Optimal Decentralized State-Feedback Control with Sparsity and Delays

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Submitted to Automatica

Abstract

This work presents the solution to a class of decentralized linear quadratic state-feedback control problems, in which the plant and controller must satisfy the same combination of delay and sparsity constraints. Using a novel decomposition of the noise history, the control problem is split into independent subproblems that are solved using dynamic programming. The approach presented herein both unifies and generalizes many existing results.

I Introduction

While optimal decentralized controller synthesis is difficult in general [25, 28], much progress has been made toward identifying tractable subclasses of problems. Two closely related conditions, partial nestedness and quadratic invariance, guarantee, respectively, that the optimal solution for LQG control problems is linear [3], and that optimal synthesis can be cast as convex optimization [16, 19]. These results alone do not necessarily guarantee that the optimal controller can be efficiently computed, since the associated convex programs are large. Indeed, the problem of [3] reduces to a system of linear equations that grows with the time horizon. In [16], a sequence of convex programs of increasing size converges to the optimal solution, and [19] uses vectorization to reduce decentralized problems to much larger centralized ones. More efficient computational tools have been developed for linear quadratic problems, in particular linear matrix inequalities have been used to solve the state-feedback [17] and output-feedback [1, 18] cases. Output-feedback problems with delays can also be solved using a combination of spectral factorization and quadratic programming [7].

A general drawback of purely computational approaches is that they give little insight into the structure of the optimal controller. When finding the best linear controller, for example, we must often restrict our search to controllers of a fixed dimension, and there is no intuition for the physical meaning of the controller states. Explicit solutions, which provide efficient computation in addition to a physical interpretation for the states of the controller, have been found for the specific constraint classes of sparsity and delays.

In the delay case, it is assumed all controllers eventually measure the global state, but not necessarily simultaneously. Problems with a one-timestep delay between controllers were found by extending classical dynamic programming arguments [4, 20, 29]. In the linear quadratic setting, the state-feedback problem with delays characterized by a graph is solved in [6]. The solution is similar in spirit and complexity to computing the optimal centralized LQR controller, but the optimal policy turns out to be dynamic rather than static. Controller states have interpretations as delayed estimates of the global state.

In the sparsity case, all measurements are transmitted instantly, but some controllers never gain access to certain measurements. This amounts to a delayed system where each delay is either zero or infinite. The first explicit solution solved the two-player case using a spectral factorization approach [24]. The results were later extended to a general class of quadratically invariant, also called poset-causal, sparsity patterns [21, 22]. A dynamic programming argument for the two-player problem was also given in [23]. Again, the controller states can be interpreted as estimates of the global states conditioned on the particular subsets of the available information.

This paper unifies the treatment of sparsity and delay constraints by considering an information flow characterized by a directed graph. Each edge may be labeled with either a 0 for instantaneous information transfer, or with a 1 for one-timestep delayed transfer. We use a generalization of the dynamic programming approach of [6] that does not require the graph to be strongly connected. Therefore, this work gives a general method for solving both the delay and sparsity problems described in the previous two paragraphs. In addition, our framework can treat problems that contain a mixture of sparsity and delay constraints.

A preliminary version of this work appeared in the conference paper [8]. The present work differs substantially from [8]. Specifically, the present work includes complete proofs to all results, new illustrative examples, and extensive discussion in Sections IV and V. We also have a new result, Theorem 3, which gives a distributed message-passing implementation of the optimal controller.

In the remainder of this section, we explain the notations and other conventions used in the paper. In Sections II, we state the main problem, and we present our solution in Section III. We discuss how our work unifies existing results in Section IV and we discuss limitations.
and possible extensions in Section V. We provide two numerical examples in Section VI. The proof to our main result appears in Section VII. Finally, we conclude in Section VIII.

I-A Notation

Lower-case letters are random vectors unless otherwise indicated. Subscripts are time indices and superscripts are subsystem labels. For example, the state of subsystem 2 at time $t$ is $x^2_t$. The global state of all subsystems is indicated by omitting the subsystem, and time sequences are denoted using the colon operator. For example,

$$x_t = \begin{bmatrix} x^1_t \\ \vdots \\ x^N_t \end{bmatrix} \quad \text{and} \quad x_{0:t} = \begin{bmatrix} x_0 \\ \vdots \\ x_t \end{bmatrix}.$$

For matrices, subscripts denote subblocks, while superscripts are used to select sets of subblocks. For example, a block $2 \times 3$ matrix $M$ is written as

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \end{bmatrix}.$$

To select row $r = \{2\}$ and columns $s = \{1, 3\}$, we write $M^{rs} = [M_{21} \ M_{23}].$ The symbol $I$ denotes a block-identity matrix whose dimensions are to be inferred by context. In the example above, we could have written $M^{rs} = r^{(1,2)} [I_{1,2,3}]^s.$ If $\mathcal{Y} = \{ y^1, \ldots, y^M \}$ is a set of random vectors (possibly of different sizes), we say that $z \in \text{lin} \mathcal{Y}$ if there are appropriately sized real matrices $C^1, \ldots, C^M$ such that $z = \sum_{i=1}^M C^i y^i.$

In general, all matrices in this paper are time-varying but we will often omit time indices if they are clear from context. We use a new notation to indicate a family of equations. When we write $x_+ = Ax + Bu + w$, we mean that $x_{t+1} = Ax_t + Bu_t + w_t$ for $t = 0, \ldots, T - 1$. We will use the same time horizon $T$ throughout this paper, so there is no ambiguity. Note that the subscript “+” means that the time index of the corresponding symbol is incremented by 1. We use a similar notation for other binary relations such as inequalities, by writing a $t$ over the symbol. Finally, a similar notation is used to modify summations. For example, we write

$$\sum_t x^T_t Q x_t$$

instead of writing

$$\sum_{t=0}^{T-1} x^T_t Q x_t.$$

II Problem statement

We begin with some basic definitions. A network graph $G(\mathcal{V}, \mathcal{E})$ is a directed graph where each edge is labeled with a 0 if the associated link is delay-free, or a 1 if it has a one-timestep delay. The vertices are $\mathcal{V} = \{1, \ldots, N\}$. If there is an edge from $j$ to $i$, we write $(j, i) \in \mathcal{E}$, or simply $j \rightarrow i$. When delays are pertinent, they are denoted as $j \overset{\Delta}{\rightarrow} i$ or $j \overset{\varepsilon}{\rightarrow} i$. Associated with the network graph $G(\mathcal{V}, \mathcal{E})$ is the delay matrix $D$. Each entry $D_{ij}$ is the sum of the delays along the directed path from $j$ to $i$ with the shortest aggregate delay. We assume $D_{ij} = 0$ for all $i$, and if no directed path exists, we set $D_{ij} = \infty$. Delays are assumed to be fixed for all time. Directed cycles are permitted in the network graph, but we assume there are no directed cycles with a total delay of zero. In our framework, all nodes belonging to such a delay-free cycle can be collapsed into a single node and treated as such without any loss of generality. See Fig. 1 for an example of a network graph. We now state the general class of problems that can be solved using the methods developed in this paper.

**Problem 1.** Let $G(\mathcal{V}, \mathcal{E})$ be a network graph with a time-invariant delay matrix $D$. Suppose the following time-varying state-space equations are given:

$$x^i_t = \sum_{j \in \mathcal{V}} (A_{ij} x^j_t + B_{ij} u^j_t) + w^i_t \quad \text{for all } i \in \mathcal{V}. \tag{1}$$

Stacking the various vectors and matrices, we obtain the more compact representation

$$x_+ = Ax + Bu + w. \tag{2}$$

All random disturbances are assumed to be jointly Gaussian and independent from one another. Specifically, the random vectors $\{x^0_t, w^1_t, \ldots, w^{n-1}_t\}_{t \in \mathcal{V}}$ are mutually independent. Their means and covariances are given by

$$x^i_0 \sim N(\mu^i_0, \Sigma^i_0) \quad \text{and} \quad w^i_t \sim N(0, W^i_t) \quad \text{for all } t \geq 0. \tag{3}$$

The inputs $u_{0:T-1}$ are controlled using state feedback subject to an information constraint. The information set for controller $i$ at time $t$ is as follows.

$$T^i_t = \{ x^j_{t-k} : j \in \mathcal{V}, \ 0 \leq k \leq t - D_{ij} \}. \tag{4}$$

In other words, $T^i_t$ is the set of states belonging to nodes that have had sufficient time to reach node $i$ by time $t$. Each decision measures its corresponding information set

$$u^i_t = \gamma^i_t (T^i_t). \tag{5}$$

The goal is to choose the set of policies $\gamma = \{ \gamma^i_{0:T-1} \}_{i \in \mathcal{V}}$ that minimize the expected quadratic cost

$$\min_{\gamma} \mathbb{E}^\gamma \left( \sum_{t} \begin{bmatrix} x^T_t & Q & S & x^T \end{bmatrix} \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & Q \end{bmatrix} \begin{bmatrix} x^T_t & Q & S & x^T \end{bmatrix} + x^T_t Q x_t \right), \tag{6}$$

where the expectation is taken with respect to the joint probability measure on $(x_{0:T}, u_{0:T-1})$ induced by the choice of $\gamma$. We make the standard assumptions that

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \succeq 0, \quad R \succ 0, \quad Q \succ 0. \tag{7}$$

In Problem 1, we assume that all decision-makers know the system parameters $A$, $B$, $Q$, $R$, $S$, $Q_f$, as well as the
topology of the underlying network graph \( G(\mathcal{V}, \mathcal{E}) \). The problem is cooperative in nature; we are to jointly design the set of policies \( \gamma \) to optimize the cost \( (6) \).

As mentioned in Section I-A, the parameters of the system in Problem 1 can be time-varying. In fact, their sizes can also vary with time. For example, the input \( u_t \) may change dimensions at every timestep, and the matrices \( B_t, R_t, S_t \) would change size accordingly.

We now illustrate how the general problem statement given by \( (1) \)–\( (7) \) specializes in a simple three-node case.

**Example 1.** Consider the network graph of Fig. 1.

![Network graph for Example 1](image)

Figure 1: Network graph for Example 1. Each node represents a subsystem, and the edge labels indicate the propagation delay from one subsystem to another.

The delay matrix for Example 1 is given by

\[
D = \begin{bmatrix} 0 & 1 & \infty \\ 1 & 0 & \infty \\ 0 & 0 & 0 \end{bmatrix}. \tag{8}
\]

While the delay matrix for Example 1 only contains 0’s, 1’s, and \( \infty \)’s, a general delay matrix may contain arbitrarily large nonnegative integers, provided the associated network graph is sufficiently large. The state-space equations for Example 1 are of the form

\[
\begin{bmatrix} x^1_+ \\ x^2_+ \\ x^3_+ \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix}
+ \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \begin{bmatrix} w^1 \\ w^2 \\ w^3 \end{bmatrix} \tag{9}
\]

Note that whenever \( D_{ij} > 1 \), we have \( A_{ij} = 0 \) and \( B_{ij} = 0 \). The policies of the three decision-makers in Example 1 are constrained as follows.

\[
\begin{align*}
u^1_t &= \gamma^1_t (x^1_{0:t}, x^2_{0:t-1}) \\
u^2_t &= \gamma^2_t (x^3_{0:t-1}, x^2_{0:t}) \\
u^3_t &= \gamma^3_t (x^1_{0:t-1}, x^2_{0:t}, x^3_{0:t})
\end{align*}
\tag{10}
\]

There is a combination of sparsity and delay constraints; some states may never be available to a particular controller, while other states might be available but delayed. Our goal is to choose the policies \( \gamma \) such that we minimize the cost given by \( (6) \).

Note that according to our definitions, both the plant \( (1) \) and the controller \( (5) \) share the same sparsity constraints. In Example 1 for instance, \( x^3 \) does not influence \( x^1 \) or \( x^2 \) via the dynamics, nor can it affect \( u^1 \) or \( u^2 \) via the controller. As explained in Subsection VII-A, this condition is sufficient to guarantee that the optimal control policies \( \gamma \) are linear, a powerful fact.

Based on our formulation, the controller may be any function of the past information history, which grows with the size of the time horizon \( T \). We will show by construction that there exists an optimal policy that has a finite memory that is independent of \( T \).

### III Main results

This section presents the main results: explicit state-space solutions for Problem 1. Two equivalent forms of the controller are presented. The first form has states which are functions of the primitive random variables \( x_0 \) and \( w_{0:t-1} \). The second form uses state feedback only and implements the controller in a distributed fashion by message passing.

To describe the controller coordinates, the network graph is transformed into an alternate representation, the *information graph*. This graph tracks the propagation of the noise signals \( w^i \) as they propagate through the network graph. Formally, we define the information graph as follows. Let \( s^i_k \) be the set of nodes reachable from node \( j \) within \( k \) steps:

\[
s^i_k = \{ i \in \mathcal{V} : D_{ij} \leq k \}. \tag{11}
\]

The information graph \( \tilde{G}(\mathcal{U}, \mathcal{F}) \), is given by

\[
\mathcal{U} = \{ s^i_k : k \geq 0, \ j \in \mathcal{V} \}
\quad \mathcal{F} = \{ (s^i_k, s^j_{k+1}) : k \geq 0, \ j \in \mathcal{V} \}.
\]

Consider Example 1, and track each of the noise signals \( w^1, w^2, w^3 \) as they are propagated through the network graph of Fig. 1.

\[
\begin{align*}
w^1 &\rightarrow \{1\} \rightarrow \{1, 2, 3\} \rightarrow \{1, 2, 3\} \rightarrow \ldots \\
w^2 &\rightarrow \{2, 3\} \rightarrow \{1, 2, 3\} \rightarrow \{1, 2, 3\} \rightarrow \ldots \\
w^3 &\rightarrow \{3\} \rightarrow \{3\} \rightarrow \ldots
\end{align*}
\tag{12}
\]

Each chain in \( (12) \) is of the form \( w^i \rightarrow s^i_0 \rightarrow s^i_1 \rightarrow s^i_2 \rightarrow \ldots \) and tracks the nodes reached by \( w^i \) after some number of timesteps. Assembling the paths in \( (12) \) and aggregating duplicate nodes, we obtain the information graph for Example 1, shown in Fig. 2. The additional labels \( w^i \) are not counted amongst the nodes of \( \tilde{G} \) as a matter of convention, but are shown as a reminder of which noise signal is being tracked. We will often write expressions such as \( \{ s \in \mathcal{U} : w^i \rightarrow s \} \) to denote the set of root nodes of \( \tilde{G} \). The following proposition gives some useful properties of the information graph.
Proposition 1. Given an information graph \( \tilde{G}(U, F) \), the following properties hold.

(i) Every node in \( \tilde{G} \) has exactly one descendant. In other words, for every \( r \in U \), there is a unique \( s \in U \) such that \( r \rightarrow s \).

(ii) Every path eventually hits a node with a self-loop.

(iii) If the network graph satisfies \( |V| = n \), the number of nodes in \( \tilde{G} \) is bounded by \( n \leq |U| \leq n^2 - n + 1 \).

The first two properties are immediate by construction. The lower bound on \( |U| \) is achieved by directed acyclic network graphs with zero delay on all edges. The upper bound on \( |U| \) is achieved by network graph consisting of one large cycle: \( 1 \rightarrow 2 \rightarrow \cdots \rightarrow n \rightarrow 1 \), and each link has a one-timestep delay. Note that the information graph may have several connected components. This happens whenever the network graph is not strongly connected. For example, Fig. 2 has two connected components because there is no path 3 \( \rightarrow 2 \) in Fig. 1.

In (12), we characterized how noise injected at the current timestep propagates to the other nodes at future timesteps. Alternatively, one may consider the history of all past injected noises, and ask which of these noise terms have affected a particular node at the present time. Define the noise history at time \( t \) to be the set of all noise terms injected into the system prior to time \( t \),

\[
H_t = \{ x_0, w_{0:t-1} \}.
\]

The noise partition diagram is formed by arranging the elements of \( H_t \) in a table and grouping them based on which nodes at the current time \( t \) they can affect. We call these groups label sets, and they are discussed in greater detail in Section VII-B. The noise partition diagram for Example 1 is shown in Fig. 3. Note that as we traverse each row of the noise partition diagram from right to left in Fig. 3, we recover (12).

We are now ready to present the main result of this paper, which expresses the optimal controller as a function of new coordinates induced by the information graph.
node uses its locally available state information to compute the optimal control decisions. To illustrate how this will work, recall the network graph of Example 1 and its corresponding information graph, Fig. 2. We implement the optimal controller of Theorem 2 by using a combination of local measurement, local computation, and message passing, as follows. Assume that at time $t$, each node has a copy of the controller states required for computing its local $u_t^i$ using (15). Namely,

$$\begin{align}
1 : \zeta_t^{(1)}, \zeta_t^{(1,2,3)} & \\
2 : \zeta_t^{(2,3)}, \zeta_t^{(1,2,3)} & \\
3 : \zeta_t^{(1,2,3)}, \zeta_t^{(3)}.
\end{align}$$

At time $t$, messages are passed along the delay-1 edges of the network graph as follows: node 1 sends $\zeta_t^{(1)}$ to node 2 and node 2 sends $\zeta_t^{(2,3)}$ to node 1. These messages arrive one timestep later, due to the delays associated with these edges of the network graph. Now advance to timestep $t + 1$, such that all these messages have been received. Nodes 1 and 2 now have copies of $\zeta_t^{(1)}, \zeta_t^{(2,3)},$, and $\zeta_t^{(1,2,3)}$, and thus, nodes 1 and 2 can compute $\zeta_{t+1}$ via (15) using their current controller state information.

We cannot use (15) to compute $\zeta_t^{(2,3)}$ or $\zeta_t^{(1,2,3)}$ since it would require knowledge of $w^1$, $w^2$, or $w^3$ respectively. However, we will see later that the $\zeta$-coordinates combine to form the global state: $x_t = \sum_{s \in U} I_t^{s} x_t^{s}$, for $t = 0, \ldots, T - 1$. The need to know the $w^i$ can thus be avoided. For example, once node 1 measures $x_{t+1}^{1}$, we can compute $\zeta_{t+1}^{1} = x_{t+1}^{1} - (\zeta_{t+1}^{(1,2,3)})^{(1)}$ and similarly for $\zeta_{t+1}^{(2,3)}$. Finally, node 2 sends $\zeta_{t+1}^{(2,3)}$ and $\zeta_{t+1}^{(1,2,3)}$ to node 3. These messages arrive instantly because the associated edge has delay 0. Node 3 then measures $x_{t+1}^{3}$ and computes $\zeta_{t+1}^{3} = x_{t+1}^{3} - (\zeta_{t+1}^{(1,2,3)})^{(3)} - (\zeta_{t+1}^{(2,3)})^{(3)}$. At this point, each node has copies of the states indicated by (17) at the incremented time $t + 1$. Therefore, $u_{t+1}$ can be computed and we have come full-circle. Fig. 4 shows all the messages passed along the network graph.

![Figure 4: Network graph for Example 1 with messages.](image)

Our next main result, given below, generalizes the above procedure to any instance of Problem 1.

**Theorem 3.** A distributed implementation of the optimal controller in Theorem 2 is given as follows. Each node $i \in V$ is responsible for storing and updating a local copy of the states $\{\zeta_t^s\}_{s \ni i}$. Each node $i \in V$ performs the following steps for $t = 0, \ldots, T - 1$.

1) Receive and store all inbound delay-0 and delay-1 messages. This provides the $\zeta_t^s$ and $\zeta_{t-1}^s$ values respectively that will be used for local updates in the coming steps. Note that if $t = 0$, there are no delay-1 messages to receive.

2) For non-root nodes $s$, so $w^i \rightarrow s$, if $\zeta_t^s$ is available from a delay-0 message, do nothing. Otherwise compute it (or initialize it if $t = 0$) using (15).

3) For the root node $s$ such that $w^i \rightarrow s$, measure $x_t^i$ and update as follows.

$$\begin{align}
(\zeta_t^s)^{(1)} & = \sum_{r \rightarrow s} I_t^{s} (A_t^{r} + B_t^{r} K_r) \zeta_{t-1}^r \\
(\zeta_t^s)^{(2,3)} & = x_t^i - \sum_{r \ni s, r \neq s} I_t^{i} \zeta_t^r.
\end{align}$$

4) Compute input $u_t^i$ according to (15).

5) Send all outbound messages $M_t^{ij}$, defined as follows.

$$\begin{align}
\text{If } i & \rightarrow j : M_t^{ij} = \{\zeta_t^s : s \in U, i, j \in s\}. \\
\text{If } i & \rightarrow j : M_t^{ij} = \{\zeta_t^s : s \in U, i \in s, j \notin s\}.
\end{align}$$

**Proof.** Proven in Subsection VII-D.

### IV Existing results

In this section, we explain how Theorem 2 specializes to the existing results mentioned in Section I. For each such case, we show the associated network graph and information graph.

**Centralized case.** The first related result is classical state-feedback LQR control. This corresponds to the trivial case where the network graph and information graphs are single nodes, as in Fig. 5. The recursion (14) becomes a standard Riccati difference equation, and $K_r^{(1)}$ is the classical LQR gain. Furthermore, the algorithm of Theorem 3 reduces to the classical state-feedback policy. No messages are passed, none of the update steps are executed, and $\zeta_t^{(1)} = x_t$.

![Figure 5: Classical centralized state-feedback control.](image)

**Sparsity constraints.** For sparsity constraints with no delays, the simplest problem, the two-player problem, is shown in Fig. 6. The information graph consists of two disconnected self-loops, implying that the optimal
controller depends on the solution of two Riccati difference equations and their associated \( K \)-gains. For sparsity over a general graph with \( N \) nodes, the information graph has \( N \) disconnected self-loops and so \( N \) Riccati difference equations must be solved.

\[
\begin{align*}
1 & \quad 0 \quad 2 \\
\{1, 2\} & \quad w^1 & \quad \{2\} & \quad w^2
\end{align*}
\]

(a) network graph  (b) information graph

Figure 6: The two-player problem, sparsity only.

**Delayed sharing.** The simplest problem with delays but no sparsity, the one-step information sharing pattern, is shown in Fig. 7. For general strongly connected graphs with one-timestep delays on every edge, the information graph is connected, and all edges eventually lead to the self-loop \( \mathcal{V} \rightarrow \mathcal{V} \). Thus, a single Riccati difference equation must be solved, which corresponds to the centralized Riccati difference equation, and its solution is propagated according to (14).

\[
\begin{align*}
1 & \quad 1 \quad 2 \\
\{1, 2\} & \quad w^1 & \quad \{1\} & \quad w^2
\end{align*}
\]

(a) network graph  (b) information graph

Figure 7: One-step delay information sharing pattern.

**V Further extensions**

We now discuss selected topics exploring the limitations of our work and directions for possible future research.

**Infinite horizon.** While the problem formulation in this paper considers optimization over a finite time horizon, our solution extends naturally to an infinite horizon. To this end, we assume all system parameters are time-invariant, and we seek a stabilizing controller that minimizes the average finite-horizon cost as the length of the horizon tends to infinity. In this limit, the Riccati difference equations (14) for the nodes of the information graph with self-loops \( r \rightarrow r \) become Algebraic Riccati Equations (AREs). Under classical conditions [2], each \( X^r \) approaches a steady-state value as \( T \rightarrow \infty \), the associated steady-state gain \( K^r \) will be stabilizing, and the corresponding \( \zeta^r \) state will be stable. One can easily show that the remaining \( X^1 \) and \( K^1 \) matrices for the other nodes of the information graph also approach steady-state limits and their corresponding states are also stable. For a numerical example and further discussion, see Example 2 in Section VI.

**Output feedback.** In output feedback problems, the decision-makers have access to noisy measurements of states rather than the states themselves. Solutions are known for two-player sparsity [12, 13], \( n \)-player broadcast [9], and one-timestep delays [4, 20, 29], but it is not clear how to extend the results of this paper to output feedback over a general graph with mixed delays. Difficulties arise because determining appropriate sufficient statistics for dynamic programming becomes subtle when the delay between any two nodes is at least 2 timesteps [15, 27, 30]. This phenomenon occurs for all sparsity patterns and most complex delay patterns. For systems in this paper, sufficient statistics are computed by projecting the state onto orthogonal subspaces of the information sets. It is unclear if such a decomposition is possible for output-feedback problems. We mention one promising exception [14], in which sufficient statistics are derived along with a solution to the finite-horizon version of the two-player problem. The output-feedback solutions found so far are significantly more complicated than their state-feedback counterparts. For instance, in Theorem 2, the noise covariances \( W_{ij}^l \) only appear in the expression for the optimal cost. In output-feedback however, the policy itself depends explicitly on \( W_{ij}^l \).

**Correlated noise.** We assume in Problem 1 that the noises injected into the various nodes are independent. This independence is used to show that the \( \zeta^r \) states are mutually independent, thus enabling a critical simplification of the value function used in the dynamic programming argument. If the noises are correlated, for example \( E(w^1 w^2 T) \neq 0 \), our approach fails. Such problems are still partially nested, as defined in Section VII-A, so we expect the optimal controller to be unique and linear, but we do not know how many states it will have. A simple version of this problem with two players and decoupled dynamics was recently solved [10], and the solution is surprisingly complex. If each player has \( n \) states, the optimal controller may have a number of states proportional to \( n^2 \).

**Realizability.** In general, causal linear time-invariant system may be equivalently represented using either state-space or transfer functions. However, the two representations are not equivalent when we impose an underlying graph structure and associated sparsity for the state-space matrices [11, 26]. Specifically, every structured state-space realization that is stabilizable and detectable corresponds to a structured transfer function, but the opposite is not true. A counterexample is given in [11]. We avoid the issue of realizability in this paper by only considering plants for which a structured state-space
realization exists, as implied by (1). Under this assumption, we produce a structured state-space realization for the optimal controller. As discussed earlier in this section, all internal modes $\zeta^r$ of this controller are stable by construction, so our realization is stabilizing. For graphs with pure sparsity (no delays), there is no loss in assuming a realizable plant, because non-realizable plants can never be stabilized using structured controllers [11]. However, no analogous result is known for systems with delays, as discussed in [26].

VI Examples

In this section, we give two examples to illustrate both the generality and versatility of our results.

Example 2. Consider the four-node system depicted in Fig. 8. We show the network graph and its associated messages from Theorem 3 and the information graph and noise partition diagram derived in Section III.

As a supplement to the discussion on infinite time horizon solutions in Section V, we now present a numerical simulation of this four-node example that shows convergence of the optimal control gains as the time horizon grows. We use the following time-invariant parameters,

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 2 & 2 & 1 \\ 0 & 1 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & \infty & \infty \\ 1 & 0 & \infty \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$$

$$Q = R = \begin{bmatrix} 8 & -1 & -1 & -1 \\ -1 & 8 & -1 & -1 \\ -1 & -1 & 8 & -1 \\ -1 & -1 & -1 & 8 \end{bmatrix}, \quad S = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

and noise covariance $W = I$. Note that the $A$ and $B$ matrices have a sparsity pattern that conforms to the delay matrix $D$. As explained in Section II, we must have $A_{ij} = 0$ and $B_{ij} = 0$ whenever $D_{ij} \geq 2$.

To guarantee that the Riccati equations for the self-loops converge to stabilizing solutions, it is sufficient that $(Q^{rr} - S^{rr}R^{rr-1}S^{rrT}, A^{rr} - B^{rr}R^{rr-1}S^{rrT}, B^{rr})$ be stabilizable and detectable for $r = \{3, 4\}$, $\{2, 3, 4\}$, and $\{1, 2, 3, 4\}$. A detailed discussion of the convergence of Riccati equations may be found in [2]. A direct calculation shows that $Q^{rr} - S^{rr}R^{rr-1}S^{rrT}$ and $B^{rr}$ are invertible for all self-loops, and so stabilizability and detectability are guaranteed. It follows that the $X^r$ matrices, which are solutions to (14), should converge to steady-state values as we get farther from the terminal timestep. This fact is supported by the plot of Fig. 9.

Figure 9: Plot of $\text{trace}(X_t^r)$ as a function of time for Example 2. A curve is shown for each $r \in \mathcal{U}$.

Example 3. To investigate the effect of increased delays on total cost, we consider the 10-node system defined by the linear network graph of Fig. 10. Depending on the value of $k$, we can vary the end-to-end delay $D_{90} = k$.

Figure 10: Network graph for Example 3.
Suppose the system matrices are bi-diagonal, as follows

\[
A = \begin{bmatrix}
2 & 4 & \cdots & 4 \\
4 & 2 & \cdots & 2 \\
& \ddots & \ddots & \ddots \\
& & 4 & 2 \\
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
& \ddots & \ddots & \ddots \\
& & 1 & 1 \\
\end{bmatrix}.
\]

The cost matrices are \( Q = R = I, \ S = 0 \), and the noise covariance is \( W = 10^{-3}I \). Because of their bi-diagonal structure, these matrices are admissible for any choice of \( k \). In Fig. 11, the total cost is plotted as a function of \( k \) for the fixed time horizon of \( T = 20 \). As expected, the cost increases monotonically as we increase the delay.

![Figure 11: Plot of the optimal cost as a function of the total delay for Example 3, with a time horizon of \( T = 20 \).](image)

### VII Proof of main results

This section contains proofs of Theorems 2 and 3. The proof of Theorem 2 requires three steps.

1) **Linearity.** We show that the information constraint for Problem 1 is partially nested \([3]\). Thus the optimal policy is unique, and consists of linear functions of the information history.

2) **Feasibility.** In (15), \( u_i^t \) depends on \( w_i^t \). In spite of this, we prove that \( u_i^t \) need only measure its associated information set \( I_i^t \). So the policy is feasible.

3) **Optimality.** Finally, we must show that the proposed policy is optimal. Our approach uses dynamic programming in the \( \zeta^t \) coordinates, with an optimization over policies rather than actions.

#### VII-A Linearity

Linearity of the optimal policy follows from **partial nestedness**, a concept first introduced by Ho and Chu in \([3]\). We state the main definition and result below.

**Definition 4.** A dynamical system (2) with information structure (5) is partially nested if for every admissible policy \( \gamma \), whenever \( w_i^t \) affects \( I_i^t \), then \( I_i^t \subset I_i' \).

**Lemma 5** (see \([3]\)). Given a partially nested structure, the optimal control law that minimizes a quadratic cost of the form (6) exists, is unique, and is linear.

In other words, an information structure is partially nested if whenever the decision of Player \( j \) affects the information used in Player \( i \)'s decision, then Player \( i \) must have access to all the information available to Player \( j \). When this is the case, the optimal policy is linear. Using partial nestedness, the following lemma shows that the optimal state and input may be expressed as linear functions of terms from the information sets \( I_i^t \) and the noise history \( \mathcal{H}_t \), which was defined in (13).

**Lemma 6.** The information structure described in Problem 1 is partially nested. Furthermore, the optimally controlled states and inputs belong to their associated information subspaces, and the global state and input belong to the noise history subspace. Namely,

\[
x_i^t, u_i^t \in \text{lin} \ I_i^t \quad \text{for all } i \in \mathcal{V}, \quad \text{and } x_t, u_t \in \text{lin} \ \mathcal{H}_t.
\]

**Proof.** See Appendix B.

#### VII-B Feasibility

It is clear from Lemma 6 that \( \text{lin} \ I_i^t \subset \text{lin} \ \mathcal{H}_t \). The inclusion is typically strict, since the information constraints may prevent some noise terms in \( \mathcal{H}_t \) from having an immediate influence on all nodes of the network. In this subsection, we begin by characterizing the subspaces of \( \mathcal{H}_t \) that are associated with each of the \( I_i^t \). This will eventually lead to an intuitive definition for the new \( \zeta^t \) coordinates.

**Lemma 7.** Consider an information graph \( \hat{G}(\mathcal{U}, \mathcal{F}) \) and define the corresponding label sets \( \{ \mathcal{L}^t_{0:U} \} \) recursively by

\[
\mathcal{L}^t_0 = \bigcup_{s \in \mathcal{U}} \{ x_s^t \}
\]

\[
\mathcal{L}^t_{+ r} = \bigcup_{s \in \mathcal{U}} \{ w_s^t \} \cup \bigcup_{r \rightarrow s} \mathcal{L}^r. \tag{20}
\]

The following properties of the label sets hold.

(i) For every \( t \geq 0 \), the label sets are a partition of the noise history:

\[
\mathcal{L}^t_{+ r} \cap \mathcal{L}^t_s = \emptyset \quad \text{when } r \neq s, \quad \text{and } \mathcal{H}_t = \bigcup_{s \in \mathcal{U}} \mathcal{L}^s. \tag{21}
\]

(ii) For all \( i \in \mathcal{V} \),

\[
\text{lin} \ I^i \overset{\Delta}{=} \text{lin} \bigcup_{s \ni i} \mathcal{L}^s. \tag{22}
\]

**Proof.** See Appendix C.

The label sets specify the groupings in the noise partition diagram, and thus we can use the noise partition
We now define the $\zeta$ in the statement of Theorem 2, provided that we set
\[ u^2_t \in \text{lin} \mathcal{Z}_t^2 = \text{lin}\{\mathcal{L}_t^{(2,3)}, \mathcal{L}_t^{(1,2,3)}\}. \] (23)

Referring to Fig. 3, we may expand the contents of these label sets and obtain
\[ u^2_t \in \text{lin}\{x_1^2, x_2^2, w_2^2, w_1^2\}. \]

We now define the $\zeta^s$ coordinates used in the statement of Theorem 2. Combining (19) and (22), we have that
\[ x^i_t, u^i_t \in \bigcup_{s \geq i} \mathcal{L}_t^s \quad \text{for all } i \in \mathcal{V}. \] (24)

Thus, we may write
\[ x^i_t = \sum_{s \leq i} I^{V,s} x^s_t \quad \text{and} \quad u^i_t = \sum_{s \leq i} I^{V,s} \varphi^s_t. \] (25)

where $\zeta^s \in \text{lin} \mathcal{L}_t^s$ and $\varphi^s \in \text{lin} \mathcal{L}_t^s$. As explained in Section I-A, $I^{V,s}$ denotes a submatrix of a large identity matrix with block rows and columns of appropriate dimensions. For example, the input in Example 1 is
\[ u_t = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \varphi_t^{(1)} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \varphi_t^{(2,3)} + \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \varphi_t^{(3)} + \varphi_t^{(1,2,3)}, \]

where each of the $\varphi^s_t$ are mutually independent. The reason for this structure is due to (23), or more generally (24). For example, if we examine the second row of the above equations, $u^2_t$ only depends on $\varphi_t^{(2,3)}$ and $\varphi_t^{(1,2,3)}$, which depend on $\mathcal{L}_t^{(2,3)}$ and $\mathcal{L}_t^{(1,2,3)}$, and this agrees with what we found in (23).

Since we assumed in Problem 1 that the random vectors in $\mathcal{H}_t$ are mutually independent and the $\{\mathcal{L}_t^s\}_{s \leq t}$ are a partition of $\mathcal{H}_t$, it follows that the vectors in the set
\[ \left\{ \begin{bmatrix} \zeta^s \\ \varphi^s \end{bmatrix} \right\}_{s \leq t} \]

are mutually independent. (26)

Just as (2) characterizes the evolution of $x_t$, we may also derive state equations for the $\zeta_t$ coordinates. We state the result in the following lemma.

**Lemma 8.** The coordinates $\{\zeta^s\}_{s \leq t}$ and $\{\varphi^s\}_{s \leq t}$ satisfy the recursive equations
\[ \zeta^s_0 = \sum_{w^s \rightarrow s} I^{s,(i)} x^i_0 \] (27)
\[ \zeta^s_t = \sum_{r \rightarrow s} (A^{sr} \zeta^r + B^{sr} \varphi^r) + \sum_{w^s \rightarrow s} I^{s,(i)} w^i. \] (28)

**Proof.** See Appendix D.

Note that (27)–(28) agrees with the formula (15) given in the statement of Theorem 2, provided that we set $\varphi^s_t = K^s_t \zeta^s_t$. This choice of policy is feasible because
\[ u^i_t = \sum_{s \geq i} I^{s,(i)} K^s_t \zeta^s_t \in \text{lin} \bigcup_{s \geq i} \mathcal{L}_t^s = \text{lin} \mathcal{T}^s_t, \]

where the inclusion follows by definition, and the last equality follows from Lemma 7. This completes the proof of feasibility.

**Remark 9.** We may interpret $\zeta^s_t$ and $\varphi^s_t$ as conditional estimates of $x_t$ and $u_t$, respectively. Namely,
\[ \zeta^s_t = I^{s,V} \mathbb{E}(x_t \mid \mathcal{L}^s_t) \quad \text{and} \quad \varphi^s_t = I^{s,V} \mathbb{E}(u_t \mid \mathcal{L}^s_t). \]

**VII-C Optimality**

We now prove the controller is optimal, and derive an expression for the corresponding minimal expected cost. Our proof uses a dynamic programming argument, and we optimize over policies rather than actions. Let $\gamma_t = \{\gamma^i_t\}_{i \in \mathcal{V}}$ be the set of policies at time $t$. By Lemma 5, we may assume the $\gamma^i_t$ are linear. Define the cost-to-go
\[ V_t(\gamma_{t-1}) = \min_{\gamma_t} \mathbb{E}^\gamma \left( \sum_{k=t}^{T-1} \left[ x_k \right]^T \begin{bmatrix} Q_k & S_k \\ S_k^T & R_k \end{bmatrix} \left[ x_k \right] + x_T^T Q_f x_T \right), \]

which the expectation is taken with respect to the joint probability measure on $(x_{t:T}, u_{t:T-1})$ induced by the choice of $\gamma = \gamma_{0:T-1}$. These functions are the minimum expected future cost from time $t$, given that the policies up to time $t-1$ have been fixed. We allow $V_t$ to be a function of past policies, but it turns out that $V_t$ will not depend on them explicitly. By causality, we may iterate the minimizations and write a recursive formulation for the cost-to-go,
\[ V_t(\gamma_{0:t-1}) = \min_{\gamma_t} \mathbb{E}^\gamma \left( \sum_{k=t}^{T-1} \left[ x_k \right]^T \begin{bmatrix} Q_k & S_k \\ S_k^T & R_k \end{bmatrix} \left[ x_k \right] + V_{t+1}(\gamma_{0:t-1}, \gamma_t) \right). \] (29)

Our objective is to find the optimal cost (6), which is simply $V_0$. Consider the terminal timestep, and use the decomposition (25),
\[ V_T(\gamma_{0:T-1}) = \mathbb{E}^\gamma (x_T^T Q_f x_T) = \mathbb{E}^\gamma \sum_{s \leq t} (\zeta^s_T)^T Q_f^s (\zeta^s_T). \]

In the last step, we used the fact that the $\zeta^s$ coordinates are independent (26). Note that $V_T$ depends on the policies up to time $T-1$ because the distribution of $\zeta^s_T$ depends on past policies implicitly through (28). We will prove by induction that the value function always has a similar quadratic form. Specifically, suppose that for some $t \geq 0$, we have
\[ V_{t+1}(\gamma_{0:t}) = \mathbb{E}^\gamma \sum_{s \leq t} (\zeta^s_{t+1})^T X^s_{t+1} (\zeta^s_{t+1}) + c_{t+1}, \]

where $\{X^s_{t+1}\}_{s \leq t}$ is a set of matrices and $c_{t+1}$ is a scalar. Now compute $V_t(\gamma_{0:t-1})$ using the recursion (29). Substituting the definitions for $\zeta^s_t$ and $\varphi^s_t$ from (25) and using
the independence result (26), we obtain

\[ V_t(\gamma_{0:t-1}) = \min_{\gamma_t} \mathbb{E}\left[ \sum_{s \in \mathcal{U}} \begin{bmatrix} \phi_s^T \\ \phi_s^R \end{bmatrix}^T \begin{bmatrix} Q^s \\ S^s \end{bmatrix} \begin{bmatrix} \phi_s^T \\ \phi_s^R \end{bmatrix} + c_t \right] + (\zeta_{t+1}^s)^T X_{t+1}^s (\zeta_{t+1} + c_{t+1}). \]

Substituting the state equations (28), using the independence result once more and rearranging terms, we obtain

\[ V_t(\gamma_{0:t-1}) = \min_{\gamma_t} \mathbb{E}\left[ \sum_{r \in \mathcal{U}} \begin{bmatrix} \phi_t^T \\ \phi_t^R \end{bmatrix} \Gamma_r \begin{bmatrix} \phi_t^T \\ \phi_t^R \end{bmatrix} + c_t \right] \]

where \( \Gamma_r \) and \( c_{0:T-1} \) are given by:

\[ \Gamma_r \equiv \begin{bmatrix} Q^r \\ S^r \\ R^r \end{bmatrix} + \begin{bmatrix} A^r \\ B^r \\ C^r \end{bmatrix}^T X_s^r \begin{bmatrix} A^r \\ B^r \\ C^r \end{bmatrix} \]

(31)

\[ c_t = c_t + \sum_{i \in \mathcal{V}} \text{trace} \left( \left( X_s^r \right)^{(i)} \right) W^i. \]

(32)

The terminal conditions are \( \Gamma_r = Q_{TT}^r \) and \( c_T = 0 \), and \( s \) is the unique node in \( \hat{G}(\mathcal{U}, \mathcal{F}) \) such that \( r \rightarrow s \), see Proposition 1. In (30), the terms in the sum are independent, so they may be optimized separately. A lower bound on the cost-to-go is found by relaxing the information constraints and performing an unconstrained optimization over the actions \( \phi_t = \{ \phi_t^i \}_{i \in \mathcal{U}} \),

\[ V_t(\gamma_{0:t-1}) \geq \mathbb{E}\left[ \sum_{r \in \mathcal{U}} \begin{bmatrix} \phi_t^T \\ \phi_t^R \end{bmatrix} \Gamma_r \begin{bmatrix} \phi_t^T \\ \phi_t^R \end{bmatrix} + c_t \right], \]

where the first inequality follows from Fatou’s lemma applied to (30), and the second inequality follows from the relaxation mentioned above. Each minimization is a simple quadratic optimization, and the optimal cost and action are given by (14). Substitution yields

\[ V_t(\gamma_{0:t-1}) \geq \mathbb{E}\left[ \sum_{s \in \mathcal{U}} (\zeta_s^r)^T X_t^s (\zeta_s^r) + c_t \right]. \]

This lower-bound is in fact tight, because the optimal unconstrained actions are \( \phi_t^i = K_t \zeta_t^i \in \text{lin } \mathcal{L}_t^i \), which is precisely the admissible set for \( \phi_t^i \). This completes the induction argument as well as the proof that the specified policy is optimal. The optimal cost is given by

\[ V_0 = \mathbb{E}\left[ \sum_{s \in \mathcal{U}} (\zeta_0^s)^T X_0^s (\zeta_0^s) + c_0 \right], \]

(33)

where \( c_0 \) may be evaluated by starting with \( c_T = 0 \) and recursing backwards using (32). Finally, (33) evaluates to the desired expression (16) because \( x_0^i \sim \mathcal{N}(\mu_0, \Sigma_0^i) \). This completes the proof of Theorem 2.

VII-D Proof of Theorem 3

First we show that the message passing algorithm executes without deadlock. At \( t = 0 \), deadlock cannot occur, since there are no delay-0 directed cycles. Therefore, the delay-0 edges induce a partial order on the nodes by \( i \preceq j \) if there is a directed delay-0 path from \( i \) to \( j \). Any execution ordering that satisfies the partial order will be feasible. Now for \( t > 0 \), assuming that all delay-1 messages were sent at time \( t - 1 \), the same reasoning shows that no deadlock could occur at time \( t \).

Each node computes \( u_i^j \) using (15), so we must check that all \( \zeta_r^i \) states with \( s \preceq i \) are available to node \( i \) at time \( t \). It is straightforward to show that the claim holds for \( t = 0 \). Suppose it holds at timestep \( t - 1 \). Computing \( \zeta_r^i \) in step 2 or the first part of step 3 requires knowledge of \( \{ \zeta_r^{i-1} \}_{r,s} \). If \( i \in r \), then \( \zeta_r^{i-1} \) is already available, by assumption. If \( i \notin r \), then there exists \( j \in r \) with \( j \rightarrow i \). Then (18) implies that \( \zeta_r^{i-1} \in M_j^{i-1} \). Thus, \( \zeta_r^{i-1} \) is made available via an inbound message. For the second part of step 3, we must know all \( \zeta_r^i \) for which \( r \preceq i \) and \( r \neq s \). This happens as long as we evaluate the updates while respecting the partial order explained above. Then all missing \( \zeta_r^i \) are provided by delay-0 messages, as defined in (18). This completes the proof of Theorem 3.

VIII Conclusion

This paper uses dynamic programming to derive optimal policies for a general class of decentralized linear quadratic state feedback problems. As noted in Section IV, the solution generalizes many existing works on decentralized state-feedback control [6, 21, 22].

The key concept in this paper is the information graph. Its nodes are a particular subsets of the set of all nodes in the network and its edges show how available information evolves over time. This turns out to be closely related to the structure of the optimal decentralized controller. For example, the number of Riccati difference equations that must be propagated is equal to the number of connected components in the information graph.

As discussed in Section V, many possible avenues for future research remain open. For example, some special cases with noisy measurements or correlated noise have been solved, but extensions to general directed graphs with mixed sparsity and delays have yet to be found.

IX Acknowledgements

The first author thanks John Doyle for very helpful discussions. The second author would like to thank Ashutosh Nayyar for some very helpful discussions.
References


A Information Set Properties

We now prove some useful properties of information sets.

Lemma 10. The information sets (4) may be expressed recursively as follows,

\[ T_{i+1} = \emptyset \]
\[ T_i = \left\{ x_i^j : j \in V, D_{ij} = 0 \right\} \cup \bigcup_{j \in V} T_{i-1} \quad \text{for } t \geq 0. \]

Furthermore, suppose \( i, j \in V \) and \( 0 \leq k \leq t \). The following are equivalent.

(i) \( T_k^i \subset T_i \)

(ii) \( x_k^i \in T_i \)

(iii) \( D_{ij} \leq t - k \)

Proof. We first prove that (i)–(iii) are equivalent.

• (i) \( \Rightarrow \) (ii): It is immediate from (4) that \( x_k^i \in T_k^i \).

So it follows that \( x_k^i \subset T_i \).

• (ii) \( \Rightarrow \) (iii): It follows from (4) that if \( x_k^i \in T_i \), then we must have \( 0 \leq k \leq t - D_{ij} \). Therefore \( D_{ij} \leq t - k \) as required.

• (iii) \( \Rightarrow \) (i): By the triangle inequality, \( D_{ia} - D_{ja} \leq D_{ij} \) for any \( \alpha \in V \). Therefore, if \( D_{ij} \leq t - k \), then \( k - D_{ja} \leq t - D_{ia} \). So for any \( \ell \) that satisfies \( 0 \leq \ell \leq t - D_{ia} \), we must also have \( 0 \leq \ell \leq t - D_{ja} \). It follows from (4) that \( T_k^i \subset T_i \).

We now derive the recursive expression for \( T_i^j \). Start with (4), which we rewrite here for convenience,

\[ T_i = \left\{ x_i^j : j \in V, 0 \leq k \leq t - D_{ij} \right\}. \]

Partition into two cases; when \( k = t \) (which implies \( D_{ij} = 0 \)), and when \( k \neq t \). Then, partition further based on the value of \( D_{ij} \).

\[ T_i = \left\{ x_i^j : D_{ij} = 0 \right\} \cup \left\{ x_k^j : 0 \leq k \leq t - D_{ij}, k \neq t \right\} \]
\[ = \left\{ x_i^j : j \in V, D_{ij} = 0 \right\} \]
\[ \cup \left\{ x_k^j : j \in V, 0 \leq k \leq t - 1 - D_{ij}, D_{ij} = 0 \right\} \]
\[ \cup \left\{ x_k^j : j \in V, 0 \leq k \leq t - D_{ij}, D_{ij} \geq 1 \right\}. \]

In the last term, when \( D_{ij} \geq 1 \), it means that there is a path \( j \rightarrow i \) with an aggregate delay of at least 1; so there exists an intermediate node \( \ell \neq i \) where \( D_{i\ell} = 1 \) and \( D_{ij} + 1 = D_{i\ell} \). Therefore,

\[ T_i = \left\{ x_i^j : j \in V, D_{ij} = 0 \right\} \]
\[ \cup \left\{ x_k^j : j \in V, 0 \leq k \leq t - 1 - D_{ij}, D_{ij} = 0 \right\} \]
\[ \cup \bigcup_{D_{i\ell} = 1} \left\{ x_k^j : 0 \leq k \leq t - 1 - D_{ij} \right\} \]
\[ \cup \bigcup_{D_{ij} \geq 1} \left\{ x_k^j : j \in V, 0 \leq k \leq t - D_{ij} \right\}. \]

Thus the first part of (19) is verified.

B Proof of Lemma 6

Suppose that \( u^j_\alpha \) affects \( T_i^j \) in the simplest way possible; namely that \( u^j_\alpha \) affects \( x_k^i \) at a future timestep \( \sigma > \tau \) via recursive applications of the state equations (1), and \( x^j_\sigma \in T_i \). Then we have

\[ u^j_\alpha \text{ affects } x^j_\sigma \implies D_{ij} \leq \sigma - \tau \]
\[ x^j_\sigma \in T_i \implies D_{i\ell} \leq \tau - \sigma. \]

Adding (35)–(36) together and using the triangle inequality, we obtain \( D_{ij} \leq t - \tau \). By Lemma 10, it follows that \( T_i^j \subset T_i \), as required. If \( u^j_\alpha \) affects \( T_i^j \) via a more complicated path, apply the above argument to each consecutive pair of inputs along the path to obtain the chain of inclusions \( T_i^j \subset \cdots \subset T_i^j \).

With partial nestedness established, Lemma 5 implies that there is a unique linear optimal controller. In particular, the optimal \( u^j_\alpha \) is a linear function of \( T_i^j \). The same is trivially true of \( x_k^i \) since we have \( x_k^i \in T_i^j \) from (4). Thus the first part of (19) is verified.

We proceed by induction to prove the second part of (19). At \( t = 0 \), we clearly have \( x_0, u_0 \in \text{lin } H_0 \), since

\[ \bigcup_{i \in V} T_i^j = \{ x_0^1, \ldots, x_0^N \} = H_0. \]

Now suppose that for some \( t \geq 0 \), we have \( x_t, u_t \in \text{lin } H_t \). Applying the state equations (2), it follows that

\[ x_{t+1} \in \text{lin} \{ H_t \cup \{ w_t \} \} = \text{lin } H_{t+1} \]
\[ u_{t+1} \in \text{lin} \bigcup_{i \in V} T_{i+1} \bigcup_{i \in V} T_{i+1} = \text{lin} \left\{ \{ x_{t+1} \} \cup H_t \right\} = \text{lin } H_{t+1}. \]

and the proof is complete.
C Proof of Lemma 7

Part (i). We proceed by induction. At \( t = 0 \), we have \( \mathcal{H}_0 = \{ x^1_0, \ldots, x^n_0 \} \). Since each \( w^i \) points to exactly one element \( s \in \mathcal{U} \), it is clear from (20) that \( \{ L^s_i \}_{s \in \mathcal{U}} \) partitions \( \mathcal{H}_0 \). Now suppose that \( \{ L^s_i \}_{s \in \mathcal{U}} \) partitions \( \mathcal{H}_t \) for some \( t \geq 0 \). By Proposition 1, for each \( r \in \mathcal{U} \) there exists a unique \( s \in \mathcal{U} \) such that \( r \to s \). Therefore each element \( w^i \in \mathcal{H}_t \) is contained in exactly one label set \( L^s_{t+1} \). It follows from (20) that \( \{ L^s_{t+1} \}_{s \in \mathcal{U}} \) must partition \( \mathcal{H}_{t+1} \) and the proof is complete.

Part (ii). Again, by induction. At \( t = 0 \),

\[
\mathcal{I}_0^t = \{ x^j_0 : D_{ij} = 0 \} = \{ x^j_0 : s^j \ni i \} = \{ x^j_0 : w^j \to s, s \ni i \} = \bigcup_{s \ni i} L^s_0.
\]

So the identity holds at \( t = 0 \). Now suppose it holds for some \( t \geq 0 \). By Lemma 10,

\[
\mathcal{I}^{t+1} = \{ x^j_{t+1} : j \in \mathcal{V}, D_{ij} = 0 \} \cup \bigcup_{D_{ij} \leq 0} \mathcal{I}^j_t.
\]

Applying the state equations (1), we have:

\[
x^j_+ = \sum_{\ell \in \mathcal{V}} \left( A_j x^\ell + B_j u^\ell \right) + w^j.
\]

By Lemma 6, the \( x^j_t \) and \( u^j_t \) terms belong to \( \text{lin} \mathcal{I}^j_t \) where \( D_{ij} \leq 1 \). Therefore, taking the \( \text{lin} \) of both sides of (37),

\[
\text{lin} \mathcal{I}^{t+1} = \text{lin} \{ w^j : j \in \mathcal{V}, D_{ij} = 0 \} + \sum_{D_{ij} \leq 0} \text{lin} \mathcal{I}^j_t
\]

\[
= \text{lin} \{ w^j : j \in \mathcal{V}, D_{ij} = 0 \} + \bigcup_{D_{ij} \leq 0} \text{lin} \bigcup_{r \geq j} L^s_t
\]

\[
= \text{lin} \bigcup_{s \ni i} L^s_{t+1},
\]

where the second equality follows from the induction hypothesis and the third equality follows from the label set recursion (20).

D Proof of Lemma 8

Suppose \( \zeta^t_i \) satisfies (27)–(28). The recursive formulation of the label sets (20) implies that \( \zeta^t_i \in \text{lin} L^t_i \) for all \( t \geq 0 \) and all \( s \in \mathcal{U} \). All that remains to be shown is that \( \zeta^t_i \) indeed provides decomposition of \( x_t \) as in (25). The decomposition of \( x_t \) is satisfied at \( t = 0 \), since for all \( i \in \mathcal{V} \) there is a unique \( s \in \mathcal{U} \) with \( w^i \to s \). Now assume inductively that (25) holds for some \( t \geq 0 \). Therefore,

\[
\sum_{s \in \mathcal{U}} I^v,s \zeta^t_i = \sum_{s \in \mathcal{U}} I^v,s \left( \sum_{r \to s} (A^{sr} \zeta^r + B^{sr} \varphi^r) + \sum_{w^i \to s} I^w,(i) w^i \right)
\]

\[
= \sum_{s \in \mathcal{U}} \left( \sum_{r \to s} (A^{V,s} \zeta^r + B^{V,s} \varphi^r) + \sum_{w^i \to s} I^V,(i) w^i \right)
\]

\[
= \sum_{s \in \mathcal{U}} (A^{V,s} \zeta^s + B^{V,s} \varphi^s) + w^i
\]

\[
= Ax + Bu + w
\]

where we have substituted the dynamics (27) in the first step, and the induction hypothesis in the fourth step. In the second step, we took advantage of the particular sparsity structures of \( A \) and \( B \), that imply

\[
AI^{V,r} = I^V,s A^{sr} \quad \text{and} \quad BI^{V,r} = I^V,s B^{sr}.
\]

It follows that (25) holds for \( t + 1 \).