The Structured Singular Value for Linear Input/Output Operators

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Abstract

In this paper, we employ a lifting method introduced by the authors in order to study the structured singular value applied to input/output operators of control systems. We moreover give a new criterion which guarantees that the structured singular value equals its upper bound defined by D-scalings.

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1 Introduction

Let A be a linear operator on a Hilbert space $\mathcal{E}$, and let Δ be an algebra of operators on $\mathcal{E}$. The structured singular value of A (relative to Δ) is the number

$$\mu_\Delta(A) = 1/\inf\{\|X\| : X \in \Delta, \ -1 \in \sigma(AX)\}\,.$$

This quantity was introduced by Doyle and Safonov [6, 12] under a more restrictive context, and it has proved to be a powerful tool in robust system analysis and design. In system analysis, the structured singular value gives a measure of robust stability with respect to certain perturbation measures. Unfortunately, $\mu_\Delta(A)$ is very difficult to calculate, and in practice an upper bound for it is used. This upper bound is defined by

$$\widehat{\mu}_\Delta(A) := \inf\{\|XAX^{-1}\| : X \in \Delta', X \text{ invertible}\},$$

where Δ' is the commutant of the algebra Δ.

In [1, 5], we formulated a lifting technique for the study of the structured singular value. The basic idea is that $\widehat{\mu}_\Delta(A)$ can be shown to be equal to the structured singular value of an operator on a bigger Hilbert space. (In [1] this was done for finite dimensional Hilbert spaces, and then in [5] this was extended to the infinite dimensional case.) The problem with these results is that the size of the ampliation necessary to get $\widehat{\mu}_\Delta(A)$ equal to a structured singular value, was equal to the dimension of the underlying Hilbert space. Hence in the infinite dimensional case we needed an infinite ampliation. In this work, we will show that in fact, one can always get by with a finite lifting. (Note that in this paper we will be using the terms "ampliation" and "lifting" interchangeably.) For the block diagonal algebras of interest in robust control, the ampliation only depends on the number of blocks of the given perturbation structure. (See Theorem 1 below.) We moreover, give a new result when $\widehat{\mu}_\Delta(A) = \mu_\Delta(A)$, that is, when no lifting is necessary and so $\widehat{\mu}_\Delta(A)$ gives a nonconservative measure of robustness. (See Theorem 2.) This is then used to derive an elegant result of Shamma [13, 14] on Toeplitz operators. See also [7, 9, 10] for related work in this area.

We now briefly sketch the contents of this paper. In Section 2, we give some background results which will be needed in the proof of Theorem 1. In Section 3, we derive a number of useful facts about the relative numerical range. Then in Section 4, we state and prove our new version of the lifting

theorem relating the structured singular value and its upper bound. In Section 5, we give new conditions when $\mu = \hat{\mu}$. These are applied in Section 6, to give a new proof of the aforementioned result of Shamma. Finally, in Section 7, we give a system-theoretic interpretation of our lifting methodology.

2   Preliminary Results

Denote by $\mathcal{L}(\mathcal{E})$ the algebra of all bounded linear operators on the (complex, separable) Hilbert space $\mathcal{E}$. Fix an operator $A \in \mathcal{L}(\mathcal{E})$ and a subalgebra $\Delta \subset \mathcal{L}(\mathcal{E})$. The numbers $\mu_\Delta(A)$ and $\hat{\mu}_\Delta(A)$ have already been defined in the Introduction. Observe that $\Delta \subset \Delta''$ and $\Delta''' = (\Delta'')' = \Delta'$ so that we have the inequalities

$$\mu_\Delta(A) \leq \mu_{\Delta''}(A), \quad \hat{\mu}_\Delta(A) = \hat{\mu}_{\Delta''}(A) \ .$$

Observe that the algebras Δ considered in [6] consisted of block diagonal matrices, so our approach is more general in this respect. In the following proposition we summarize some of the elementary properties of μ_Δ; see Doyle [6] or [1] for proofs. We will denote by $\|T\|_{\text{sp}}$ the spectral radius of the operator T.

Lemma 1   *(i)* $\mu_\Delta(A) = \sup\{\|AX\|_{\text{sp}} : X \in \Delta, \ \|X\| \leq 1\}$;

(ii) μ_Δ *is upper semicontinuous;*

(iii) *If $\mathcal{E}$ is finite dimensional, then μ_Δ is continuous;*

(iv) $\mu_\Delta(A) \leq \hat{\mu}_\Delta(A)$.

In our study we will need further singular values which we now define. For $n \in \{1, 2, \ldots, \infty\}$ we denote by $\mathcal{E}^{(n)}$ the orthogonal sum of n copies of $\mathcal{E}$, and by $T^{(n)}$ the orthogonal of n copies of $T \in \mathcal{L}(\mathcal{E})$. Operators on $\mathcal{E}^{(n)}$ can be represented as $n \times n$ matrices of operators in $\mathcal{L}(\mathcal{E})$, and $T^{(n)}$ is represented by a diagonal matrix, with diagonal entries equal to T.

Denote by Δ_n the algebra of all operators on $\mathcal{E}^{(n)}$ whose matrix entries belong to Δ, and observe that $(\Delta_n)'' = (\Delta'')_n$, and $(\Delta_n)' = (\Delta')^{(n)} = \{T^{(n)} : T \in \Delta'\}$. Therefore we will denote these algebras by Δ_n'' and Δ_n', respectively.

Lemma 2 *For every finite number n we have*

$$\mu_\Delta(A) \le \mu_{\Delta_n}(A^{(n)}) \le \mu_{\Delta_{n+1}}(A^{(n+1)}) \le \mu_{\Delta_\infty}(A^{(\infty)}) \le \widehat{\mu}_\Delta(A) \ ,$$

and

$$\mu_{\Delta''}(A) \le \mu_{\Delta''_n}(A^{(n)}) \le \mu_{\Delta''_{n+1}}(A^{(n+1)}) \le \mu_{\Delta''_\infty}(A^{(\infty)}) \le \widehat{\mu}_\Delta(A) \ .$$

Proof. It clearly suffices to prove the first sequence of inequalities. Observe that for every $X \in \Delta_n$ and for $m > n$ we can define an operator $Y \in \Delta_m$ by $Y = X \oplus 0$. Clearly $\sigma(A^{(n)}X) = \sigma(A^{(m)}Y) \cup \{0\}$ and hence $-1 \in \sigma(A^{(n)}X)$ implies $-1 \in \sigma(A^{(m)}Y)$. Since $\mu_\Delta(A) = \mu_{\Delta_1}(A^{(1)})$, this proves the first three inequalities. The last one follows because $\mu_{\Delta_\infty}(A^{(\infty)}) \le \widehat{\mu}_{\Delta_\infty}(A^{(\infty)}) = \widehat{\mu}_\Delta(A)$.
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We will now state (without proof) several results from [1, 2, 3, 4, 5] which we will need in the sequel.

Lemma 3 ([5]) *Let A be finite dimensional $\mathbf{C}^*$–algebra. Then A has only finitely many equivalence classes of cyclic representations.*

Lemma 4 ([5]) *Let the sequence Y_j of operators on $\mathcal{H}$, and the sequence $h_j \in \mathcal{H}$ satisfy*

(i) $\sup_j \ \mathrm{rank} \ Y_j < \infty, \ \ \sup_j \|Y_j\| < \infty;$

(ii) $\lim_{j\to\infty} \|(Y_j - I)h_j\| = 0;$

(iii) $\lim_{j\to\infty} \|h_j\| = 1.$

Then $\liminf_{n\to\infty} \|Y_j\|_{\mathrm{sp}} \ge 1.$

Lemma 5 ([4]) . *Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{L}(\mathcal{H})$, and $D_j \in \mathcal{L}(\mathcal{H})$ invertible so that*

$$T_0 = \lim_{j\to\infty} D_j T D_j^{-1}.$$

If the set $\{D_j, D_j^{-1} : j = 1, 2, \ldots\}$ is contained in a finite dimensional subspace, then $\|T_0\|_{\mathrm{sp}} = \|T\|_{\mathrm{sp}}$.

3 Relative Numerical Range

We will also need some results in what follows about the relative numerical range. Let $\mathcal{H}$ be a complex separable Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the set of bounded linear operators on $\mathcal{H}$. Let $T_1, \ldots, T_m, Q \in \mathcal{L}(\mathcal{H})$. Then we define the following *relative numerical ranges*:

$$W_Q(T_1, \ldots, T_m) := \{\lambda \in \mathbf{C}^n, \ \lambda = \lim_{n\to\infty} \left(\langle T_j h_n, h_n \rangle\right)_{j=1}^m :$$

$$h_n \in \mathcal{H}, \|h_n\| = 1, \lim_{n\to\infty} \|Q h_n\| = 0\},$$

and

$$W_Q^0(T_1, \ldots, T_m) := \{\lambda \in \mathbf{C}^n, \ \lambda = \lim_{n\to\infty} \left(\langle T_j h_n, h_n \rangle\right)_{j=1}^m :$$

$$h_n \in \mathcal{H}, \|h_n\| = 1, \lim_{n\to\infty} \|Q h_n\| = 0, h_n \to 0 \text{ weakly}\}.$$

Lemma 6 $W_Q^0(T_1, \ldots, T_m)$ *is a compact convex subset of* $\mathbf{C}^m$.

Proof. The compactness is immediate since $W_Q^0(T_1, \ldots, T_m)$ is a closed bounded subset of $\mathbf{C}^m$. As for the convexity, let $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\mu = (\mu_1, \ldots, \mu_m) \in W_Q^0(T_1, \ldots, T_m)$, and let the sequences of unit vectors

$$\{h_n\}_{n=1}^\infty, \ \{k_n\}_{n=1}^\infty \subset \mathcal{H}$$

satisfy

$$\lambda_j = \lim_{n\to\infty} \langle T_j h_n, h_n \rangle,$$

$$\mu_j = \lim_{n\to\infty} \langle T_j k_n, k_n \rangle, \ j = 1, \ldots, m,$$

$$\lim_{n\to\infty} \|Q h_n\| = 0 = \lim_{n\to\infty} \|Q k_n\|, \ h_n \to 0, \ k_n \to 0 \text{ weakly}.$$

Next for n fixed choose $N_n \geq n$ such that

$$|\langle h_n, k_{N_n} \rangle| \leq \frac{1}{n}, \ |\langle T_j h_n, k_{N_n} \rangle| + |\langle T_j^* h_n, k_{N_n} \rangle| \leq \frac{1}{n}, \ j = 1, 2, \ldots, m.$$

Then for any $\theta \in [0, 1]$

$$g_n := \sqrt{\theta} h_n + \sqrt{1 - \theta} k_{N_n}, \ n \geq 2,$$

satisfies the following conditions:

$$\|g_n\|^2 = 1 + 2\sqrt{\theta(1-\theta)}\Re\langle h_n, k_{N_n}\rangle \to 1, \ \|g_n\|^2 \geq \frac{1}{2},$$

$$g_n \to 0 \text{ weakly}, \ \|Qg_n\| \to 0,$$

$$|\langle T_j g_n, g_n\rangle - \theta\lambda_j - (1-\theta)\mu_j| \leq \theta|\langle T_j h_n, h_n\rangle - \lambda_j| + (1-\theta)|\langle T_j k_{N_n}, k_{N_n}\rangle - \mu_j| +$$

$$\sqrt{\theta(1-\theta)}\frac{1}{n} \to 0 \ \text{ as } n \to \infty.$$

Thus replacing the g_n by $g_n/\|g_n\|$ we immediately conclude that

$$\theta\lambda + (1-\theta)\mu \in W_Q^0(T_1, \ldots, T_m),$$

as required. 2

Lemma 7 $W_Q(T_1, \ldots, T_m)$ *is the union of all segments*

$$\{\theta\lambda + (1-\theta)\mu : 0 \leq \theta \leq 1\},$$

where $\lambda \in W_Q^0(T_1, \ldots, T_m)$ *and* $\mu = (\langle T_j h, h\rangle)_{j=1}^m$ *for some* $h \in \ker Q, \ \|h\| = 1.$

Proof. Let $\theta, \lambda = (\lambda_j)_{j=1}^m, \mu = (\mu_j)_{j=1}^m$ be as above and let the sequence $\{h_n\}_{n=1}^\infty \subset \mathcal{H}$ satisfy $\|h_n\| = 1$, $h_n \to 0$ weakly, $Qh_n \to 0$ strongly, and $\langle T_j h_n, h_n\rangle \to \lambda_j$ for $j = 1, \ldots, m$. Then as in the proof of Lemma 6, we obtain that

$$g_n = \sqrt{\theta}h_n + \sqrt{1-\theta}h, \ n = 1, 2, \ldots$$

satisfies the conditions

$$\|g_n\| \to 1, \ \|Qg_n\| \to 0,$$

and

$$\langle T_j g_n, g_n\rangle \to \theta\lambda_j + (1-\theta)\mu_j, \ j = 1, \ldots, m.$$

Therefore

$$\theta\lambda + (1-\theta)\mu \in W_Q(T_1, \ldots, T_m).$$

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Conversely, if $\psi = (\psi_j)_{j=1}^m \in W_Q(T_1, \ldots, T_m)$, then

$$\psi_j = \lim_{n \to \infty} \langle T_j g_n, g_n \rangle, \ j = 1, \ldots, m,$$

for some sequence $\{g_n\}_{n=1}^\infty \subset \mathcal{H}$ such that $\|g_n\| = 1$, $\|Q g_n\| \to 0$. Without loss of generality we can assume that g_n converges weakly to some $h' \in \ker Q$.

If $\|h'\| = 1$ then g_n converges strongly to h' and $\psi_j = \langle T_j h', h' \rangle$, $j = 1, 2, \ldots, m$. Clearly then

$$\psi = \theta \lambda + (1 - \theta)(\langle T_j h, h \rangle)_{j=1}^m$$

with $h = h'$, $\theta = 0$, and $\lambda \in W_Q^0(T_1, \ldots, T_m)$. If $\|h'\| = 0$ then ψ belongs to $W_Q^0(T_1, \ldots, T_m)$ and hence $\psi = \theta \psi + (1 - \theta)\mu$ with $\theta = 1$ and μ arbitrary. Finally we consider the case when $h' \neq 0$ and $\|h'\| \neq 1$. The vectors $h_n = (g_n - h')/\|g_n - h'\|$ converge weakly to zero, $\|Q h_n\| \to 0$, and $h = h'/\|h'\|$ is a unit vector in $\ker Q$. Clearly then

$$\psi_j = (1 - \|h\|^2) \lim_{n \to \infty} \langle T_j h_n, h_n \rangle + \|h\|^2 \langle T_j h, h \rangle, \ j = 1, \ldots, m,$$

and therefore

$$\psi \in \{\theta \lambda + (1 - \theta)(\langle T_j h, h \rangle)_{j=1}^m : 0 \leq \theta \leq 1\}$$

where

$$\lambda := \lim_{n \to \infty} (\langle T_j h_n, h_n \rangle)_{j=1}^m \in W_Q^0(T_1, \ldots, T_m).$$

This concludes the proof. 2

Corollary 1 *For all $T, Q \in \mathcal{L}(\mathcal{H})$, the set*

$$W_Q(T) = \{\lambda = \lim_{n \to \infty} \langle T h_n, h_n \rangle : h_n \in H \ , \|h_n\| = 1 \ , \lim_{n \to \infty} \|Q h_n\| = 0\}$$

is a compact convex set.

Proof. First notice that by an application of the classical Toeplitz-Hausdorff theorem to $T_Q := PT \,|\, \ker Q$ where P denotes orthogonal projection of $\mathcal{H}$ onto $\ker Q$, we see that the set

$$W(T_Q) = \{\langle T h, h \rangle : \|h\| = 1, \ h \in \ker Q\}$$

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is compact and convex. Therefore the convex hull of

$$W_Q^0(T) \cup W(T_Q)$$

is the union of all segments

$$\{\theta\lambda + (1-\theta)\mu : 0 \leq \theta \leq 1\},$$

where λ and μ run over $W_Q^0(T)$ and $W(T_Q)$, respectively. But according to Lemma 7, this union is precisely $W_Q(T)$. 2

Remark. Corollary 1 was proven in [3] using a completely different argument, based on an approximation lemma which is of independent interest.

Finally, for the proof of our lifting theorem (to be given in Section 4), we will need the following elementary fact:

Lemma 8 *Let $\mathcal{Z}$ denote a finite dimensional normed space, and let S be a set of linear functionals on $\mathcal{Z}$. Suppose that for every $z \in \mathcal{Z}$ there exists a sequence $\ell_n \in S$ such that $\lim_{n\to\infty} \ell_n(z) = 0$. Then there exists a sequence ℓ_n in the convex hull of S such that $\lim_{n\to\infty} \|\ell_n\| = 0$.*

Proof. Since $\mathcal{Z}$ is finite dimensional, S is contained in the dual $\mathcal{Z}'$ of $\mathcal{Z}$. We may also assume that S is a convex set. To prove the lemma we must show that the closure of S contains zero. If it did not then the Hahn-Banach theorem would imply the existence of a vector $z \in \mathcal{Z}$ and of a number $\varepsilon > 0$ such that $\Re\ell(z) > \varepsilon$ for all $\ell \in S$. This is contrary to the assumption of the lemma. 2

4   Ampliations of Perturbations

In this section, we will formulate and prove a new lifting result relating $\mu_\Delta(A)$ and $\widehat{\mu}_\Delta(A)$. For finite dimensional $\mathcal{E}$, a lifting result of this type was first proven in [1]. The result was then generalized to the infinite dimensional case in [5]. (For another proof of this type of lifting result in finite dimensions, see [7].) In these theorems, the lifting or ampliation of the operator A and perturbation structure Δ depends on the dimension of $\mathcal{E}$. Thus if $\mathcal{E}$ is infinite

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dimensional, we get an infinite lifting. In the new result proven below, we only have to lift up to the dimension of Δ' which in the cases of interest in the control applications of this theory only depends on the number of blocks of the given perturbation structure.

The notation will be that used in Section 2.

Theorem 1 *Assume that Δ' is a $*$-algebra of finite dimension n. Then*

$$\widehat{\mu}_\Delta(A) = \mu_{\Delta''_n}(A^{(n)})$$

for every $A \in \mathcal{L}(\mathcal{E})$.

Proof. The argument starts as in the proof of Theorem 3 in [1], and of Theorem 1 of [5]. Without loss of generality, we may assume that $\widehat{\mu}_\Delta(A) = 1$. We must show that $\mu_{\Delta''_n}(A^{(n)}) \geq 1$. Choose a sequence of invertible operators $X_j \in \Delta'$ such that $\|X_j A X_j^{-1}\| \to \widehat{\mu}_\Delta(A)$. Since $X_j A X_j^{-1}$ belongs to the finite dimensional space generated by $\Delta' A \Delta'$, we may assume that the sequence $X_j A X_j^{-1}$ converges to some operator A_0 such that $\|A_0\| = 1$. Obviously $\|X A_0 X^{-1}\| \geq \|A_0\|$ for every invertible operator $X \in \Delta'$. In particular we have

$$\|(I - X)A_0(I + X + X^2 + \cdots)\| \geq 1$$

for $X \in \Delta'$ with $\|X\| < 1$. Fix an operator $X \in \Delta'$ and a sequence $\varepsilon_j > 0$ converging to zero. There exist vectors $h_j \in \mathcal{E}$ with $\|h_j\| = 1$, such that

$$\|(I - \varepsilon_j X)A_0(I + \varepsilon_j X + \varepsilon_j^2 X^2 + \cdots)h_j\|^2 \geq 1 - \varepsilon_j^2 \ .$$

This can be rewritten as,

$$\langle A_0^* A_0 h_j, h_j \rangle + 2\varepsilon_j \Re\langle A_0^*(A_0 X - X A_0)h_j, h_j \rangle + O(\varepsilon_j^2) \geq 1 - \varepsilon_j^2$$

or equivalently,

$$2\varepsilon_j \Re\langle A_0^*(A_0 X - X A_0)h_j, h_j \rangle + O(\varepsilon_j^2) \geq \langle (I - A_0^* A_0)h_j, h_j \rangle - \varepsilon_j^2 \geq -\varepsilon_j^2.$$

Dividing by ε_j and letting $\varepsilon_j \to 0$ as $j \to \infty$, we see from the last equation that

$$\langle (I - A_0^* A_0)h_j, h_j \rangle \quad \to \quad 0, \tag{1}$$

$$\liminf_{j\to\infty} \Re\langle A_0^*(A_0 X - X A_0)h_j, h_j \rangle \quad \geq \quad 0. \tag{2}$$

We easily conclude that

$$\liminf_{j \to \infty} \Re \langle (X - A_0^* X A_0) h_j, h_j \rangle \geq 0. \tag{3}$$

Set

$$Q = I - A_0^* A_0, \ T = X - A_0^* X A_0.$$

Then from (1,3), we see that

$$Q h_j \to 0, \ \liminf_{j \to \infty} \Re \langle T h_j, h_j \rangle \geq 0. \tag{4}$$

Applying the above argument to ζX for any $\zeta \in \partial \mathbf{D}$ (the unit circle), we see that there exists a sequence $h_j^{(\zeta)}$, $\|h_j^{(\zeta)}\| = 1$ such that

$$Q h_j^{(\zeta)} \to 0, \ \liminf_{j \to \infty} \Re \zeta \langle T h_j, h_j \rangle \geq 0. \tag{5}$$

We claim that $0 \in W_{Q,0}(T)$. Indeed, if this were not the case, Corollary 1 would imply the existence of $\zeta \in \partial \mathbf{D}$ such that

$$\liminf_{j \to \infty} \Re \zeta \langle T h_j, h_j \rangle < 0$$

for all sequences of unit vectors h_j such that $Q h_j \to 0$ contradicting (5).

Thus, we have shown that for each $X \in \Delta'$, there exists a sequence of unit vectors $h_j \in \mathcal{E}$ such that

$$(I - A_0^* A_0) h_j \to 0, \ \text{and} \ \langle (X - A_0^* X A_0) h_j, h_j \rangle \to 0. \tag{6}$$

Let

$$\Delta'_{\mathrm{sa}} := \{ X = X^* : X \in \Delta' \}.$$

Consider now a subspace $D \subset \Delta'_{\mathrm{sa}}$ of real dimension $n - 1$ such that $\Delta'_{\mathrm{sa}} = D + \mathbf{R} I$. Set $\mathcal{Z} = \{ X - A_0^* X A_0 : X \in D \}$, and for every unit vector $h \in \mathcal{E}$ define a linear functional $\ell(h)$ on $\mathcal{Z}$ by $\ell(h)(T) = \langle Th, h \rangle$, $T \in \mathcal{Z}$. Then Lemma 8 applied to the set $S_k = \{ \ell(h) : \|(I - A_0^* A_0)h\| \leq 1/k \}$ implies the existence of linear functionals ℓ_k in the convex hull of S_k such that $\|\ell_k\| \leq 1/k$. Observe furthermore that the real dimension of $\mathcal{Z}$ is at most $n - 1$. Then from a standard result (see e.g., [11], page 73), each ℓ_k is a

convex combination of at most n functionals $\ell(h)$, say $\ell_k = \sum_{j=1}^{n} \alpha_j^{(k)} \ell(h_j^{(k)})$, where $\alpha_j^{(k)} \geq 0$, $\sum_{j=1}^{n} \alpha_j^{(k)} = 1$, and the $h_j^{(k)}$ are unit vectors in $\mathcal{E}$, such that

$$\|(I - A_0^* A_0) h_j^{(k)}\| \leq 1/k.$$

Let us define unit vectors vectors $u_k \in \mathcal{E}^{(n)}$ by

$$u_k = \oplus_{j=1}^{n} (\alpha_j^{(k)})^{1/2} h_j^{(k)}, \tag{7}$$

and observe that $\lim_{k\to\infty} \langle (X^{(n)} - A_0^{*(n)} X^{(n)} A_0^{(n)}) u_k, u_k \rangle = 0$ for every $X \in \Delta'$. Taking $X = Y^* Y$ we obtain

$$\lim_{k\to\infty} (\|Y^{(n)} A_0^{(n)} u_k\| - \|Y^{(n)} u_k\|) = 0 \tag{8}$$

for every $Y \in \Delta'$.

Consider now the spaces $\mathcal{H}_k = \Delta'_n A_0^{(n)} u_k$ and $\mathcal{K}_k = \Delta'_n u_k$. Lemma 3 implies that, by passing to appropriate subsequences, we may assume that all the representations $X \to X^{(n)} | \mathcal{H}_k$ (resp. $X \to X^{(n)} | \mathcal{K}_k$) are unitarily equivalent. It follows that we can find partial isometries U_k, V_k in Δ''_n such that $U_k \mathcal{H}_k = \mathcal{H}_1$ and $V_k \mathcal{K}_k = \mathcal{K}_1$. Dropping again to appropriate subsequences, we may assume that the limits $u = \lim_{k\to\infty} U_k A_0^{(n)} u_k$ and $v = \lim_{k\to\infty} V_k u_k$ exist. Then (8) implies that

$$\|Y^{(n)} u\| = \|Y^{(n)} v\|$$

for every $Y \in \Delta'$. Therefore there exists a partial isometry $W \in \Delta''_n$ such that
$$W Y^{(n)} u = Y^{(n)} v$$

for every $Y \in \Delta'$. Of course, W can be chosen equal to zero on the orthogonal complement of $\Delta'_n u$ and thus to have finite rank at most n. The partial isometries $R_k := V_k^* W U_k$ are in Δ''_n, they have uniformly bounded rank, and

$$\lim_{k\to\infty} (R_k A_0^{(n)} - I) u_k = \lim_{k\to\infty} V_k^* (W U_k A_0^{(n)} u_k - V_k u_k) = 0.$$

Therefore Lemma 4 implies that

$$\liminf_{k\to\infty} \|R_k A_0^{(n)}\|_{\mathrm{sp}} \geq 1.$$

Finally, since R_k commutes with $X^{(n)}, X \in \Delta'$, and we have

$$X_j^{(n)} R_k A^{(n)} X_j^{(n)-1} \to R_k A_0^{(n)}$$

in norm as $j \to \infty$. Lemma 5 shows that

$$\|R_k A_0^{(n)}\|_{\mathrm{sp}} = \|R_k A^{(n)}\|_{\mathrm{sp}}.$$

Consequently, we have

$$\liminf_{k \to \infty} \|R_k A^{(n)}\|_{\mathrm{sp}} = \liminf_{k \to \infty} \|R_k A_0^{(n)}\| \geq 1.$$

Thus,

$$\mu_{\Delta_n''}(A^{(n)}) \geq \liminf_{k \to \infty} \|A^{(n)} X_k\|_{\mathrm{sp}} \geq 1 = \hat{\mu}_\Delta(A),$$

which completes the proof of the theorem. 2

Remark. In the cases of interest in control,

$$\Delta'' = \Delta,$$

and so one has from Theorem 1 that

$$\mu_{\Delta_n}(A) = \hat{\mu}_\Delta(A).$$

5   Conditions for $\mu = \hat{\mu}$

In this section, we will discuss some new conditions when $\mu = \hat{\mu}$ without any need for lifting or ampliation. In the finite dimensional case, there have been some results of this kind, the most famous of which is that of Doyle [6], who showed that no lifting is necessary for perturbation structures with three or fewer blocks.

We begin by noting that in the proof of Theorem 1, we established a useful property of the critical operators A_0 in the closed Δ' similarity orbit

$$\overline{\mathcal{O}_{\Delta'}(A)} = \overline{\{X A X^{-1} : X \in \Delta'\}}$$

of A. Namely, if we call *critical* any $A_0 \in \overline{\mathcal{O}_{\Delta'}(A)}$ satisfying

$$\limsup_{\epsilon \downarrow 0} \|(I - \epsilon X) A_0 (I - \epsilon X)^{-1}\| \geq \|A_0\|, \ \forall X \in \Delta',$$

then the first part of the proof of Theorem 1 establishes the following:

Lemma 9 *If A_0 is a critical operator in $\overline{\mathcal{O}_{\Delta'}(A)}$, then it enjoys the following property $(\mathcal{O})$:*

$$0 \in W_Q(\|A_0\|^2 X - A_0^* X A_0), \ X \in \Delta',$$

where $Q = \|A_0\|^2 I - A_0^* A_0$.

Indeed, property $(\mathcal{O})$ is a reformulation of equation (6) in the case in which the norm of A_0 may be different from 1.

The next lemma is the key step in adapting the proof of Theorem 1 in order to show that

$$\mu_\Delta(A) = \hat{\mu}_\Delta(A)$$

in several interesting cases.

Lemma 10 *Let A_0 be an operator on $\mathcal{E}$ which satisfies the essential version of property $(\mathcal{O})$, property $(\mathcal{O}^0)$, namely*

$$0 \in W_Q^0(\|A_0\|^2 X - A_0^* X A_0), \ X \in \Delta',$$

where $Q = \|A_0\|^2 I - A_0^* A_0$. Then there exists a sequence $\{h_k\}_{k=1}^\infty \subset \mathcal{E}$, $\|h_k\| = 1$, $k = 1, 2, \ldots$, such that

$$Qh_k \to 0 \text{ strongly and } \langle (\|A_0\|^2 X - A_0^* X A_0)h_k, h_k \rangle \to 0,$$

for all $X \in \Delta'$.

Proof. Without loss of generality we can assume that $\|A_0\| = 1$. Let $X_1, \ldots, X_n$ be an algebraic basis of Δ'. (Note that Δ' is finite dimensional.) Set $T_j := X_j - A_0^* X_j A_0$, $j = 1, \ldots, n$. Then by virtue of Lemma 6, $W_Q^0(T_1, \ldots, T_n)$ is convex and compact. If $0 \notin W_Q^0(T_1, \ldots, T_n)$, there exists $\psi = (\psi_1, \ldots, \psi_n) \in \mathbf{C}^n$ and $\epsilon > 0$ such that

$$\Re \sum_{j=1}^n \psi_j \lambda_j \geq \epsilon, \ \ \forall \lambda = (\lambda_1, \ldots, \lambda_n) \in W_Q^0(T_1, \ldots, T_n).$$

Set

$$T = \sum_{j=1}^n \psi_j T_j.$$

Property $(\mathcal{O}^0)$ implies that there exists a sequence $\{g_k\}_{k=1}^\infty \subset \mathcal{E}$, $\|g_k\| = 1$, $g_k \to 0$ weakly such that $\langle Tg_k, g_k \rangle \to 0$. Without loss of generality (by passing to a subsequence if necessary), we can assume that

$$\langle T_j g_k, g_k \rangle \to \lambda_j, \quad (j = 1, \ldots, n),$$

for $k \to \infty$. Thus

$$\lambda = (\lambda_1, \ldots, \lambda_n) \in W_Q^0(T_1, \ldots, T_n).$$

Hence

$$0 \leftarrow \Re\langle Tg_k, g_k \rangle = \Re\sum_{j=1}^n \psi_j \langle T_j g_k, g_k \rangle \to \Re\sum_{j=1}^n \psi_j \lambda_j \geq \epsilon,$$

a contradiction. We therefore conclude that $0 \in W_Q^0(T_1, \ldots, T_n)$, i.e., there exists a sequence $\{h_k\}_{k=1}^\infty \subset \mathcal{E}$ satisfying the properties $\|h_k\| = 1$, $k = 1, 2, \ldots$, $\|Qh_k\| \to 0$, $h_k \to 0$ weakly, and

$$\langle (X_j - A_0^* X_j A_0)h_k, h_k \rangle = \langle T_j h_k, h_k \rangle \to 0$$

for all $j = 1, 2, \ldots, n$. This implies that

$$\langle (X - A_0^* X A_0)h_k, h_k \rangle \to 0,$$

for all $X \in \Delta'$. 2

We can now state the second main result of this paper:

Theorem 2 *If there exists a critical operator A_0 satisfying property $\mathcal{O}^0$ in the closed Δ'−orbit of A, then*

$$\mu_{\Delta''}(A) = \hat{\mu}_\Delta(A).$$

Proof. We only have to note that because of Lemma 10, in the proof of Theorem 1, we need not take direct sums. More precisely, referring to equation (7) in the proof of Theorem 1, we can take $u_k = h_k$, where

$$\{h_k\}_{k=1}^\infty$$

is the sequence provided by Lemma 10. The proof then proceeds exactly as in Theorem 1 with A_0 replacing $A_0^{(n)}$, X replacing $X^{(n)}$, and Y replacing $Y^{(n)}$. 2

Remark. Under the hypotheses of Theorem 2, when $\Delta'' = \Delta$ (which happens in all cases of interest in control), we have that

$$\mu_\Delta(A) = \hat{\mu}_\Delta(A).$$

Let $L(\Delta' A \Delta')$ denote the linear space generated by

$$\Delta' A \Delta' = \{XAY : X, Y \in \Delta'\}.$$

Obviously $L(\Delta' A \Delta')$ is finite dimensional, and therefore closed. Hence $\overline{\mathcal{O}_{\Delta'}(A)} \subset L(\Delta' A \Delta')$.

Corollary 2 *If for every $B \in L(\Delta' A \Delta')$, $B \neq 0$, the norm of B is not attained (that is, there is no $h \in \mathcal{H}$ such that $\|Bh\| = \|B\|\|h\| \neq 0$), then*

$$\mu_{\Delta''}(A) = \hat{\mu}_\Delta(A).$$

Proof. The critical operator A_0 constructed in the first part of the proof of Theorem 1 belongs to $L(\Delta' A \Delta')$, and therefore its norm is not attained. However in equation (6), we can assume that the sequence $\{h_j\}_{j=1}^\infty$ is weakly convergent, say $h_j \to h$ weakly. Without loss of generality, we may assume that $\|A_0\| = 1$. Then (6) shows that

$$(I - A_0^* A_0)h = 0.$$

Therefore if $h \neq 0$, we would have

$$\|A_0 h\|^2 = \|h\|^2 = \|A_0\|^2 \|h\|^2 \neq 0,$$

and so the norm of A_0 would be attained. We conclude that $h_j \to 0$ weakly, and so A_0 satisfies property $(\mathcal{O}^0)$. The required result now follows by Theorem 2. 2

Remark. Note that Corollary 2 applies only to infinite dimensional Hilbert spaces $\mathcal{E}$.

Example.

We would like to give an explicit example to which Corollary 2 applies. Let A_j be an operator on a Hilbert space $\mathcal{E}_j$ ($j = 1, \ldots, n$) for which the norm is not attained. (For example, take $\mathcal{E}_j = L^2((0,1))$ and A_j the muliplication operator $f(x) \mapsto x f(x)$ for $x \in (0,1)$ and $f \in L^2((0,1))$.) Set

$$\mathcal{E} := \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n,$$

and Δ the algebra of operators on $\mathcal{E}$ of the form

$$\begin{bmatrix} X_1 & 0 & \ldots & 0 \\ 0 & X_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & X_n \end{bmatrix}$$

with $X_j \in \mathcal{L}(\mathcal{E}_j)$, $j = 1, 2, \ldots, n$. Then Δ' is formed by the diagonal operators

$$\begin{bmatrix} \lambda_1 I_{\mathcal{E}_1} & 0 & \ldots & 0 \\ 0 & \lambda_2 I_{\mathcal{E}_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n I_{\mathcal{E}_n} \end{bmatrix}$$

for $\lambda_j \in \mathbf{C}$, $j = 1, 2, \ldots, n$, and $I_{\mathcal{E}_j}$ denotes the identity operator on $\mathcal{E}_j$, $j = 1, 2, \ldots, n$. Let A be any operator on $\mathcal{E}$ the $n \times n$ block matrix representation of which has entries in the set $\{0, A_1, \ldots, A_n\}$ with only one nonzero entry in each row and column. Then it is easy to check that $L(\Delta' A \Delta')$ has the property required in Corollary 2, and therefore

$$\mu_{\Delta''}(A) = \hat{\mu}_\Delta(A).$$

6 Toeplitz Operators

In this section, we want to use our lifting methodology in order to derive a beautiful result of Shamma [13, 14] on the structured singular value of a Toeplitz operator, i.e., a linear time invariant system.

Accordingly, set $\mathcal{E} = H^2(\mathbf{C}^n)$ and let A denote the multiplication (analytic Toeplitz) operator on $\mathcal{E}$ defined by

$$(Ah)(z) = A(z)h(z), \quad |z| < 1, \ h \in \mathcal{E},$$

where

$$A(z) = [a_{jk}]_{j,k=1}^n, \quad |z| < 1,$$

has H^∞ entries. Let Δ' be any $*$–subalgebra of $\mathcal{L}(\mathbf{C}^n)$, the elements of which are regarded as multiplication operators on $\mathcal{E}$. Note that in this case, $\Delta'' = \Delta$ is the algebra generated by operators of the form

$$(Bh)(z) = B(z)h(z), \quad |z| < 1, \ h \in \mathcal{E}$$

with $B(z)X = XB(z)$, $|z| < 1$, $X \in \Delta'$ as well as of the form

$$B \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{bmatrix} = \begin{bmatrix} Yh_1 \\ Yh_2 \\ \vdots \\ Yh_n \end{bmatrix},$$

with $Y \in \mathcal{L}(H^2(\mathbf{C}))$ arbitrary. We can now state:

Lemma 11 *Let A_0 be an analytic Toeplitz operator. Then if A_0 has property $(\mathcal{O})$, it also has property $(\mathcal{O}^0)$.*

Proof. Without loss of generality we may assume $\|A_0\| = 1$. Let $X \in \Delta'$ and let $h_j, j = 1, 2, \ldots$ be a sequence of unit vectors satisfying

$$\|(I - A_0^* A_0)h_j\|^2 \to 0, \quad \langle (X - A_0^* X A_0)h_j, h_j \rangle \to 0. \tag{9}$$

Note that since $I - A_0^* A_0 \geq 0$ the first condition in (9) is equivalent to

$$\langle (I - A_0^* A_0)h_j, h_j \rangle \to 0.$$

Let U denote the canonical unilateral shift on $\mathcal{E} = H^2(\mathbf{C}^n)$, that is,

$$(Uh)(z) := zh(z), \quad |z| < 1, \ h \in \mathcal{E}.$$

As is well-known, we can view $H^2(\mathbf{C}^n)$ as a subspace of $L^2(\mathbf{C}^n)$. In particular, in this representation the relations (9) are equivalent to

$$\int_0^{2\pi} (\|h_j(e^{it})\|^2 - \|A_0(e^{it})h_j(e^{it})\|^2)dt \quad \to \quad 0$$

$$\int_0^{2\pi} [\langle Xh_j(e^{it}), h_j(e^{it}) \rangle - \langle XA_0(e^{it})h_j(e^{it}), h_j(e^{it}) \rangle]dt \quad \to \quad 0. \tag{10}$$

Note that X is an $n \times n$ matrix with constant coefficients. Therefore in (10), h_j can be replaced by $U^k h_j$ for any $k \geq 0$ without changing the values of the integrals. We infer that

$$\|(I - A_0^* A_0) U^{k_j} h_j\|^2 \to 0, \quad \langle (X - A_0^* X A_0) U^{k_j} h_j, U^{k_j} h_j \rangle \to 0,$$

for any sequence $\{k_j\}_{j=1}^{\infty}$ of natural numbers. Since for any $g, h \in \mathcal{E}$

$$|\langle U^k g, h \rangle| = |\langle g, U^{*k} h \rangle| \leq \|g\| \|U^{*k} h\| \to 0 \quad (k \to \infty),$$

we can choose k_j sufficiently large in order to guarantee that

$$|\langle U^{k_j} h_j, h \rangle| \leq \frac{1}{2^j},$$

for any h of the form

$$h = (z^m \delta_{pk})_{k=1}^n, \ 0 \leq m \leq j, \ 1 \leq p \leq n, \tag{11}$$

where δ_{pk} is the Kronecker delta. Thus

$$\langle U^{k_j} h_j, h \rangle \to 0, \quad \text{as } j \to \infty$$

for all vectors of the form (11). Since these vectors form an orthonormal basis of $\mathcal{E}$, we see that $U^{k_j} h_j \to 0$ weakly, which concludes the proof of the lemma. 2

Corollary 3 ([13, 14]) *For A and Δ' as above, we have that*

$$\mu_\Delta(A) = \hat{\mu}_\Delta(A).$$

Proof. First, note that any operator B in $L(\Delta' A \Delta')$ is also an analytic Toeplitz operator. In particular, the critical operator A_0 obtained in the proof of Theorem 1 is a multiplication operator given by

$$A_0(z) = [a_{jk}^0(z)]_{j,k=1}^n, \ |z| < 1.$$

By Lemma 9, the operator A_0 has property $(\mathcal{O})$, and thus also property $(\mathcal{O}^0)$, by virtue of Lemma 11. The conclusion now follows from Theorem 2. 2

7 Structured Singular Value of Input-Output Operators

In this section, we will put some of the above results into a system-theoretic framework. Accordingly, let ℓ_+^2 be the space of square summable one-sided sequences in $\mathbf{C}$, let $\mathcal{C}$ denote the set of all bounded linear operators on ℓ_+^2. Further, let $A : \ell_+^2(\mathbf{C}^n) \to \ell_+^2(\mathbf{C}^n)$ be an arbitrary bounded linear operator. Thus A defines a (possibly) time-varying system. (Here $\ell_+^2(\mathbf{C}^n)$ the space of of square summable sequences in $\mathbf{C}^n$, i.e., the space of finite energy vector-valued signals with n components.) Then we want to interpret $\widehat{\mu}_\Delta(A)$ as a structured singular value on an extended space with an enhanced perturbation structure. Note $\mathcal{E}$ in this case is the Hilbert space $\ell_+^2(\mathbf{C}^n)$.

Define the algebra of perturbations

$$
\Delta := \{ \begin{bmatrix} \delta_1 & 0 & \ldots & 0 \\ 0 & \delta_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \delta_n \end{bmatrix} : \delta_i \in \mathcal{C}, i = 1, \ldots, n \}.
$$

Then the commutant of Δ is the finite dimensional C^*-algebra,

$$
\Delta' := \{ \begin{bmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_n \end{bmatrix} : d_i \in \mathbf{C}, i = 1, \ldots, n \}.
$$

Note that a constant $d \in \mathbf{C}$ defines an operator on ℓ_+^2 via multiplication.

From Theorem 1, it follows that the μ upper bound given by the infimum of $\|XAX^{-1}\|$ over all constant $X-$scales equals $\bar{\mu}_\Delta(A)$. We now have the following interpretation of $\widehat{\mu}_\Delta(A)$. We lift A to $A^{(n)} : \mathcal{E}^{(n)} \to \mathcal{E}^{(n)}$. Then

$$
(\Delta_n)'' \cong \{ \begin{bmatrix} \tilde{\Delta}_1 & 0 & 0 & 0 \\ 0 & \tilde{\Delta}_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \tilde{\Delta}_n \end{bmatrix} : \tilde{\Delta}_j \in \Delta \}.
$$

$(\Delta_n)''$ is a space of time-varying perturbations and we have from Theorem 1 that

$$
\widehat{\mu}_\Delta(A) = \mu_{\Delta_n''}(A^{(n)}).
$$

This is true for *arbitrary time-varying systems A*. In case A is Toeplitz, i.e., the system is time-invariant, then as we have seen (Corollary 3, [13, 14]),

$$\widehat{\mu}_\Delta(A) = \mu_\Delta(A).$$

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