^2 & \cdots & x_1^n & 1 \\ x_2^1 & x_2^2 & \cdots & x_2^n & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_n^1 & x_n^2 & \cdots & x_n^n & 1 \end{vmatrix}. \end{aligned} \tag{2.3}$$

Note particularly that, in an n-dimensional affine space, the respective bracket expressions (2.1), (2.3), depend on n displacement vectors, but $n+1$ points. Restricting the group of affine transformations to those that preserve volume produces the so-called *equiaffine*, or *unimodular affine* transformation group, denoted by $\mathrm{SA}(n) \simeq \mathrm{SL}(n) \quad {}^n$, consisting of all pairs (A, b) where $\det A = 1$, and $b \in TE$. The associated equiaffine geometry in E and TE will form the principle subject of this paper.

In Euclidean geometry, one endows the displacement vector space TE with the additional structure, determined by a *norm* $v \mapsto |v|$. The geometric properties of the Euclidean norm come from the fact that it is characterized as the square root, $|v| = \sqrt{v \cdot v}$, of a positive definite quadratic form, associated to a symmetric, bilinear, scalar product $v \cdot w$. The norm on the displacement space TE induces the Euclidean distance† $\mathbf{d}_e(P, Q) = |Q - P|$ between pairs of points in E. The group of Euclidean motions is the set of all transformations of E that preserve the norm in TE. It has the form $\mathrm{E}(n) \simeq \mathrm{O}(n) \quad {}^n$, being a semidirect prodcut between the orthogonal group, consisting of rotations and reflections, along with the translations. Choosing an orientation, which amounts to a choice of an orthonormal basis $\{e_1, \ldots, e_n\}$ of TE, restricts us to the proper (or unimodular) Euclidean motions of E, which excludes the reflections, and so is given by $\mathrm{SE}(n) \simeq \mathrm{SO}(n) \quad {}^n$.

In general, given a group G acting on a space M, by an *invariant* of G we mean a real-valued function $I\colon M \to \quad$ which is unaffected by the group action: $I(g \cdot x) = I(x)$

† We shall consistently employ the subscript $_e$ for Euclidean invariant quantities, so as to distinguish them from the affine and equiaffine invariants that are the primary focus of this paper.

for all $x \in M$, $g \in G$. For example, the norm $|v|$ defines an invariant for the Euclidean group action on the displacement space TE. On the other hand, since the action on space of points includes the translations, there are no (non-constant) invariants of the Euclidean group action on E itself. In this case, we must look at invariants depending on more than one point. In general, a *joint invariant* of a group action is an invariant of the product action of G on the m-fold Cartesian product $M \times \cdots \times M$. Thus $I(x^1, \ldots, x^m)$ is a joint invariant if and only if $I(g \cdot x^1, \ldots, g \cdot x^m) = I(x^1, \ldots, x^m)$ for all $g \in G$. The simplest joint invariant of the Euclidean group acting on E is the distance function $\mathbf{d}_e(P, Q)$. In fact, according to [**30**], every joint invariant of the Euclidean group can be written in terms of the distances between pairs of points. For example, the inner product $v \cdot w = (P - P') \cdot (Q - Q')$ between two displacement vectors can be re-expressed via the Law of Cosines: $v \cdot w = \frac{1}{2}\left\{|v - w|^2 - |v|^2 - |w|^2\right\}$. Further, since the Euclidean group is a subgroup of the affine group, any (equi-)affine invariant is automatically a Euclidean invariant, and hence can also be rewritten in terms of Euclidean distances. Thus, the volume $|\Pi(w_1, \ldots, w_n)|$ of the parallelotope $\Pi(w_1, \ldots, w_n)$ spanned by n displacement vectors $\{w_1, \ldots, w_n\} \in TE$ has its square rationally determined by the mutual scalar products:

$$|\Pi(w_1, \ldots, w_n)|^2 = \det(w_i \cdot w_j). \tag{2.4}$$

In the case of the unimodular affine group, there are no non-constant invariants on either E or TE. The simplest joint invariant associated with the equiaffine group action on TE is the fundamental bracket (2.1) governing the volume element. See Weyl [**30**], for a proof that the brackets constitute a complete set of joint affine invariants for displacement vectors, meaning that any equiaffine joint invariant can be written as a function of the various brackets between sets of n displacement vectors. We note that the brackets are *not* algebraically independent; their functional inter-relationships are completely governed by the fundamental system of syzygies

$$\sum_{i=0}^{n} (-1)^k [v_0, \ldots, \widehat{v_k}, \ldots, v_n]\, [v_k, w_1, \ldots, w_{n-1}] = 0, \tag{2.5}$$

valid for any set of displacement vectors $v_0, \ldots, v_n$, $w_1, \ldots, w_{n-1}$. Similarly, the fundamental joint invariants of the action of $\mathrm{SA}(n)$ on E itself are the simplex volumes (2.2) prescribed by $n+1$ points in E. Besides the syzygies induced by the displacement bracket syzygies (2.5), the point bracket expressions are subject to an additional linear syzygy

$$[P_0, P_1, \ldots, P_n] = \sum_{i=0}^{n} [P_0, \ldots, P_{k-1}, Q, P_{k+1}, \ldots, P_n], \tag{2.6}$$

valid for any $n+2$ points $P_0, P_1, \ldots, P_n, Q$. Finally, in the case of the full affine group $\mathrm{A}(n)$, relative ratios of brackets (or volumes) provide the required joint invariants.

3. Euclidean Curvature and Curve Flows.

We now specialize to Euclidean geometry of the plane, so that E denotes the two-dimensional Euclidean space, with displacement space TE. If we introduce coordinates on

E via the choice of an origin $O \in E$ and orthonormal basis e_1, e_2 of TE, then each point $A \in E$ can be identified with its coordinates $(x_A, y_A) \in {}^2$, such that $A - O = x_A e_1 + y_A e_2$. The basic equi-affine invariant geometric quantity is the area of a displacement parallelogram

$$[v, w] = v \wedge w = \det \begin{vmatrix} x_v & y_v \\ x_w & y_w \end{vmatrix}. \tag{3.1}$$

We note that, in accordance with the general theory, the affine-invariant area of the triangle having vertices A, B, C, which is

$$\Delta(A, B, C) = \tfrac{1}{2}[A, B, C] = \tfrac{1}{2}(B - A) \wedge (C - A) = \frac{1}{2} \det \begin{vmatrix} x_A & y_A & 1 \\ x_B & y_B & 1 \\ x_C & y_C & 1 \end{vmatrix}, \tag{3.2}$$

cf. equations (2.3), (2.2), can be written in terms of their Euclidean distances $a = \mathbf{d}_e(A, B)$, $b = \mathbf{d}_e(B, C)$, $b = \mathbf{d}_e(C, A)$, via the well-known semi-perimeter formula:

$$[A, B, C] = \sqrt{s(s-a)(s-b)(s-c)}, \qquad \text{where} \qquad s = \tfrac{1}{2}(a + b + c). \tag{3.3}$$

Consider a regular, smooth plane curve $\subset E$ of class C^2. The Euclidean curvature of at a point $B \in$, is defined as the reciprocal $\kappa_e = 1/r$ of the radius of the osculating circle to at B. Let us choose an affine coordinate system (x, y) on E, and parametrize the curve by a pair of smooth functions $\mathbf{x}(r) = (x(r), y(r))$, where the parameter r ranges over an interval $I \subset$. In terms of the parametrization, then, the Euclidean curvature has the well-known formula

$$\kappa_e = \frac{\mathbf{x}_r \wedge \mathbf{x}_{rr}}{|\mathbf{x}_r|^3}, \tag{3.4}$$

in which subscripts denote derivatives. In particular, if we choose a coordinate system such that the part of near B is represented by the graph of a function $y = u(x)$, then

$$\kappa_e = \frac{u_{xx}}{(1 + u_x^2)^{3/2}}. \tag{3.5}$$

In this form, κ_e describes the simplest differential invariant of the Euclidean group in the plane, [**17**]. The Euclidean arc length parameter is defined as $ds_e = \sqrt{1 + u_x^2}\, dx$, the right hand side representing the simplest invariant one-form for the Euclidean group. The arc length integral $\int_\Gamma ds$ determines the Euclidean distance traversed along the curve. Higher order differential invariants are provided by the successive derivatives of curvature with respect to arc length. In fact, the functions

$$\kappa_e, \quad \frac{d\kappa_e}{ds_e}, \quad \frac{d^2\kappa_e}{ds_e^2}, \quad \frac{d^3\kappa_e}{ds_e^3}, \quad \ldots, \tag{3.6}$$

provide a complete list of differential invariants for the Euclidean group, in the sense that any other differential invariant can be (locally) expressed as a function of the fundamental differential invariants (3.6).

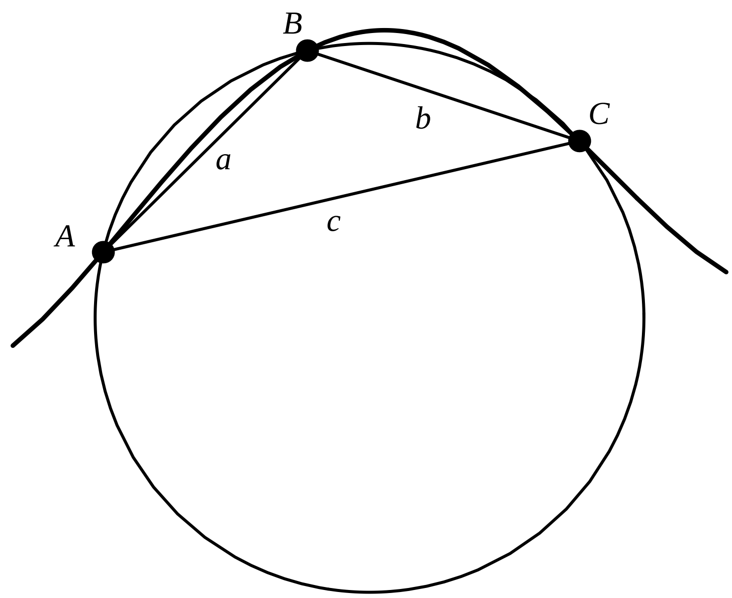

Figure 1. Euclidean Curvature Approximation.

As a first illustration of our general philosophy of approximating differential invariants by joint invariants, we describe how to use standard geometrical constructions to obtain a numerical approximation to the Euclidean curvature that is unaffected by rigid motions, so that any translated or rotated version of the curve will provide precisely the same numerical approximation for its curvature. We first approximate the parametrized curve $\mathbf{x}(r) = (x(r), y(r))$ by a sequence of mesh points $P_i = \mathbf{x}(r_i)$, not necessarily equally spaced. Our goal is to approximate the Euclidean curvature of , by a Euclidean invariant numerical approximation based on the mesh points. Clearly, because the curvature is a second order differential function, the simplest approximation will require three mesh points. (A deeper, but related reason for this is because the joint invariants of the Euclidean group are the distances between two points, so that one can only produce numerical joint invariant approximations by comparing the joint invariants involving three or more points.)

With this in mind, we now derive the basic approximation formula for the Euclidean curvature. Let A, B, C be three successive points on the curve , such that the Euclidean distances are $a = \mathbf{d}_e(A, B)$, $b = \mathbf{d}_e(B, C)$, $c = \mathbf{d}_e(A, C)$, which are assumed to be small; see Figure 1. The key idea is to use the circle passing through the points A, B, C as our approximation to the osculating circle to the curve at B. Therefore, the reciprocal of its radius $r = r(A, B, C)$ will serve as an approximation to the curvature of the curve at B. We can apply Heron's formula to compute the radius of the circle passing through the points A, B, C, leading to the exact formula

$$\widetilde{\kappa}_e(A, B, C) = 4\,\frac{\Delta}{abc} = 4\,\frac{\sqrt{s(s-a)(s-b)(s-c)}}{abc}, \tag{3.7}$$

cf. (3.3), for its curvature. Since formula (3.7) only depends on the Euclidean distances between the three points, it provides us with a completely Euclidean invariant numerical approximation to the curvature of Γ at the middle point B. In other words, the approximation for two curves related by a Euclidean motion will be *identical*.

We now need to analyze how closely the numerical approximation $\widetilde{\kappa}_e(A,B,C)$ is to the true curvature $\kappa_e(B)$ at the point B. Our analysis is based on the following series expansion of the distance c in terms of the other two distances a and b, which are assumed small.

Theorem 3.1. *Let A, B, C be three successive points on the curve Γ, and let $a = \mathbf{d}_e(A,B)$, $b = \mathbf{d}_e(B,C)$, $c = \mathbf{d}_e(A,C)$ be their Euclidean distances. Let $\kappa_e = \kappa_e(B)$ denote the Euclidean curvature of Γ at the middle point B. Then the following expansion is valid:*

$$\begin{aligned} c^2 = (a+b)^2 - \frac{1}{4}ab(a+b)^2\kappa_e^2 + \frac{1}{6}ab(a+b)^2(a-b)\kappa_e\frac{d\kappa_e}{ds_e} - \\ - \frac{1}{24}ab(a+b)(a^3+b^3)\kappa_e\frac{d^2\kappa_e}{ds_e^2} - \frac{1}{36}ab(a+b)^2(a-b)^2\left(\frac{d\kappa_e}{ds_e}\right)^2 - \\ - \frac{1}{64}ab(a+b)^2(a-b)^2\kappa_e^4 + \cdots . \end{aligned} \tag{3.8}$$

The omitted terms involve powers of the distances a, b of order ≥ 7.

Proof: This is found by a direct, albeit complicated, Taylor series expansion. We represent the curve between A and C as the graph of $y = u(x)$, which, assuming the three points are sufficiently close, can always be arranged via a Euclidean motion. The points can be assumed to be $A = (h, u(h))$, $B = (0, 0 = u(0))$, and $C = (k, u(k))$, with $h < 0 < k$ if B is the middle point. We then expand $c = \sqrt{(k-h)^2 + (u(k)-u(h))^2}$ as a Taylor series in powers of h, k. Then we substitute for h and k their expansions in powers of a, b, obtained by inverting the Taylor series for $a = \sqrt{h^2 + u(h)^2}$, and $b = \sqrt{k^2 + u(k)^2}$. (The computations are quite complicated, and were done with the aid of the computer algebra system MATHEMATICA.) *Q.E.D.*

Remark: Since a, b, and c are Euclidean invariants, *every* coefficient of the powers $a^m b^n$ in the full expansion of c must be a Euclidean differential invariant, and hence a function of κ_e and its arc length derivatives. The precise formulas for the coefficients were found by inspection — we do not know the general term in the expansion (3.8).

We now substitute the expansion (3.8) into Heron's fomula (3.7) to obtain the following expansion for the numerical approximation.

Theorem 3.2. *Let A, B, C be three successive points on the curve Γ, and let a, b, c be their Euclidean distances. Let $\kappa_e = \kappa_e(B)$ denote the Euclidean curvature at B. Let $\widetilde{\kappa}_e = \widetilde{\kappa}_e(A,B,C)$ denote the curvature of the circle passing through the three points. Then the following expansion is valid:*

$$\widetilde{\kappa}_e = \kappa_e + \frac{1}{3}(b-a)\frac{d\kappa_e}{ds_e} + \frac{1}{12}(b^2 - ab + a^2)\frac{d^2\kappa_e}{ds_e^2} + \cdots . \tag{3.9}$$

In particular, if we choose the points to be equally spaced, meaning that $a = b$ (not that their x coordinates are equally spaced), then the first error term in the approximation (3.9) is of second order.

Remark: The same general method can also be used to find Euclidean-invariant numerical approximations for computing the higher order differential invariants $d\kappa_e/ds_e$, etc., using more points and more distances, as needed.

In recent years, the analysis and geometrical and image processing applications of curve flows based on curvature has received a lot of attention. We consider a one-parameter family of curves $\mathbf{x}(\cdot, t)$ that satisfy a geometric evolution equation. Here t represents either the time, or, in computer vision applications, a scale parameter. The partial differential equation governing the time evolution of the curve family is assumed to be geometric, meaning that it does not depend on the precise mode of parametrizing the family of curves, but, rather, on purely intrinsic geometric quantities associated with the curve at a give time. The most fundamental of these geometric flows is the Euclidean curve shortening flow, in which one moves in the normal direction to the curve according to its Euclidean curvature:

$$\frac{d\mathbf{x}}{dt} = \kappa_e \mathbf{n}_e. \tag{3.10}$$

Here $\mathbf{n}_e$ denotes the Euclidean inward normal. When the curve is given as the graph of a function $y = u(x, t)$, the Euclidean curve flow takes the form:

$$u_t = \frac{u_{xx}}{1 + u_x^2}. \tag{3.11}$$

This flow has the effect of shrinking the Euclidean arc length of the curve as rapidly as possible, cf. [**10**]. The Euclidean curve shortening flow is of great interest in differential geometry, computer vision, and other fields, and has been studied by many authors. See [**3**] for applications to image enhancement, and [**13**] for applications to the theory of shape in computer vision. Clearly the flow (3.10) is invariant under the Euclidean group acting on the plane, and so a fully invariant numerical integration must rely on Euclidean joint invariants, meaning intermesh distances.

Two particular types of solutions are of immediate interest. First, if the initial curve is a circle, with contstant curvature, then it remains circular, with its radius satisfying $r_t = 1/r$, so that the curve shrinks to a point in a finite time. The results of Gage and Hamilton [**9**], and Grayson [**10**] show that *any* smooth, embedded, closed curve converges to a round point when deforming according to the flow (3.10). This means that, first, if the intial curve is not convex, it becomes convex, and then the resulting convex curve shrinks to a point, asymptotically becoming circular before disappearing.

A second class of solutions are the "grim reapers" which are found by assuming that the curve has constant velocity. Taking the velocity to be in the vertical direction and using the graphical form (3.11) means that we assume that $u_t = c$ where c is a constant. The resulting Euclidean-invariant ordinary differential equation

$$\frac{u_{xx}}{1 + u_x^2} = c$$

can be readily integrated, leading to the general form

$$u(x, t) = -\frac{1}{c} \log[\cos c(x - x_0)] + c(t - t_0),$$

for constants x_0 and t_0, for the grim reaper. At this point, we conclude our brief survey of Euclidean curve flows, and turn to our main subject of interest.

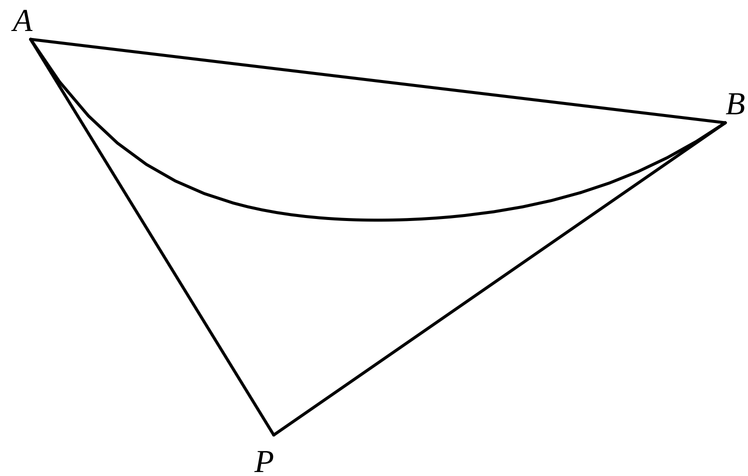

Figure 2. Support Point and Support Triangle.

4. The Equiaffine Length Integral.

We now turn to our primary focus: the affine geometry of curves in the plane. Affine invariants are not suited for the study of curves with inflection points; therefore we shall deal only with *strongly locally convex* curves. In this section, all curves will be assumed to have not only no inflection points, but to be continuously differentiable with respect to suitable parameters of order up to 5, although in the next few paragraphs derivatives of orders at most 3 will appear. Many of our constructions will refer to a sufficiently short piece of the convex curve, in the following precise sense.

Definition 4.1. Let , be a smooth plane curve without inflection points. A compact arc , $(A, B) \subset$, , i.e., with both end points A, B included, will be called a *short arc* if no two tangent lines to , (A, B) are mutually parallel, including the tangents at the end points.

This condition, in Euclidean geometry, is equivalent to the statement that the total turning angle of the tangential direction of , (A, B) is less than half a revolution; in terms of purely affine invariants of , , the property means that the arc , (A, B) may be inscribed in a *support triangle*, which is bounded by the segment joining the endpoints A, B, and by the tangent lines at the two endpoints.

Definition 4.2. Let , a strongly convex curve, and let , (A, B) be a short arc of , with end points A and B. The *support point* of , (A, B) is the point P where the two end-point tangent lines intersect. The *support triangle* of , (A, B) is defined as the triangle $\mathbf{T}(A, B) = APB$; see Figure 2.

Note that, by convexity, the support triangle circumscribes the short arc , . We regard the (positive) area of the support triangle,

$$\mathbf{A}(A, B) = |\Delta(A, P, B)| = \tfrac{1}{2}\,|\,[A, P, B]\,|\,,$$

cf. (3.2), as an equiaffinely invariant "indicator" of the distance between the (non-oriented) *tangent line elements* (A, AP) and (B, BP). More precisely, we want to introduce an

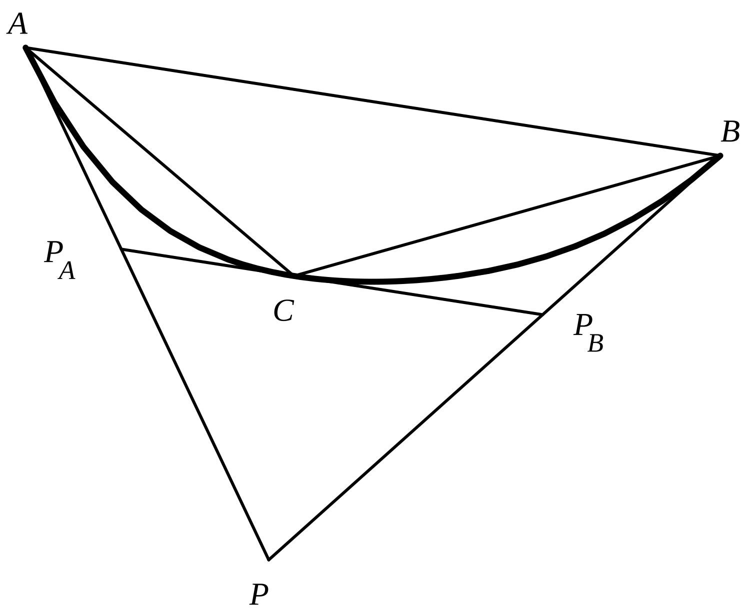

Figure 3. The Affine Anti-Triangle Inequality.

equiaffinely invariant distance function between the two line segments, so that, if we break the arc , (A, B) in two at any intermediate point C and compare the distance from the tangent line element at A to the one at C, with the distance from the tangent line element at C to the one at B, we obtain, asymptotically for very short smooth arcs, the original distance defined from $\mathbf{A}(A, B)$. It is obvious that the area $\mathbf{A}(A, B)$ itself does not have this asymptotic property. However, its cube root does, as the following theorem of Blaschke [**6**], shows.

Theorem 4.3. *Let , (A, B) be a short arc of a strongly convex curve , , joining a point A to a point B, and let $C \in$, be another point, interior to the arc , (A, B). Draw tangent lines to , at each of the three points, as well as the three chords joining them, as shown in Figure 3. We let P denote the support point for the arc , (A, B), and P_A, P_B, the respective support points for the respective sub-arcs , (A, C), , (C, B); thus P_A, P_B, are the points where the tangent line at C intersects the tangents AP and BP. Each of the three support triangles $\mathbf{T}(A, B) = APB$, $\mathbf{T}(A, C) = AP_AC$, $\mathbf{T}(C, B) = CP_BB$ circumscribes the corresponding arc of , . Let $\mathbf{d}(A, B) = 2\sqrt[3]{\mathbf{A}(A, B)}$, $\mathbf{d}(A, C) = 2\sqrt[3]{\mathbf{A}(A, C)}$, $\mathbf{d}(C, B) = 2\sqrt[3]{\mathbf{A}(C, B)}$, denote twice the cube roots of their respective areas. (The factor of 2 is merely included for later convenience.) Then the following* anti-triangle inequality *is true:*

$$\mathbf{d}(A, B) \geq \mathbf{d}(A, C) + \mathbf{d}(C, B). \tag{4.1}$$

Equality is achieved if and only if the following affinely invariant length relations (length ratios among pairs of segments in the same line) hold:

$$\frac{AP_A}{AP} = \frac{PP_B}{PB} = \frac{P_AC}{P_AP_B}. \tag{4.2}$$

Furthermore, if one fixes the two boundary line elements (A, AP) and (B, BP), then the set of line elements (C, CP_B) that satisfy (4.2), with C in the interior of the triangle $\mathbf{T}(A, B)$, constitute a one-parameter family of tangent line elements of the unique arc of the parabola having the prescribed tangent elements at the end points.

Proof: Since any two (non-degenerate) triangles, with their vertices in a given order, are affinely equivalent in a unique way, we may fix the two boundary line elements (A, AP) and (B, BP) in such a way that the area of the resulting triangle $\mathbf{T}(A, B)$ equals unity. Then the set of line elements (C, CP_B) that may occur as tangent line elements of any strongly convex, short arc joining the given boundary elements is in a natural correspondence with the triple of real numbers (u, v, w) with $0 < u < 1$, $0 < v < 1$, $0 < w < 1$, according to the following recipe.

First, choose the point P_A on the line segment AP according to the vector relation $A - P_A = u(A - P)$. Then choose P_B on the segment PB so that $P_B - P = v(B - P)$, and, finally, C on the segment $P_A P_B$ so that $C - P_A = w(P_B - P_A)$. One readily verifies that

$$\mathbf{A}(A, C) = |\Delta(A, P_A, C)| = u\, |\Delta(A, P, C)|\,,$$
$$|\Delta(A, P, C)| = w\, |\Delta(A, P, P_B)|\,, \qquad |\Delta(A, P, P_B)| = v\, |\Delta(A, P, B)| = v,$$

whence $\mathbf{A}(A, C) = uvw$. Similarly,

$$\begin{aligned}\mathbf{A}(C, B) &= |\Delta(C, P_B, B)| = (1 - v)\, |\Delta(C, P, B)| \\ &= (1 - v)(1 - w)\, |\Delta(P_A, P, B)| = (1 - v)(1 - w)(1 - u).\end{aligned}$$

Thus the "distances" between the line elements in question satisfy the relations

$$\mathbf{d}(A, C) = \sqrt[3]{uvw}\, \mathbf{d}(A, B), \qquad \mathbf{d}(C, B) = \sqrt[3]{(1 - u)(1 - v)(1 - w)}\, \mathbf{d}(A, B).$$

It is well known that the geometric mean of any finite family of positive real numbers is strictly smaller than their arithmetic mean; applying this to the identities above, and adding, one sees that

$$\mathbf{d}(A, C) + \mathbf{d}(C, B) \le \tfrac{1}{3}(u + v + w)\mathbf{d}(A, B) + \tfrac{1}{3}[(1 - u) + (1 - v) + (1 - w)]\mathbf{d}(A, B) = \mathbf{d}(A, B),$$

with equality achieved only when $u = v = w$. This proves the first part of our assertion.

In order to constructively verify the second assertion, we take the circumscribed triangle $\mathbf{T}(A, B)$ as before and adapt an affine coordinate system (x, y) to it with origin at P, so that the points A and B have respective coordinates $(0, 1)$ and $(1, 0)$. Setting $u = v = w = r$, where $0 < r < 1$ is a parameter, the line $P_A P_B$ has equation $(1 - r)x + ry = 1$, and the point C on that line is defined parametrically by its coordinates

$$C = \mathbf{x}(r) = (x(r), y(r)) = \left(\tfrac{1}{2}r^2, \tfrac{1}{2}(1 - r)^2\right). \tag{4.3}$$

This shows that the point $\mathbf{x}(r)$ traces the arc of the parabola $y = x + \frac{1}{2} - \sqrt{2x}$ bounded between the points $P_A = \left(0, \frac{1}{2}\right)$ and $P_B = \left(\frac{1}{2}, 0\right)$, with the corresponding axes as tangents. This completes the proof of the theorem. *Q.E.D.*

The construction of the parametric equation (4.3) of the parabola and the statement of Theorem 4.3 show, in addition, that for any two values $r_1 < r_2$ of the parameter r, the area of the triangle circumscribed to the arc corresponding to $[r_1, r_2]$ equals $\frac{1}{8}(r_2 - r_1)^3$. We recall here that the usual, formal definition of the equiaffinely invariant arc length for locally convex smooth curves $\mathbf{x}(r)$ is expressed by the invariant integral

$$s = \int \sqrt[3]{\left| \left[\frac{d\mathbf{x}}{dr} \, \frac{d^2\mathbf{x}}{dr^2} \right] \right|} \; dr, \tag{4.4}$$

where we are considering the derivatives of $\mathbf{x}(r)$ as displacement vectors in TE, and using the notation of (3.1). In the case of the parametric representation (4.3) of a parabola, the parameter r describes affine arc length.

More generally, let , be a convex curve of class C^2 traced by $\mathbf{x}(r)$ for r in a closed interval $I = [r_0, r_1]$. Subdivide I into a finite sequence of n small subintervals using a mesh $r_0 < r_1 < \cdots < r_{n-1} < r_n$, and let $P_k = \mathbf{x}(r_k)$ be the corresponding points on , . Inscribe each subarc $\gamma_k =\, , (P_{k-1}, P_k)$ in a corresponding support triangle $\mathbf{T}_k = \mathbf{T}(P_{k-1}, P_k)$. Let $\mathbf{d}_k$ equal twice the cube root of the area of $\mathbf{T}_k$. Then, on the one hand, the sum of the quantities $\mathbf{d}_k$ is non-increasing under successive refinements of the subdivision, while, on the other hand, the sum converges downward to the value of the integral (4.4). With this observation, we make the following definition of the *pseudo-distance* between any two (non-oriented) line elements in general position in the equiaffine plane.

Definition 4.4. Suppose the two line elements (A, AX) and (B, BY) are in general position, meaning that the lines AX and BY are not parallel, intersecting at a point P, and that the three points A, B, and P are distinct. Then the *distance* (or *pseudo-distance*) between (A, AX) and (B, BY) is defined to be twice the cube root of the area of the triangle $\mathbf{T} = APB$.

There are two easy, alternative geometrical interpretations of the equiaffine arc length of a convex curve. One can replace the cube root of eight times the area of the small triangles by either the cube root of twelve times the area of the region between the small arcs of the subdivision of , and the corresponding chords, or that of 24 times the area between the small arcs and their endpoint tangents. Either of these two definitions are easier to adapt to the case of convex hypersurfaces in $\ ^n$ than the one presented here; however, the approximation of the true affine length by subdivision is no longer monotone in either of the two modified cases.

The geometric interpretation of the equiaffine arc length just described admits two natural generalizations to higher dimensions. One generalization pertains to curves in n-dimensional space. Here the equiaffinely invariant arc length of an arc of class C^n parametrized by $\mathbf{x}(r)$ is formally defined by the integral

$$s = \int \left| \left[\frac{d\mathbf{x}}{dr}, \frac{d^2\mathbf{x}}{dr^2}, \ldots, \frac{d^n\mathbf{x}}{dr^n} \right] \right|^{\frac{2}{n(n+1)}} dr.$$

The other generalization deals with hypersurfaces (mainly in the strongly locally convex case) of class C^2 in $\ ^n$. In this case, the easiest description of the formally equiaffinely invariant metric structure is in terms of a Euclidean structure on $\quad^n$ defined by a positive definite quadratic form $\langle\cdot\,,\cdot\rangle$, inducing the familiar Euclidean invariants: the first fundamental form ds_e^2, the element of surface area dA_e, the unit normal vector N_e, the second fundamental form $\mathrm{II}_e = \langle N_e\,, d^2X\rangle$, assumed to be positive definite, and the Gaussian curvature $K_e > 0$. Then the positive definite quadratic form $ds^2 = K_e^{-1/(n+1)}\mathrm{II}_e$ and the corresponding $(n-1)$-dimensional area form $dA = K_e^{1/(n+1)}\,dA_e$ may easily be shown to be invariant under the equiaffine group. The content of the geometrization of these formulas is to enable us to "see", from the shape of the surface, the length of paths and areas of subdomains.

5. The Equiaffine Structure Equations.

Having introduced the element of equiaffine arc length for a smooth curve , without inflection points, the remaining equiaffine invariants are best described, analytically, in terms of the derivatives of the parametric representation of the curve, when the oriented, equiaffine arc length s itself is used as a parameter. However, one should observe that the existence and continuity of $d^k\mathbf{x}(s)/ds^k$ for any $k \geq 1$ require existence and continuity of the $(k+1)^{\mathrm{st}}$ derivative of $\mathbf{x}$ with respect to a general parameter. The formal definition of s implies that the first two derivatives $\mathbf{x}_s = d\mathbf{x}(s)/ds$ and $\mathbf{x}_{ss} = d^2\mathbf{x}(s)/ds^2$ are linearly independent and, indeed, satisfy the identity

$$[\mathbf{x}_s, \mathbf{x}_{ss}] = \pm 1. \tag{5.1}$$

If necessary, one may replace the parameter s by $-s$ in order to reduce the right hand side of (5.1) to $+1$. Either way, $\mathbf{x}_{ss}$ points toward the concave side of the curve , , while the positive sign in (5.1) indicates that, as s increases, the curve turns towards the left. For each point $\mathbf{x}(s_0)$ one defines what É. Cartan called the "moving frame" (repère mobile) of , , namely the affine coordinate system with origin at $\mathbf{x}(s_0)$, such that the coordinate pair (u, v) corresponds to the point $\mathbf{x}(s_0) + u\mathbf{x}_s(s_0) + v\mathbf{x}_{ss}(s_0)$. The two "unit" coordinate vectors $\mathbf{x}_s(s_0)$, $\mathbf{x}_{ss}(s_0)$ are then called the *(affine) unit tangent* and *unit normal* respectively, and accordingly denoted by $\mathbf{t}(s_0)$ and $\mathbf{n}(s_0)$ respectively.

Differentiating both sides of (5.1) with respect to to s, we see that $[\mathbf{x}_s, \mathbf{x}_{sss}] = 0$, implying that $d\mathbf{n}(s)/ds$ is a scalar multiple of $\mathbf{t}(s)$. One is thus led to the formal definition of the *(equi-)affine curvature* $\kappa(s)$ via the equation

$$\frac{d\mathbf{n}(s)}{ds} = -\kappa(s)\,\mathbf{t}(s). \tag{5.2}$$

The seemingly capricious choice of sign in the above equation is contrived so that, in the case of non-singular conic sections (in which case κ is constant), κ is positive, zero, or negative, according to whether the conic is, respectively, an ellipse, a parabola, or a hyperbola. See Theorem 6.4 below.

The data consisting of the equiaffine arc length parameter s and the affine curvature κ furnish the total generating system of equiaffine invariants of a curve , . In fact, the structure equations for , may be deduced from (5.1), (5.2), and can be written in Cartan's notation as the evolution of the moving frame $(\mathbf{x}(s), \mathbf{t}(s), \mathbf{n}(s))$ as follows:

$$d\begin{pmatrix} \mathbf{x}(s) \\ \mathbf{t}(s) \\ \mathbf{n}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \end{pmatrix} ds. \tag{5.3}$$

The initial conditions $(\mathbf{x}(s_0), \mathbf{t}(s_0), \mathbf{n}(s_0))$ consist of an arbitrary unimodular affine coordinate frame, and the solution $(\mathbf{x}(s), \mathbf{t}(s), \mathbf{n}(s))$ is unique, meaning that the frame corresponding to any s to which the solution of (5.3) may be extended is related to the initial frame by a unique equiaffine motion. However, since the system reduces to a scalar third order equation, namely

$$\frac{d^3\mathbf{x}(s)}{ds^3} + \kappa(s)\,\frac{d\mathbf{x}(s)}{ds} = 0, \tag{5.4}$$

it is not easy to estimate the geometric shape of the solution. For instance, when does a periodic curvature function $\kappa(s)$ produce a closed curve solution?

A suggested exercise at this point is to compute the equiaffine arc length, the moving frame, and the affine curvature for the closed, convex curve defined as follows:

$$\mathbf{x}(s) = (\cos t - \tfrac{1}{10}\cos 3t, \sin t + \tfrac{1}{10}\sin 3t).$$

6. Local Coordinates.

Let , be a short, compact arc of a convex curve. One can choose, in many ways, an equiaffine coordinate system (x, y) such that , is the graph of a convex function $y = u(x)$, with x ranging over a compact interval $[x_0, x_1]$. We now rewrite the affine arc length, normal, and curvature in the given coordinate system. First, the element of equiaffine arc length of , is given by $ds = \sqrt[3]{u_{xx}}\,dx$, where the subscripts indicate successive differentiations with respect to x. It follows that the affine tangent and normal vectors at the point corresponding to x are

$$\mathbf{t} = (u_{xx})^{-1/3}(1, u_x), \qquad \mathbf{n} = \tfrac{1}{3}(u_{xx})^{-5/3}(-u_{xxx}, 3u_{xx}^2 - u_x u_{xxx}). \tag{6.1}$$

In particular, we have

Lemma 6.1. *The y-axis is parallel to the affine normal at a point $(x, u(x))$ if and only if $u_{xxx} = 0$.*

Finally, one deduces the formula

$$\kappa = \frac{3u_{xx}u_{xxxx} - 5u_{xxx}^2}{9(u_{xx})^{8/3}} \tag{6.2}$$

from the structure equations (5.3). As in the Euclidean case, the element of equiaffine arc length

$$ds = \sqrt[3]{u_{xx}}\;dx \tag{6.3}$$

is the simplest invariant one-form, and the curvature κ the simplest differential invariant for the equiaffine group in the plane. Every other differential invariant can be expressed as a function of κ and its successive derivatives with respect to to arc length. Since the equiaffine curvature is a fourth order differential invariant, the following equiaffine version of the definition of the Euclidean curvature via an osculating circle is immediate.

Definition 6.2. Let , be a smooth, convex curve, and let $A \in$, . The *osculating conic* to , at A is the unique conic passing through A having fourth order contact with , at A.

Theorem 6.3. *Two smooth, convex curves passing through a common point A have the same equiaffine curvature at A if and only if they have fourth order contact at A. In particular, the curvature to a curve , at A equals the (constant) curvature of its osculating conic at A.*

In particular, we need to know the explicit formula for the curvature of a general conic.

Theorem 6.4. *Consider a nondegenerate conic $\mathcal{C}$ defined by the quadratic equation*

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0. \tag{6.4}$$

The equiaffine curvature of $\mathcal{C}$ is given by

$$\kappa = \frac{S}{T^{2/3}}, \tag{6.5}$$

where

$$S = AC - B^2 = \det \begin{vmatrix} A & B \\ B & C \end{vmatrix}, \qquad T = \det \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}. \tag{6.6}$$

Both S and T are equi-affine invariants of the conic. The invariant S vanishes if and only if the five points lie on a parabola. The invariant T vanishes if and only if the conic degenerates to a pair of lines, and hence fails our convexity hypothesis.

Corollary 6.5. *The equiaffine curvature of an ellipse in the plane is given by $\kappa = (\pi/\mathbf{A})^{2/3}$, where $\mathbf{A}$ is the area of the ellipse.*

7. The Affine Normal.

We now begin our discussion of geometric approximations to the affine geometric quantities associated with a convex plane curve. Let , be, as before, a short arc, with end points A, B. Let M be the midpoint of the chord AB. Let the tangents to , at A and B intersect at a point P, so that , is inscribed in the support triangle $\mathbf{T} = APB$.

Theorem 7.1. *The direction of the median PM of the triangle $\mathbf{T}$ is a mean affine normal direction of , , in the sense that if , is of class C^3, then there exists at least one point of , where the affine normal is parallel to PM.*

Proof: Choose an equiaffine coordinate system (x, y) such that the y-axis includes the median PM in the direction indicated. Then , is the graph of a convex function $y = u(x)$ and, since M lies on the y-axis, u is defined over a symmetric interval $-a \le x \le a$ for some $a > 0$. At the same time, since P also lies on the y-axis, u satisfies the boundary

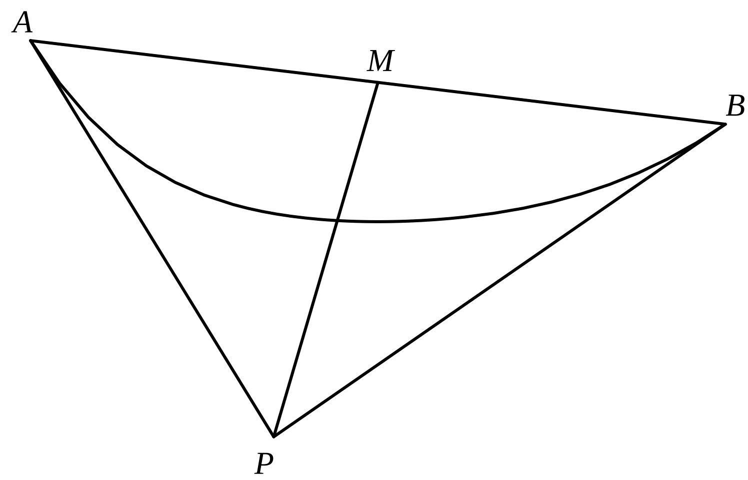

Figure 4. Median of Support Triangle.

condition $au_x(a) - u(a) = (-a)u_x(-a) - u(-a)$, which may be translated by integration by parts as follows:

$$0 = \int_{-a}^{a} d[xu_x - u] = \int_{-a}^{a} xu_{xx}\, dx = \int_{-a}^{a} \tfrac{1}{2}(a^2 - x^2)u_{xxx}\, dx.$$

Since the "weight" function $a^2 - x^2$ is positive in the interior of the interval, the third derivative u_{xxx} has a weighted mean value of zero. Lemma 6.1 completes the proof. *Q.E.D.*

There are several analogous statements, giving alternative geometric interpretations of some mean direction of the affine normal, but none are as simple to state or prove as the one just shown. However, we shall present some of these alternatives, because they may be better suited for generalizations to locally convex hypersurfaces in ^{n}. All of them deal with the support triangle APB, the midpoint M of the chord AB and various choices of an interior point $C \in$, , so that both CM and PC represent mean directions of the affine normal. We shall deal first with the case where C is the unique point of , where the tangent line is parallel to the line AB.

Theorem 7.2. *Let , be a short, strongly convex arc, inscribed in the triangle* $\mathbf{T} = APB$*, where* A*,* B *are the end points of , and the corresponding tangent lines intersect at* P*. Let* M *be the midpoint of the chord* AB*, and* $C \in$, *where the tangent line is parallel to the line* AB*. Then there exist* a) *a point* $C' \in$, *where the affine normal is in the same direction as the directed line* PC*, and* b) *a point* $C'' \in$, *where the affine normal is in the same direction as the directed line* CM*.*

Proof: We first prove the existence of C'. Let (x, y) be an affine coordinate system such that the y-axis contains the segment PC with the same orientation. Then , is the graph of a convex function $y = u(x)$ defined over a closed interval $a \leq x \leq b$ with $a < 0 < b$. The assumptions on PC correspond, in this coordinate system, to the following conditions: a) the point $C = (0, u(0))$ satisfies $u_x(0) = (u(b) - u(a))/(b - a)$, corresponding to the boundary condition

$$bu_{xx}(0) - u(b) = au_x(0) - u(a); \tag{7.1}$$

b) the point $P = (0, y_0)$ lying on the y-axis corresponds to the boundary condition

$$-y_0 = bu_x(b) - u(b) = au_x(a) - u(a). \tag{7.2}$$

The two boundary conditions (7.1), (7.2) can, in turn, be expressed in integral form, respectively, as follows:

$$\begin{aligned}
0 &= \int_a^b d[xu_x(0) - u(x)] = \int_a^b [u_x(0) - u_x(x)]\,dx = -\int_a^b \int_0^\xi u_{xx}(x)\,dx\,d\xi \\
&= -\int_0^b (b-x)u_{xx}\,dx + \int_a^0 (x-a)u_{xx}\,dx, \\
0 &= \int_a^b d[xu_x(x) - u(x)] = \int_a^b xu_{xx}(x)\,dx.
\end{aligned}$$

Adding the two equations and integrating by parts once more, we obtain:

$$\begin{aligned}
0 &= -\int_0^b (b-2x)u_{xx}(x)\,dx + \int_a^0 (2x-a)u_{xx}(x)\,dx \\
&= -\int_0^b x(b-x)u_{xxx}(x)\,dx + \int_a^0 (-x)(x-a)u_{xxx}(x)\,dx.
\end{aligned}$$

The last expression expresses the vanishing of the integral of a continuous third derivative u_{xxx}, weighted by a positive function, over the interval $[a, b]$. Therefore, we deduce the existence of an interior point $C' = (x', u(x'))$ such that $u_{xxx}(x') = 0$, meaning that the affine normal at C' is vertical.

In order to prove assertion b), we choose an affine coordinate system (x, y) such that the y-axis contains the segment CM. We prove the assertion, at first, under the additional assumption that , is the graph of a convex function $y = u(x)$. In this case, the assumptions translate into the following two statements: i) since M lies on the y-axis, u is defined on a symmetric interval $[-a, a]$ for some $a > 0$; ii) since the tangent to , at C is parallel to AB, $u_x(0) = (u(a) - u(-a))/(2a)$, leading to the following argument, similar to the previous ones:

$$\begin{aligned}
0 &= \int_{-a}^a d[xu_x(0) - u(x)] = \int_{-a}^a [u_x(0) - u_x(x)]\,dx = -\int_{-a}^a \int_0^\xi u_{xx}(x)\,dx\,d\xi \\
&= -\int_0^a (x-a)u_{xx}\,dx + \int_{-a}^0 (a+x)u_{xx}\,dx = \int_{-a}^a \tfrac{1}{2}(a-|x|)^2 u_{xxx}\,dx.
\end{aligned}$$

Once more, the vanishing of the last integral implies the existence of a point C'' where the affine normal is vertical — under the extra condition that , is a graph over $[-a, a]$.

Assume now that , is no longer a graph, and let A and B have coordinates $(-a, y_0)$ and (a, y_1) respectively. Then , includes either a point $A' = (-a, y_0')$ with $y_0' < y_0$, in which case the subarc from A' to B is the graph of a function, or else there exists a

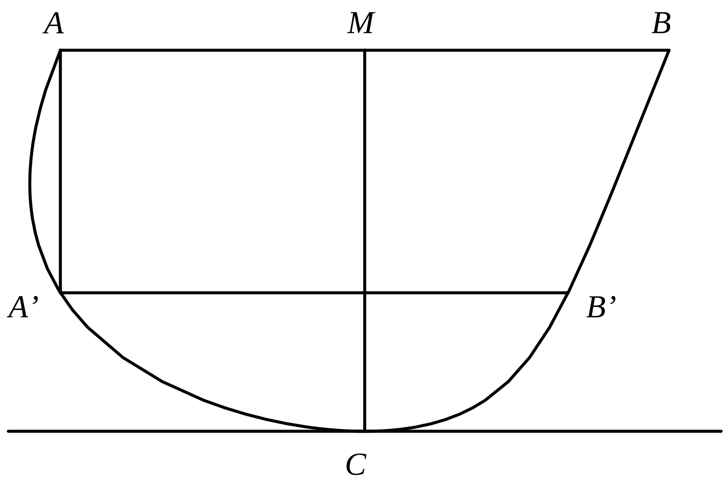

Figure 5. Mean Affine Normal.

point $B' = (a, y_1')$ with $y_1' < y_1$, in which case the subarc from A to B' is the graph of a function (but not both, since , is a short arc). Without loss of generality, we assume the former case, as in Figure 5. In this case, a vertical line drawn downwards (i.e., in the same direction as MC) from A meets , at another point A', and a line from A' in the direction of AB (i.e., to the *right*) meets , at a point B'. It is clear that the midpoint M' of the segment $A'B'$ is to the *left* of the segment CM. Therefore, replacing , by the subarc $,'$ from A' to B', one may apply the previous argument, whereby there is a point in $,'$ where the affine normal is in the same direction as CM', that is to say to the *left* of CM. On the other hand, the arc of , from A to A', by a similar argument, contains a point where the affine normal points to the right of the direction of $A'A$, or, equivalently, CM. By continuity, there exists a point C'' in the subarc of , from A to B' where the affine normal is in the same direction as CM. This concludes the proof of Theorem 7.2. *Q.E.D.*

A somewhat different geometrical construction of mean affine normals for a short convex arc is described by the following theorem.

Theorem 7.3. *Let , be a short, smooth, convex arc, inscribed in the triangle* $\mathbf{T} = APB$, *as above, and let* M *be the midpoint of the chord* AB. *For any point* $C \in ,$, *not an end point, denote by* P_A *and* P_B *the points of intersection of the tangent to , at* C *with the segments* PA *and* PB, *respectively. Then there exists at least one point* $C \in ,$ *which is the midpoint of the associated segment* $P_A P_B$. *Furthermore, for any such point, there exist points* $C', C'' \in ,$ *where the affine normal is in the same direction as* PC *or* CM, *respectively.*

Proof: To show the existence of the point C, consider the area of the triangle $P_A P P_B$ as C varies between A and B. The area is always positive, continuously dependent on C, and approaches zero as C approaches either A or B. The desired point C occurs when this area attains a (local) maximum value. Note that if , were to include a sub-arc $,'$ of a hyperbola having PA and PB as asymptotes, then *each* point $C \in ,'$ would have the desired property.

Assuming, then that C is the midpoint of its associated segment $P_A P_B$, we proceed to prove, first, that the direction PC occurs as a direction of an affine normal to , . Let (x, y) be an affine coordinate system such that the y-axis contains the segment PC with the same orientation. The assumptions imply, first of all, that , is the graph of a convex function $y = u(x)$ defined over a closed interval $a \leq x \leq b$ withe $a < 0 < b$. The assumptions that P lies on the y-axis is equivalent to the boundary condition (7.2). It is now convenient to choose the direction of the x-axis to be parallel to $P_A P_B$, which means that $u_x(a) = -u_x(b)$ and $u_x(0) = 0$. Our assumption that C is the midpoint of $P_A P_B$ means that

$$u_x(b) - 2u_x(0) + u_x(a) = 0. \tag{7.3}$$

We now introduce a Legendre transform of the function u: choose the strictly monotone function $u_x(x) - u_x(0)$ of x as the new independent variable $\hat{x}$ and the transformed function $\hat{y} = \hat{u}(\hat{x}) = x u_x(x) - u(x)$ as the new dependent variable. Then

$$d\hat{x} = u_{xx}\, dx, \qquad d\hat{y} = x u_{xx}\, dx = x\, d\hat{x}.$$

Therefore

$$\frac{d^2 \hat{u}}{d\hat{x}^2} = \frac{1}{u_{xx}} > 0.$$

The boundary condition (7.3), in terms of the transformed function, sets the interval of definition of $\hat{u}$ to be $[-\hat{a}, \hat{a}]$, where $\hat{a} = u_x(b) - u_x(0) = u_x(0) - u_x(a)$, while (7.2) becomes $\hat{u}(-\hat{a}) = \hat{u}(\hat{a})$. In addition, we have $\hat{u}_{\hat{x}}(0) = 0$. These conditions on the transformed variables and the function $\hat{u}$ include the properties of the function u in the proof of assertion $b)$ of Theorem 7.2, namely the graph $\widehat{\ }$ of $\hat{u}$, with end points $\widehat{A} = (-\hat{a}, \hat{u}(-\hat{a}))$, $\widehat{B} = (\hat{a}, \hat{u}(\hat{a}))$, and the point $\widehat{C} = (0, \hat{u}(0))$ such that $2\hat{a}\hat{u}(0) = \hat{u}(\hat{a}) - \hat{u}(-\hat{a})$ (both sides here being zero). Therefore there exists an intermediate value $\hat{x}'$ corresponding to $x' = \hat{u}_{\hat{x}}(\hat{x}') \in [a, b]$, for which $\hat{u}_{\hat{x}\hat{x}\hat{x}}(\hat{x}') = -u_{xxx}(x')/[u_{xx}(x')]^2 = 0$. This shows the existence of $C' = (x', u(x')) \in$, where the affine normal is in the same direction as PC†.

To show the existence of a point $C'' \in$, whose affine normal is in the same direction as CM, we arrange the y axis of our equiaffine coordinate system to include the segment CM in the positive direction, as in Figure 6. Introduce the chords AC and CB, and let M_A and M_B be their respective midpoints, such that the corresponding affine normals are positive scalar multiples of the vectors $M_A - P_A$ and $M_B - P_B$ respectively. On the other hand, taking into account the identities

$$M_A - P_A = \tfrac{1}{2}(A - P_A) + \tfrac{1}{4}(P_B - P_A), \qquad M_B - P_B = \tfrac{1}{2}(B - P_B) + \tfrac{1}{4}(P_A - P_B),$$

† It is possible, of course, to give the same proof without using Legendre transforms; however the steps to deduce, by two integrations by parts, from the assumptions on u the corresponding identity

$$0 = \int_a^b du(x) = \frac{1}{2}\int_a^b \Big[u_x(b) - u_x(0) - |u_x(x) - u_x(0)|\Big]^2 \frac{u_{xxx}}{(u_{xx})^2}\, dx$$

would seem much more opaque than the proof presented here.

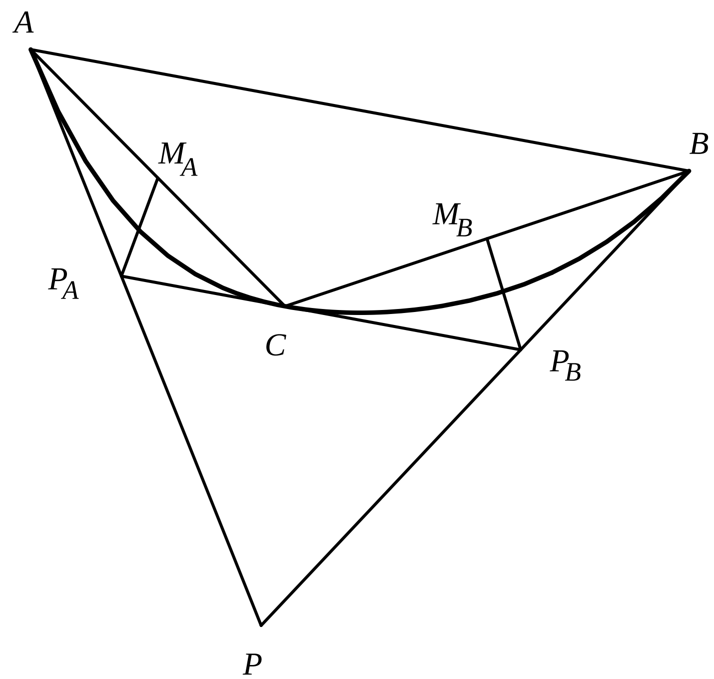

Figure 6. Mean Affine Normal.

we see that the vector

$$M - C = \tfrac{1}{2}(A + B) - \tfrac{1}{2}(P_A + P_B) = (M_A - P_A) + (M_B - P_B).$$

It follows that the direction of $M - C$ is intermediate between those of $M_A - P_A$ and $M_B - P_B$. Therefore, by continuity, there exists a point $C'' \in$, between C''_A and C''_B where the affine normal is a positive scalar multiple of the vector $M - C$. This completes the proof of Theorem 7.3. *Q.E.D.*

Having shown various ways to represent an "average" direction of the affine normal, one naturally seeks a corresponding "average" normalization in agreement with the formal definition of the equiaffine invariants. Recalling the definition of the equiaffine length of a short arc , (A, B), cf. Definition 4.4, we denote by $\mathbf{d}(A, B)$ the equiaffine distance between the line elements (A, AP) and (B, PB). Then the vector $B - A$ on the one hand is the "average" direction of the oriented tangents over , ; on the other hand it represents the integral $\int_A^B \mathbf{t}\, ds$, where $\mathbf{t}$ is the equiaffine tangent vector and ds is the element of equiaffine arc length. The estimation of the average value of $\mathbf{t}$ by $(B - A)/\mathbf{d}(A, B)$ is then obvious.

We choose now the average direction of the affine normal of the arc , according to Theorem 7.1, represented by the median vector $M - P$ of the support triangle APB. Observe that the alternating product $[B - A, M - P]$ is twice the area of the triangle APB, which is $\frac{1}{4}\mathbf{d}(A, B)^3$. Since $B - A$ is approximately equal to $\mathbf{d}(A, B)\mathbf{t}$, and, from the structure equations $[\mathbf{t}, \mathbf{n}] = 1$, it follows that the equiaffine normal vector $\mathbf{n}$ should, in the mean, be represented by $4\mathbf{d}(A, B)^{-2}(M - P)$. We formally summarize these estimates in the following theorem.

Theorem 7.4. *Let , be a short arc and let* $\mathbf{d}(A,B)$ *be the equiaffine distance between its endpoints* A *and* B*, i.e., twice the cube root of the area of its support triangle* APB*. Let* M *be the midpoint of the chord* AB*. Then a mean value for the equiaffine frame* $(\mathbf{t},\mathbf{n})$*, consisting of the tangent and normal to , , is represented by*

$$\mathbf{t}_{av} = \frac{B-A}{\mathbf{d}(A,B)}, \qquad \mathbf{n}_{av} = 4\,\frac{M-P}{\mathbf{d}(A,B)^2}.$$

8. The Affine Curvature.

The structure equation (6.2) has two obvious consequences that serve to interpret it in geometrical terms. In the first place, under infinitesimal displacements of a point on the curve, the equiaffine normal shifts parallel to the tangent. Secondly, the sign of the equiaffine curvature κ tells us which way the affine normal varies over small arcs of a convex curve. More precisely, if κ is everywhere positive in a short arc, then the affine normals at its endpoints, both pointing into the concave side of the curve, lean towards each other, like the Euclidean normals of a convex arc, while if $\kappa < 0$ everywhere, then the affine normals lean away from each other. One can apply the results of the last section to make these statements more precise.

Proposition 8.1. *Let , be a short arc of a smooth, convex curve, and let* APB *be its support triangle. Let* $C \in$ *, be the point whose tangent line is parallel to the chord* AB*, and let the tangent line at* C *intersect the segments* PA *and* PB *at* P_A *and* P_B *respectively. Let* $t = t_{A,B}$ *be the real number,* $0 < t < 1$ *defined by the equivalent vector relations* $P_A - P = t(A-P)$ *or* $P_B - P = t(B-P)$*. Then there exists a point on , where the equiaffine curvature* κ *is positive, negative, or zero according to whether* t *is, respectively,* $> \frac{1}{2}$*, or* $< \frac{1}{2}$*, or* $= \frac{1}{2}$*.*

Proof: We refer to Figure 7. Draw the chords of , from A to C and from C to B, and let M_A and M_B be their respective midpoints. It follows that the vector $M_B - M_A$ equals $\frac{1}{2}(B-A)$. Since the tangent at C is parallel to the line AB, it follows that $P_B - P_A = t(B-A)$, where t is as in the statement of the proposition. From Theorem 7.1 we known that the directed half-lines P_AM_A and P_BM_B represent mean directions of the equiaffine normal in the respective portions of , . To compare these two directions, one immediately verifies that

$$(M_B - P_B) - (M_A - P_A) = (M_B - M_A) - (P_B - P_A) = (\tfrac{1}{2} - t)(B-A).$$

The required conclusion now follows. *Q.E.D.*

Corollary 8.2. *Let , be a smooth, closed, convex curve without inflection points in the affine plane. Let* $\mathbf{B}$ *denote the convex body bounded by , . Let* $\mathbf{B}^*$ *denote the convex body neighborhood of* $\mathbf{B}$*, obtained as the Minkowski sum* $\mathbf{B}^* = 2\mathbf{B} + (-\mathbf{B})$*.*[†] *Then, from every point on the boundary of* $\mathbf{B}^*$ (and, a fortiori *from every exterior point of* $\mathbf{B}^*$*) one can "see" at least one point on , where the equiaffine curvature is positive.*

[†] In other words, $\mathbf{B}^*$ is the set of points P for which one can find points $M, Q \in \mathbf{B}$ such that M is the midpoint of the segment PQ.

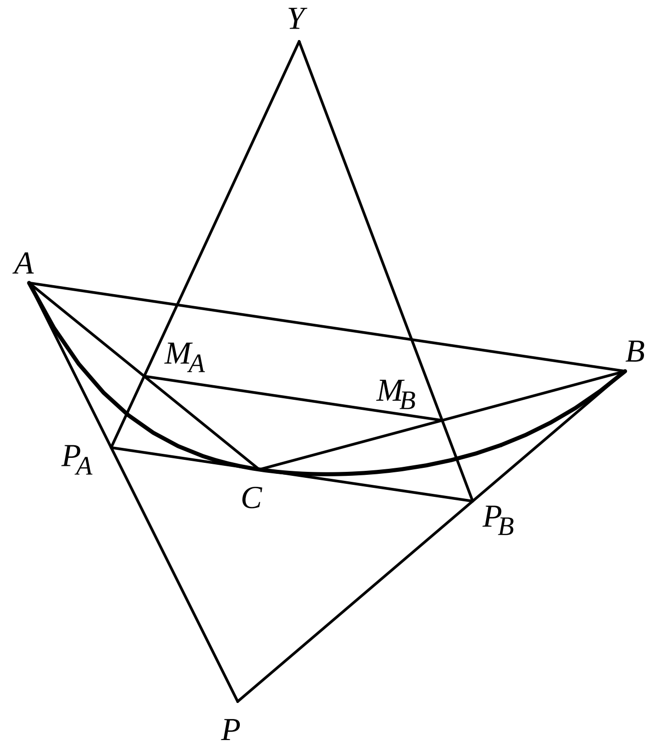

Figure 7. Affine Curvature Construction.

Remark: This statement is considerably stronger than one found in various textbooks, asserting the existence of a point with positive equiaffine curvature on any "half-oval", i.e., on any locally convex, smooth bounded arc whose tangents at the endpoints are parallel, and with no other pair of parallel tangents.

Proof: Let P be any point on the boundary of $\mathbf{B}^*$, and let PA and PB be the two tangent lines to , from P, where A and B are the respective points of contact with , . Let A' and B' be the midpoints of the respective segments PA and PB. It follows from the definition of $\mathbf{B}^*$ that the line $A'B'$ cannot meet $\mathbf{B}$. Consequently, if one draws the tangent line to the short arc of , between A and B, i.e., the set of points of , that are visible from P, then the ratio $t_{A,B}$ defined in Proposition 8.1 is strictly greater than $\frac{1}{2}$. *Q.E.D.*

To conclude this section, we shall refine the last proposition to yield a numerical approximation to the actual value of the equiaffine curvature of a short arc.

Theorem 8.3. *Let , be a short arc of a smooth, convex curve, with end points A, B, and the same construction as in Proposition 8.1. Let $\mathbf{d}(A,B)$ denote the equiaffine distance from A to B. In Figure 6, prolong the lines P_AM_A and P_BM_B to their intersection point Y (if necessary, in the projective completion of the plane), as depicted in Figure 8. The three points P, C, and Y lie on a common line. Let Q_C denote the intersection of that line with the chord AB, and consider the (negatively valued) cross ratio*

$$\rho(A,B) = [Q_C, P : Y, C] = \frac{(Q_CY : PY)}{(Q_CC : PC)}.$$

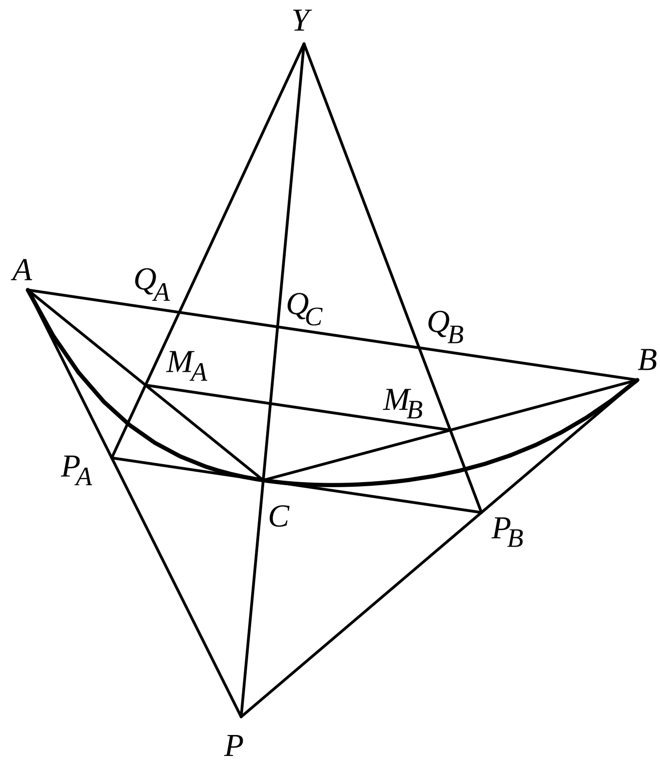

Figure 8. Equiaffine Curvature Approximation.

Then a mean value of the equiaffine curvature κ over , is represented by

$$\overline{\kappa_\Gamma} = 8\,\frac{1+\rho(A,B)}{\mathbf{d}(A,B)^2}. \tag{8.1}$$

Proof: The collinearity of the three points P, C, and Y follows from Desargues' Theorem. From the perspective point A, the four points P, C, Q_C, Y, defining the cross ratio ρ, may be projected to the corresponding points P_A, M_A, Q_A, Y in the line P_AM_A. (Equivalently, we can project from B to obtain P_B, M_B, Q_B, Y in the line P_BM_B.) Since M_A is the midpoint of P_AQ_A, the cross ratio ρ reduces to the scalar coefficient in the linear vector relation $Y - Q_A = -\rho(Y - P_A)$, whence

$$M_A - P_A = \tfrac{1}{2}(1+\rho)(Y - P_A). \tag{8.2}$$

According to Theorem 7.4,

$$M_A - P_A = \tfrac{1}{4}\mathbf{d}(A,C)^2\mathbf{n}(A,C), \tag{8.3}$$

where $\mathbf{d}(A,C)$ is twice the cube root of the area of the triangle AP_AC (and asymptotically the affine length of the arc AC in ,) and $\mathbf{n}(A,C)$ is a mean vector value of the equiaffine normal over the same arc. Furthermore, the point Y, marking the intersection of two neighboring affine normal lines, approximates, as the arc , is shortened to a point

$X \in \Gamma$, the corresponding point $X + \kappa^{-1}\mathbf{n}$ of the affine evolute of Γ at X. Combining the approximate relations $Y - P_A \approx \kappa^{-1}\mathbf{n}(A, C)$ with (8.3), we see from (8.2) that an approximate value $\overline{\kappa}$ of the equiaffine curvature is given by $\frac{1}{8}(1+\rho)\mathbf{d}(A, C)^{-2}$. Interchanging A and B, we have another approximation $\overline{\kappa} \approx \frac{1}{2}(1+\rho)\mathbf{d}(C, B)^{-2}$. Recalling Theorem 4.3, the equiaffine length of Γ is approximated by $\mathbf{d}(A, B)$, or, alternatively, by $2\mathbf{d}(A, C)$ or $2\mathbf{d}(C, B)$. Combining these formulas completes the proof of (8.1). *Q.E.D.*

Although (8.1) can in principle be used as a method for approximating the affine curvature, it has several numerical difficulties that preclude its direct use. First, the construction relies on the introduction of the tangent lines at the point A and B, and hence we need to introduce an additional numerical approximation. Moreover, the approximation needs to incorporate affine invariances, and so the standard difference quotient is not satisfactory for this purpose. More serious is the instability in the computation of the intersection point Y, which can be at infinity (and indeed is if the curve is a parabola), and is thus highly unstable from a numerical point of view. Presumably, one can overcome the latter difficulty by mutliplying the numerator and denominator in the ratio (8.1) by an appropriate factor, although we have not thoroughly investigated this as of yet.

9. Finite Difference Approximations of Affine Invariants.

In this section, we discuss a fully affine-invariant finite difference approximation to the affine curvature of a convex curve in the plane. The starting point is the result that one can approximate the (positive) affine curvature at a point of a plane curve by the affine curvature of the conic section passing through five nearby points. We will explicitly show how this may be used to produce an affine-invariant finite difference approximation to the affine curvature. The first item is to determine the formula for the affine curvature of a conic passing through five points.

Given a numbered set of points P_i, $i = 0, 1, 2, \ldots$, we let

$$[ijk] = [P_i, P_j, P_k] = (P_i - P_j) \wedge (P_i - P_k)$$

denote twice the (signed) area of the triangle with vertices P_i, P_j, P_k, cf. (3.2). See [**23**], [**28**], for a proof of the following elementary fact.

Theorem 9.1. *Let $P_0, \ldots, P_4$ be five points in general position in the plane. There is then a unique conic section $\mathcal{C}$ passing through them, whose quadratic equation has the affine-invariant form*

$$[013][024][\mathbf{x}12][\mathbf{x}34] = [012][034][\mathbf{x}13][\mathbf{x}24], \tag{9.1}$$

where $\mathbf{x} = (x, y)$ is an arbitrary point on $\mathcal{C}$.

In order to compute the affine curvature of the conic (9.1), we use formula (6.5), and thus need to compute the two equiaffine invariants S, T, as given in (6.6), in equiaffine invariant form. In other words, the resulting formula should be written in terms of the

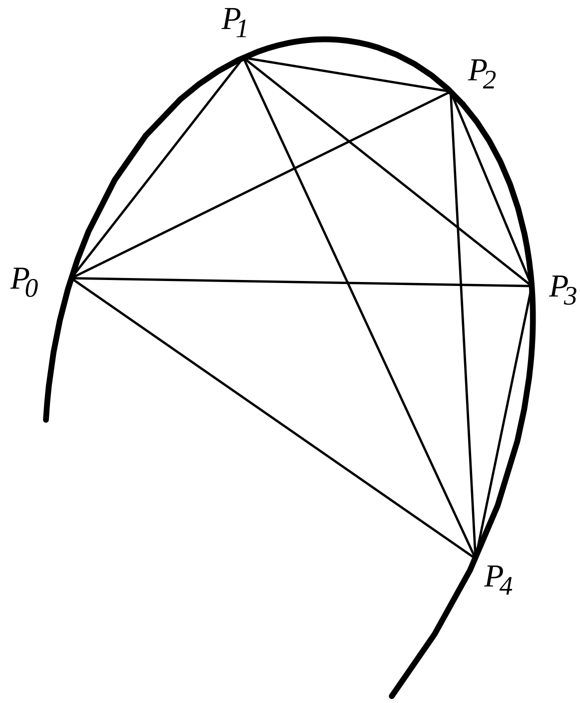

Figure 9. The Affine Pentagram.

areas of the $\binom{5}{3} = 10$ triangles determined by the points taken three at a time; see Figure 9. (Of course, only 5 of these areas are independent, due to the syzygies listed below.) Substituting the formulas for the coefficients, we find a particularly nice affine-invariant expression for our first affine invariant

$$4T = \prod_{0 \le i < j < k \le 4} [ijk]; \tag{9.2}$$

in other words, to compute T, multiply together all 10 triangular areas in the pentagram described by the 5 points. The fact that T has such a form is not so surprising, since T vanishes if and only if the conic degenerates to a pair of lines, which requires that three of the five points lie on a line, meaning that $[ijk] = 0$ for some $i < j < k$.

One affine-invariant formula for S is found directly:

$$\begin{aligned} 4S = {} & [013]^2[024]^2[1234]^2 - \\ & - \; 2[012][034][013][024]\big([123][234] + [124][134]\big) + [012]^2[034]^2[1324]^2. \end{aligned} \tag{9.3}$$

where

$$[ijkl] = (P_i - P_k) \wedge (P_j - P_l) = [ijl] - [ijk].$$

Formula (9.3) is not nearly as pleasant as (9.2), particularly because the right hand side appears to be unsymmetrical with respect to permuations of the five points. However, S

must clearly be symmetrical with respect to these permutations. Of course, the explanation lies in the syzygies among the triangular areas: these are

$$[123] = [012] + [023] + [031], \tag{9.4}$$

$$[012][034] - [013][024] + [014][023] = 0, \tag{9.5}$$

and the analogous formulas obtained by permutation of the symbols $0, \ldots, 4$, cf. (2.5), (2.6). A judicious application of (9.4), (9.5), will suffice to demonstrate that (9.3) is symmetrical under permutation. A completely symmetrical formula for S can, of course, be obtained by symmetrizing (9.3), i.e., summing over all possible permutations of the set $\{0, 1, 2, 3, 4\}$ and dividing by $5! = 120$, although the result is much more complicated than (9.3). We have been unable to find a simple yet symmetrical version of the formula for S.

As in the Euclidean case, we are interested in numerical approximations to the affine curvature of a strongly convex plane curve , which are invariant under the special affine group. As before, we approximate the parametrized curve $\mathbf{x}(r) = (x(r), y(r))$ by a sequence of mesh points $P_i = \mathbf{x}(r_i)$. Any affine-invariant numerical approximation to the affine curvature κ (as well as any other affine differential invariant $d^n\kappa/ds^n$) must be a function of the joint affine invariants of the mesh points, which means that it must be a function of the areas $[ijk] = [P_i P_j P_k]$ of the parallelograms (or triangles) described by the mesh points. Because the affine curvature is a fourth order differential function, the simplest approximation will require five mesh points, so that the approximation will depend on the ten triangular areas (or, more basically, the five independent areas) in the pentagram whose vertices are the five mesh points; see Figure 9.

With this in mind, let us number the five successive mesh points as P_0, P_1, P_2, P_3, P_4. (This is just for simplicity of exposition; of course, in general, one should replace the indices $0, \ldots, 4$ by $i, i+1, i+2, i+3, i+4$.) Since we are assuming that , is convex, the mesh points are in general position. Let $\mathcal{C} = \mathcal{C}(P_0, P_1, P_2, P_3, P_4)$ be the unique conic passing through the mesh points. Let $\widetilde{\kappa} = \widetilde{\kappa}(P_0, P_1, P_2, P_3, P_4)$ denote the affine curvature of the conic $\mathcal{C}$, which we evaluate via the basic formula (6.5), where the invariants S, T are computed in terms of the triangular areas according to (9.3), (9.2). We regard $\widetilde{\kappa}$ as a numerical approximation to the affine curvature $\kappa = \kappa(P_2)$ of , at the middle point P_2. We now need to analyze how closely the numerical approximation $\widetilde{\kappa}$ is to the true curvature κ at the point P_2. Assuming the points are close together (see the discussion below), we need to compute a Taylor series expansion of the distance $\widetilde{\kappa}$. An extensive MATHEMATICA computation produces the desired result

Theorem 9.2. *Let P_0, P_1, P_2, P_3, P_4 be five successive points on the convex curve , . Let κ be the affine curvature of , at P_2, and let $\widetilde{\kappa}$ denote the affine curvature of the conic section passing through the five points. Let*

$$L_i = \int_{P_2}^{P_i} ds, \qquad i = 0, \ldots, 4, \tag{9.6}$$

denote the signed affine arc length of the conic from P_2 to P_i; in particular $L_2 = 0$. We assume that each L_i is small. Then the following expansion is valid:

$$\widetilde{\kappa} = \kappa + \frac{1}{5}\left(\sum_{i=0}^{4} L_i\right)\frac{d\kappa}{ds} + \frac{1}{30}\left(\sum_{0\le i\le j\le 4} L_i L_j\right)\frac{d^2\kappa}{ds^2} + \cdots. \tag{9.7}$$

The higher order terms are cubic in the distances L_i.

Remark: The property of "being close" is therefore expressed in affine-invariant form as the statement that all the arc lengths $L_0, \ldots, L_4$ are small. In this way, we are able to introduce a fully affine-invariant notion of "distance", albeit one that requires knowledge of five, rather than two, points.

Proof: This is found by a direct Taylor series expansion of the affine-invariant expressions (9.3), (9.2), for the affine invariants S, T, and then substitution into the formula (6.5) for the curvature of the conic section. We represent the curve as the graph of $y = u(x)$, which, assuming the three points are sufficiently close, can always be arranged. The points can be assumed to be $P_0 = (h, u(h))$, $P_1 = (i, u(i))$, $P_2 = (0,0) = (0, u(0))$, $P_3 = (j, u(j))$, $P_4 = (k, u(k))$, where h, i, j, k are small. The areas are then given, for example, by

$$[013] = (h-i)(u(h)-u(j)) - (h-j)(u(h)-u(i)) = hu(i) - iu(h) - hu(j) + ju(h),$$

with elementary Taylor series expansion. The result is a Taylor series expansion for $\widetilde{\kappa}$ in terms of h, i, j, k, with leading term κ, as given in (6.2), the derivatives being evaluated at 0. However, h, i, j, k, being the differences of the x coordinates of the mesh points, are not affine invariant, and hence the coefficients of the expansion are not affine differential invariants. To remedy this, we must introduce the affine arc lengths (9.6) as our basic affine-invariant paramaters. Using the formula (6.3) for the affine-invariant arc length element, the expansion

$$L_0 = \int_0^h \sqrt[3]{u_{xx}}\, dx = h\sqrt[3]{u_{xx}} + \tfrac{1}{6}h^2 \frac{u_{xxx}}{(u_{xx})^{2/3}} + \cdots, \tag{9.8}$$

can be inverted to produce a Taylor series expressing h in terms of L_0. Plugging this, and the analogous series for i, j, k into the previous Taylor series produces the final result. *Q.E.D.*

Remark: An affine invariant finite difference approximation to the affine normal can also be found by computing the affine normal to the approximating conic $\mathcal{C}$ at the middle point P_2, and expressing this in terms of the triangular areas. The method can also produce invariant numerical methods for computing $d\kappa/ds$, etc., using more points.

10. A General Conjecture.

The reader has probably already noticed that the Euclidean and affine curvature approximation series (3.9), (9.7), bear a remarkable similarity. This suggests a generalization which we indicate here, albeit as a conjecture without proof. We begin by surveying the general theory of differential invariants of finite-dimensional Lie groups of transformations in the plane; see [**17**] for a detailed presentation. Let G be an r-dimensional Lie group acting on $E = \ {}^2$, with coordinates x, y, and let $\mathfrak{g}$ denote its Lie algebra of infinitesimal generators, which are vector fields $\mathbf{v} = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ on E. Curves in the plane are then (locally) represented as functions $y = u(x)$. Let J^n denote the n^{th} jet space of E, which has coordinates $(x, u^{(n)}) = (x, u, u_x, u_{xx}, \ldots, u_n)$. There exists a G-invariant arc length element $ds_G = P(x, u^{(n)})\, dx$ represented by the simplest (lowest order) G-invariant one-form, and a G-invariant curvature κ_G, which is the simplest (lowest order) differential invariant. We also assume that G determines an "ordinary" action, meaning that it acts transitively and locally effectively on E, and, moreover, its prolonged actions $G^{(n)}$are also locally transitive on a dense open subset of J^n for all $0 \le n \le r-2$, where r is the dimension of G. (In the language of [**17**], G admits no pseudo-stabilization of the prolonged orbit dimensions.) Indeed, Lie's complete classification of all finite-dimensional transformation groups on the plane, [**15**], [**17**], shows that, of the transitive groups, only the elementary similarity group $(x, u) \mapsto (\lambda x + c, \lambda u + d)$ and some minor variants thereof fail this hypothesis. Under these assumptions, the G-invariant arc length has order $n \le r-2$ and the G-invariant curvature $\kappa(x, u^{(r-1)})$ has order exactly $r-1$. The solutions to the ordinary differential equation

$$\kappa(x, u^{(r-1)}) = c, \tag{10.1}$$

for c constant determine the curves of constant curvature for the group action. In fact, one does not need to integrate the ordinary differential equation (10.1), since these curves can be found directly from the group action.

Proposition 10.1. *A curve* $\ \subset M$ *has constant G-invariant curvature if and only if it is the orbit,* $\ = \exp(t\mathbf{v})P_0$, *of some point* $P_0 \in M$ *under a one-parameter subgroup* $\exp(t\mathbf{v}) \subset G$ *determined by an infinitesimal generator* $\mathbf{v} \in \mathfrak{g}$.

Thus, for the Euclidean group, we recover the circles and straight lines as the constant curvature curves, while for the special affine group, these are the conic sections. Since (10.1) has order $r-1$, given r points $P_1, \ldots, P_r \in E$ in "general position", there exists a unique constant curvature curve $\ {}_0(P_1, \ldots, P_r)$ passing through them. Let $\kappa_0(P_1, \ldots, P_r)$ denote its curvature. Since (10.1) is a G-invariant ordinary differential equation, $\kappa_0(P_1, \ldots, P_r)$ is a joint invariant of the r points.

Let $\ \subset M$ be an arbitrary curve in the plane. We are interested in constructing a G-invariant finite difference approximation to its G-invariant curvature $\kappa(P_1)$ at a given point $P_1 \in \ $ in the curve. Choose $r-1$ nearby points $P_2, \ldots, P_r \in \ $. Then the curvature $\kappa_0 = \kappa_0(P_1, \ldots, P_r)$ of the constant curvature curve $\ {}_0 = \ {}_0(P_1, \ldots, P_r)$ passing through the points determines our approximation to $\kappa(P_1)$.

Conjecture: The following series expansion holds:

$$\kappa_0 = \kappa + \frac{1}{r}\left(\sum_{i=1}^{r} L_i\right)\frac{d\kappa}{ds} + \frac{1}{r(r+1)}\left(\sum_{1 \le i \le j \le r} L_i L_j\right)\frac{d^2\kappa}{ds^2} + \cdots, \tag{10.2}$$

where κ, $d\kappa/ds$, etc. are evaluated at P_1, and

$$L_i = \int_{P_1}^{P_j} ds, \tag{10.3}$$

denotes the G-invariant "distance" from the point P_1 to P_j, measured as the G-invariant arc length along the constant curvature curve Γ_0. (In particular, $L_1 = 0$.) The expansion assumes that all the arc lengths L_i are small.

Example 10.2. Consider the translation group $(x, u) \mapsto (x + c, u + d)$. In this case, $\kappa = du/dx$, and the constant curvature curves are the straight lines. Then $\kappa_0(p_1, p_2) = (u_2 - u_1)/(x_2 - x_1)$. Therefore, the expansion (10.2) is merely the Taylor series, and so is valid to general order! (Note that since dx is the translation-invariant arc length, the "length" of a straight line segment is $\int_{P_1}^{P_2} dx = x_2 - x_1$.)

Thus, the conjectured series expansion (10.2) is valid up to order 2 for the translation group, the Euclidean group, and the special affine group. Direct verification for other planar groups appears to be problematic because the formulas for the finite difference approximation κ_0 are not so easy to come by, because the constant curvature curves involve transcendental functions. Moreover, preliminary computations with the Euclidean group indicate that the natural generalization of (10.2) is not valid to order 3. Thus, the series should be viewed perhaps more as an interesting speculation rather than a firm conjecture.

11. Affine Curvature Flow.

In this section, we will present several new solutions to an affine invariant nonlinear heat equation which arose out of certain problems in vision and image processing. Indeed, invariant theory has recently become a major topic of study in computer vision. Since the same object may be seen from a number of points of view, one is motivated to look for shape invariants under various transformations. A closely related topic that has been receiving much attention from the image analysis community is the theory of scale-spaces or multiscale representations of shapes for object recognition and representation; see [**12**] for an extensive list of references on the subject. Initially, most of the work was devoted to linear scale-spaces derived filtering using a Gaussian kernel or equivalently running the shapes through the linear heat equation. Here the variance of the filter (or equivalently the time of the heat equation) plays the role of a scale-space parameter. The greater the variance (or time), the more the given shape is smoothed. Of course, the diffusion being isotropic, the shape will be blurred as well. To remedy such problems, in the last few years, a number of non-linear and geometric scale-spaces have been investigated as well. The idea is to introduce a nonlinear smoothing which preserves edges while smoothing on either side of the edge. Such nonlinear smoothing methods may be found in [**21**], [**4**], and [**13**]. See also the references in [**12**]. The combination of invariant theory and geometric multiscale analysis was investigated in [**24**] and [**25**]. There, the authors introduced an affine invariant geometric scale-space. Part of this work was extended to other groups as well in [**18**], [**19**]; see also [**4**]. The shape representations which we derive allow us to compute invariant signatures at different scales and in a robust way. These flows are already being used with satisfactory results in various applications [**12**].

The affine curvature flow was introduced precisely to give an affine invariant multiscale representation of planar shape. For closed convex curves, the affine curvature flow is found by evolving the curve in the direction of its affine normal, proportional to the affine curvature, and so has the form

$$\mathbf{x}_t = \kappa\, \mathbf{n}, \tag{11.1}$$

where κ is the affine curvature and $\mathbf{n}$ the affine normal. Interestingly, although the affine curvature depends on fourth order derivatives of the parametrized curve, we can replace (11.1) by a much simpler flow whose image curves are the same, differing only by an inessential reparametrization. Specifically, we rewrite the affine normal $\mathbf{n}$ in terms of the tangential and Euclidean normal directions to the curve, leading to an evolutionary flow of the form

$$\mathbf{x}_t = \sqrt[3]{\kappa_e}\, \mathbf{n}_e + F\, \mathbf{t}, \tag{11.2}$$

where κ_e is the *Euclidean* curvature, $\mathbf{n}_e$ the *Euclidean* (inward) normal, and F is some function of the curvature and its derivatives whose precise form is irrelevant. Indeed, since the tangential component $F\,\mathbf{t}$ in (11.2) only affects the reparametrization of the image curve, this term can be safely omitted (or even replaced by any other convenient tangential term $\widetilde{F}\,\mathbf{t}$ if desired). As a result, the flow (11.1) can be written in the equivalent (but non-affine-invariant) form

$$\mathbf{x}_t = \sqrt[3]{\kappa_e}\, \mathbf{n}_e. \tag{11.3}$$

Note that, in the form (11.3), the flow can be extended to nonconvex curves — inflection points have zero Euclidean curvature, but that does not cause any difficulties for either the Euclidean normal or curvature, even though the affine curvature is not well defined at such points. In particular, if the curve is given as the graph of a function $y = u(x)$, then the normal version (11.3) of the affine curvature flow has the particularly simple form

$$u_t = \sqrt[3]{u_{xx}}\,. \tag{11.4}$$

An alternative, useful formulation of the affine flow is obtained by taking the cross product of (11.2) with the tangent vector $\mathbf{x}_r$ to the parametrized curve $\mathbf{x}(r,t)$ at time t. Using the formula (3.4) for the Euclidean curvature of a parametrized curve $\mathbf{x}(r) = (x(r), y(r))$, and the fact that $\mathbf{x}_r \wedge \mathbf{n}_e = |\mathbf{x}_r|^2$, we find that the components $x(r)$, $y(r)$ of $\mathbf{x}$ must satisfy the underdetermined second order partial differential equation

$$(\mathbf{x}_r \wedge \mathbf{x}_t)^3 = \mathbf{x}_r \wedge \mathbf{x}_{rr}, \qquad \text{or} \qquad (x_r y_t - x_t y_r)^3 = x_r y_{rr} - x_{rr} y_r. \tag{11.5}$$

Conversely, since $\mathbf{x}_r$ is parallel to the unit tangent vector $\mathbf{t}$, given a solution to (11.5), it must also satisfy (11.2) for some choice of tangential component $F\mathbf{t}$, and hence the individual image curves for each value of t will describe solution to the affine curvature flow (11.1) (although they will not necessarily have the correct parametrization to satisfy the normal version (11.3)).

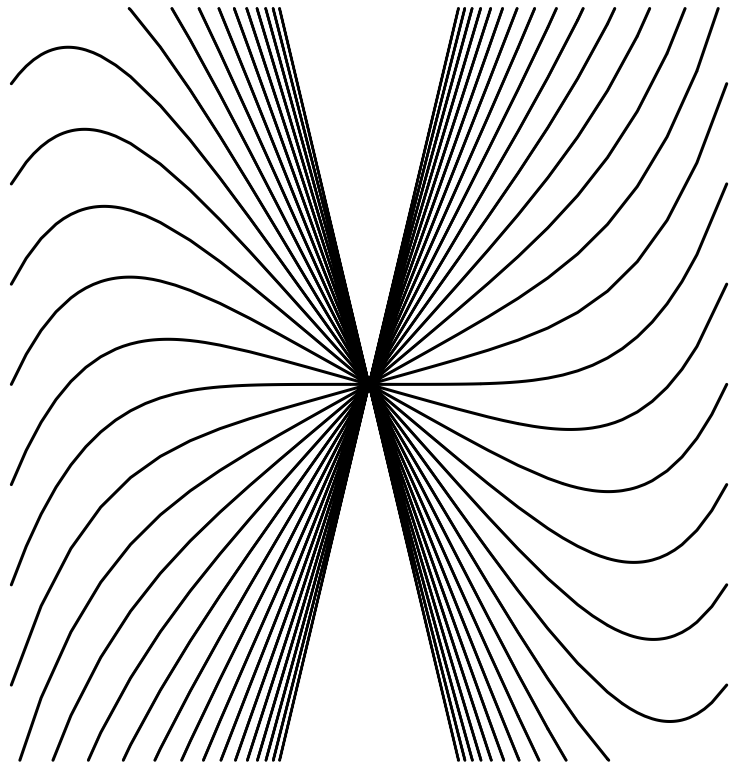

Figure 10. Quintic Fan.

Very few nontrivial particular solutions to the affine curvature flow are known. First, the curves of constant affine curvature will evolve analogously to the Euclidean circles. Thus, an initial ellipse will remain elliptical, having the same eccentricity and orientation, while collapsing to a point. Moreover, recent results of Angenent, Sapiro, Tannenbaum, [**5**], demonstrate a smooth closed embedded curve remains regular, eventually shrinking to a point.

There is an affine analogue of the grim reaper. If we choose the vertical axis to translate along, the solution is obtained by setting $u_t = c$, for constant c in (11.4). The result is a vertically translating parabolic solution

$$y = u(x,t) = \tfrac{1}{2} c^3 (x - x_0)^2 + c(t - t_0). \tag{11.6}$$

We have also constructed some examples of solutions having inflection points. The simplest example is found by assuming $u_t = cx$ in (11.4). Omitting an inessential integration constant, the resulting solution is

$$y = u(x,t) = \tfrac{1}{20} c^3 x^5 + ctx,$$

which, at each value of t, has a stationary inflection point at the origin. The successive image curves of the "quintic fan" are depicted in Figure 10.

To produce explicit solutions with non-stationary inflection point is more difficult. The following procedure allows us to describe another solution having a moving inflection

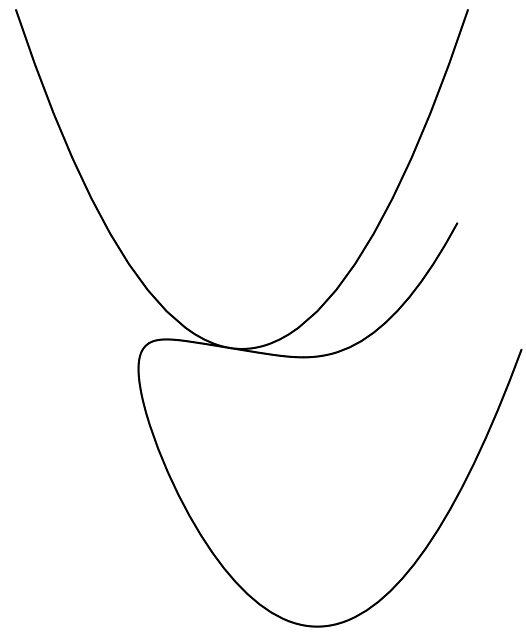

Figure 11. Parabolic Scooper with Inflection Point Locus.

point. In fact the whole curve precesses under the one-parameter affine flow $(x,y) \mapsto (x+s, y+sx+\frac{1}{2}s^2+cs)$, with infinitesimal generator $\partial_x + (x+c)\partial_y$, where c is a constant. With this in mind, we consider the parametrized family of curves

$$\mathbf{x}(r,t) = (w(r)+t, \tfrac{1}{2}(w(r)+t)^2 + br + ct), \tag{11.7}$$

where $w(r)$ is to be determined. Substituting into (11.5), we find that w must satisfy the reduced ordinary differential equation

$$bw_{rr} = bw_r^3 - (cw_r - b), \tag{11.8}$$

which can clearly be integrated by quadrature. Let us specialize to the case $b = c = 1$, whereby (11.8) reduces to the first order differential equation

$$\frac{dz}{dr} = 1 - 3z + 3z^2 \tag{11.9}$$

in $z = w_r$. Solving (11.9) (and ignoring integration constants), we find

$$z(r) = \frac{1}{2} + \frac{1}{2\sqrt{3}} \tan \frac{\sqrt{3}}{2} r, \qquad w(r) = \tfrac{1}{2} r - \frac{1}{3} \log \cos \frac{\sqrt{3}}{2} r, \tag{11.10}$$

which, upon substitution into the defining equation

$$\mathbf{x}(r,t) = (w(r)+t, \tfrac{1}{2}(w(r)+t)^2 + r + t), \tag{11.11}$$

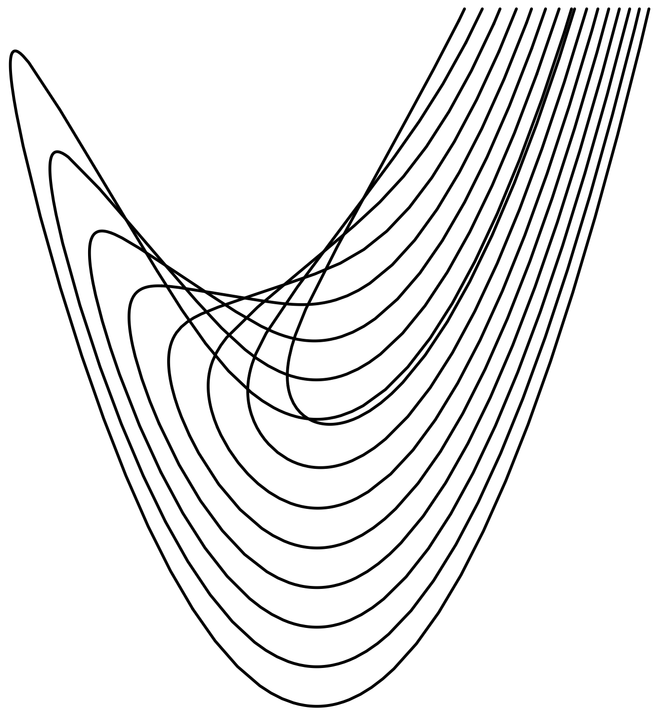

Figure 12. Parabolic Scooper.

produces the desired solution. The curve looks like a version of the Euclidean grim reaper, but bent around a parabola. It has a triple inflection point on the interior half of the curve, close to the point of maximum curvature. Since $\mathbf{x}_r \wedge \mathbf{x}_{rr} = 0$ at the inflection point, it is explicitly given by solving $z = w_r = 1$, so that $r = 2\pi/3\sqrt{3}$. Thus, the inflection point traces out a parabola

$$y = \tfrac{1}{2}x^2 + x + \frac{\pi}{3\sqrt{3}} - \frac{1}{3}\log 2;$$

see Figure 11 for a picture of the curve (at $t = 0$) and the parabolic locus traced out by the inflection points. The actual curve evolution, which we propose to call the "parabolic scooper" is depicted in Figure 12; notice the inflection parabola reappearing as an envelope to the curve family.

An important point of interest of this example is that it describes the asymptotic behavior in the small of any (triple) inflection point that is precessing. Indeed, translating the inflection point for $t = 0$ to the origin, one finds the following convenient reparametrization

$$x(r) = -\int \frac{dr}{1+r^3} = -r + \frac{1}{4}r^4 - \cdots,$$
$$y(r) = \frac{1}{2}x(r)^2 - \int \frac{r\,dr}{1+r^3} = -\frac{1}{20}r^5 + \frac{11}{112}r^8 - \cdots,$$

clearly demonstrating the nature of the inflection point.

Finally, for numerical implementations of the affine flow based on the Osher-Sethian level curve algorithm [**20**] as well as explicit applications to invariant planar curve representation, see [**25**]. Implementations based on the applications of our affine-invariant finite difference approximations are under development, and will form the subject of a subsequent work.

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