

Optimal Control with Noisy Time

Andrew Lamperski and Noah J. Cowan

Abstract—This paper examines stochastic optimal control problems in which the state is perfectly known, but the controller’s measure of time is a stochastic process derived from a strictly increasing Lévy process. We provide dynamic programming results for continuous-time finite-horizon control and specialize these results to solve a noisy-time variant of the linear quadratic regulator problem and a portfolio optimization problem with random trade activity rates. For the linear quadratic case, the optimal controller is linear and can be computed from a generalization of the classical Riccati differential equation.

Index Terms—Optimal Control; Stochastic Optimal Control; Uncertain Time

I. INTRODUCTION

Effective feedback control often requires accurate timekeeping. For example, finite-horizon optimal control problems generally result in policies that are time-varying functions of the state. However, chronometry is imperfect and thus feedback laws are inevitably applied at incorrect times. Little appears to be known about the consequences of imperfect timing on control [1]–[3]. This paper addresses optimal control with temporal uncertainty.

A stochastic process can be *time-changed* by replacing its time index by a monotonically increasing stochastic process [4]. Time-changed stochastic processes arise in finance, since changing the time index to a measure of economically relevant events, such as trades, can improve modeling [5]–[7]. This new time index is, however, stochastic with respect to “calendar” time.

We suspect that similar notions of stochastic time changing may facilitate the study of time estimation and movement control in the nervous system. Biological timing is subject to noise and environmental perturbation [8]. Furthermore, humans rationally exploit the statistics of their temporal noise during simple timed movements, such as button pushing [9] and pointing [10]. To analyze more complex movements, a theory of feedback control that compensates for temporal noise seems desirable.

Within control, the most closely related work to the present paper deals with analysis and synthesis of systems with uncertain sampling times. The study of uncertain sampling times has a long history in control [11], and is often motivated by problems of clock jitter [12], [13] or network delays [14]. In these works, control inputs are sampled at known times and held over unknown intervals. To derive the dynamic programming principle in this paper, system behavior is analyzed for control inputs held over random intervals, bearing some similarity to optimal control with random sampling [15]. Fundamentally, however, studies of sampling uncertainty assumes that an accurate clock can measure the sample times; the present work relaxes this assumption.

Other aspects of imperfect timing have been addressed in control research to a more limited extent. For example, the importance of synchronizing clocks in distributed systems seems clear [16], [17], but more work is needed to understand the implications of asynchronous clock behavior on common control issues, such as stability [18] and optimal performance [19].

A. Lamperski is with the Department of Electrical and Computer Engineering, University of Minnesota, Minneapolis, MN, USA (alampers@umn.edu).

N. J. Cowan is with the Department of Mechanical Engineering, The Johns Hopkins University, Baltimore, MD, USA (ncowan@jhu.edu).

This paper focuses on continuous-time stochastic optimal control with perfect state information, but a stochastically time-changed control process. Dynamic programming principles for general nonlinear stochastic control problems are derived, based on extensions of the classical Hamilton-Jacobi-Bellman equation. The results apply to a wide class of stochastic time changes given by strictly increasing Lévy processes. The dynamic programming principles are then specialized to give explicit solutions to a time-changed version of the finite-horizon linear quadratic regulator and a portfolio optimization problem.

Section II defines the notation used in the paper, states the necessary facts about Lévy, and defines the class of noisy clock models used. The main results on time-changed diffusions and optimal control are given in Section III. The results are proved in Sections IV with supplementary arguments given in the appendices. Sections V and VI discuss future work and conclusions, respectively.

II. PRELIMINARIES

After establishing notation and reviewing Lévy processes, this section culminates in the construction of Lévy-process-based clock models upon which the remainder of the theory of this paper is built.

A. Notation

The norm symbol, $\|\cdot\|$, is used to denote the Euclidean norm for vectors and the Frobenius norm for matrices.

For a set S , its closure is denoted by \bar{S} .

The spectrum of matrix A is denoted by $\text{spec}(A)$.

The Kronecker product is denoted by \otimes , while the Kronecker sum is denoted by \oplus : $A \oplus B = A \otimes I + I \otimes B$.

The vectorization operation of stacking the columns of a matrix is denoted by vec .

A function $h : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is in $\mathcal{C}^{1,2}$ if $h(s, x)$ is continuously differentiable in s , twice continuously differentiable in x . The function h is said to satisfy a *polynomial growth condition*, if in addition, there are constants K and q such that

$$\max \left\{ |h(s, x)|, \left| \frac{\partial h(s, x)}{\partial s} \right|, \left| \frac{\partial h(s, x)}{\partial x_i} \right|, \left| \frac{\partial^2 h(s, x)}{\partial x_i \partial x_j} \right| \right\} \leq K (1 + \|x\|^q),$$

for $i, j = 1, \dots, n$, and all $x \in \mathbb{R}^n$. In this case, $h \in \mathcal{C}_p^{1,2}$ is written.

Stochastic processes will be denoted as ζ_t , X_s , etc., with time indices as subscripts. Occasionally, processes with nested subscripts will be written with parentheses, e.g. $\zeta_{\tau_s} = \zeta(\tau_s)$. Similarly, the elements of a stochastic vector will be denoted as $X_1(s)$.

The left limit of a process X_t will be denoted by $X_{t-} = \lim_{s \uparrow t} X_s$.

Functions that are right-continuous with left-limits will be called *càdlàg*, while functions that are left-continuous with right-limits will be called *càglàd*.

B. Background on Lévy Processes

Basic notions from Lévy processes required to define the general class of clock models are now reviewed. The definitions and results can be found in [20].

A real-valued stochastic process Z_s is called a *Lévy process* if

- $Z_0 = 0$ almost surely (a.s.).
- Z_s has independent, stationary increments: If $0 \leq r \leq s$, then Z_r and $Z_s - Z_r$ are independent and $Z_s - Z_r$ has the same distribution as Z_{s-r} .
- Z_s is stochastically continuous: For all $a > 0$ and all $s \geq 0$, $\lim_{r \rightarrow s} \mathbb{P}(|Z_s - Z_r| > a) = 0$.

It will be assumed that Lévy processes in this paper are right-continuous with left-sided limits, i.e. they are càdlàg. No generality is lost since, for every Lévy process, Z_t , there is a càdlàg Lévy process, \tilde{Z}_t , such that $Z_t = \tilde{Z}_t$ for almost all t .

Some of the technical arguments rely on Poisson random measures, which will now be defined. Let \mathcal{B} be the Borel subsets of \mathbb{R} and let $(\Omega, \Sigma, \mathbb{P})$ be a probability space. A *Poisson random measure* is a function $N : [0, \infty) \times \mathcal{B} \times \Omega \rightarrow \mathbb{N} \cup \{\infty\}$, such that

- For all $s \geq 0$ and $\omega \in \Omega$, $N(s, \cdot, \omega)$ is a measure.
- For all disjoint Borel subsets $A, B \in \mathcal{B}$ such that $0 \notin \bar{A}$ and $0 \notin \bar{B}$, $N(\cdot, A, \cdot)$ and $N(\cdot, B, \cdot)$ are independent Poisson processes.

Typically, the ω argument will be dropped, and it will be implicitly understood that $N(s, A)$ denotes a measure-valued stochastic process.

The following relationship between Lévy processes and Poisson random measures will be used in several arguments. For a Lévy process, Z_s , with jumps denoted by ΔZ_s , there is a Poisson random measure that counts the number of jumps into each Borel set A with $0 \notin \bar{A}$:

$$N(s, A) = |\{\Delta Z_r \in A : 0 \leq r \leq s\}|.$$

Here, $|S|$ denotes the cardinality of a finite set, S .

Subordinators. A monotonically increasing Lévy process, τ_s , is called a *subordinator*. The following properties of subordinators will be used throughout the paper.

- **Laplace Exponent:** There is function, ψ , called the *Laplace exponent*, defined by

$$\psi(z) = bz + \int_0^\infty (1 - e^{-zt}) \lambda(dt), \quad (1)$$

such that

$$\mathbb{E}[e^{-z\tau_s}] = e^{-s\psi(z)} \quad \text{for all } z \geq 0. \quad (2)$$

Here $b \geq 0$ and the measure satisfies $\int_0^\infty \min\{t, 1\} \lambda(dt) < \infty$. The measure λ is called a *Lévy measure*. The pair (b, λ) is called the *characteristics* of τ_s .

- **Lévy-Itô Decomposition:** There is a Poisson random measure N such that

$$\tau_s = bs + \int_0^\infty tN(s, dt).$$

Furthermore, if $A \subset (0, \infty)$ is a Borel set such that $0 \notin \bar{A}$, then $\mathbb{E}[N(1, A)] = \lambda(A)$.

The function, ψ , is called the Laplace exponent because (2) is the Laplace transform of the distribution of τ_s .

For control problems, simpler formulas will often result from replacing ψ with the function $\beta(z) = -\psi(-z)$. Note then, that β has the form

$$\beta(z) = bz + \int_0^\infty (e^{zt} - 1) \lambda(dt). \quad (3)$$

Define r_{\max} by

$$r_{\max} = \sup \left\{ r : \int_1^\infty e^{rt} \lambda(dt) < \infty \right\}$$

and define the domain of β as

$$\text{dom}(\beta) = \{z \in \mathbb{C} : \text{Re } z < r_{\max}\}.$$

Note that $\int_1^\infty \lambda(dt) < \infty$ implies that $r_{\max} \in [0, \infty]$.

The function β is used to construct optimal solutions for the linear quadratic problem, as well as the portfolio problem below. The main properties are given in the following lemma, which is proved in Appendix B.

Lemma 1: For all $z \in \text{dom}(\beta)$, the function β is analytic at z , and

$$\mathbb{E}[e^{z\tau_s}] = e^{s\beta(z)}. \quad (4)$$

Furthermore, if A is a square matrix with $\text{spec}(A) \subset \text{dom}\beta$, then

$$\beta(A) = bA + \int_0^\infty (e^{At} - I) \lambda(dt) \quad (5)$$

is well defined and

$$\mathbb{E}[e^{A\tau_s}] = e^{s\beta(A)}. \quad (6)$$

Since β is analytic, several methods exist for numerically computing the matrices $\beta(A)$ [21]. In special cases, as discussed below, $\beta(A)$ may be computed using well-known matrix computation methods.

Example 1: The simplest non-trivial subordinator is the homogeneous Poisson process N_t , which is characterized by

$$\mathbb{P}(N_t = k) = e^{-\gamma t} \frac{(\gamma t)^k}{k!},$$

where $\gamma > 0$ is called the rate constant. Its Laplace exponent is given by $\psi(z) = \gamma - \gamma e^{-z}$, which is found by computing the expected value directly. The characteristics are $(0, \gamma\delta(t-1))$. In this case, $\text{dom}(\beta) = \mathbb{C}$, and $\beta(A) = \gamma e^A - \gamma I$, which can be computed from the matrix exponential.

Example 2: The gamma subordinator, which is often used to model “business time” in finance [22], [23], has increments distributed as gamma random variables. It has Laplace exponent $\psi(z) = \delta \log(1 + z/\gamma)$ with characteristics $b = 0$ and $\lambda(dt) = \delta e^{-\gamma t} t^{-1} dt$. Thus $\beta(z) = -\delta \log(1 - z/\gamma)$, $\text{dom}(\beta) = \{z \in \mathbb{C} : \text{Re } z < \gamma\}$, and matrix function $\beta(A) = -\delta \log(I - \gamma^{-1}A)$ may be computed from the matrix logarithm.¹

Why Lévy Processes? In the next subsection, the clock model in this paper will be constructed from a subordinator τ_s . The motivation for using Lévy processes will be explained. Consider a continuous-time noisy clock, c_s which is sampled with period δ . A natural model might take the form

$$c_{\delta(k+1)} = c_{\delta k} + \delta + n(k, \delta), \quad (7)$$

where $n(j, \delta)$ are random variables. In this case, the clock increments consist of a deterministic step of magnitude δ plus a random term.

If c_s is a Lévy process, then by definition, all of the increments $c_{\delta(k+1)} - c_{\delta k}$ are independent and identically distributed. Thus, the decomposition in (7) holds with $n(k, \delta) = c_{\delta(k+1)} - c_{\delta k} - \delta$. If c_s were not a Lévy process, then (7) may hold for some particular δ , but there might be another period, $\delta' < \delta$, for which the decomposition fails. The Lévy process assumption will guarantee that the clocks are well-behaved when taking continuous time limits (i.e. $\delta \downarrow 0$).

¹When $\text{spec}(M) \subset \{z \in \mathbb{C} : \text{Re } z > 0\}$, $\log(M)$ refers to the principle logarithm: the unique matrix such that $e^{\log(M)} = M$ and $\text{spec}(\log(M)) \subset \{z \in \mathbb{C} : -\pi < \text{Im } z < \pi\}$. See [21].

C. Clock Models

Throughout the paper, t will denote the time index of the plant dynamics, while s will denote the value of clock available to the controller. Often, t and s will be called *plant time* and *controller time*, respectively. The interpretation of s and t varies depending on context. In biological motor control, t would denote real time, since the limbs obey Newtonian mechanics with respect to real-time, while s would denote the internal representation of time. For the portfolio problem studied in Subsection III-B, an opposite interpretation holds. Here, the controller (an investor) can accurately measure calendar time, but price dynamics are simpler with respect a different index, “business time”, which represents the progression of economic events [5]–[7]. Thus, s would denote calendar time, while t would denote business time, which might not be observable.

The relationship between s and t will be described stochastically. Let τ_s be a strictly increasing subordinator. In other words, if $s < s'$ then $\tau_s < \tau_{s'}$ a.s. (Note that any subordinator can be made to be strictly increasing by adding a drift term bs with $b > 0$.) The process τ_s will be interpreted as the amount of plant time that has passed when the controller has measured s units of time. The process ζ_t will be an inverse process that describes how much time the controller measures over t units of plant time. Formally, ζ_t is defined by

$$\zeta_t = \inf\{\sigma : \tau_\sigma \geq t\}. \quad (8)$$

We claim that $\zeta(\tau_s) = s$ a.s. Indeed, $\zeta(\tau_s) = \inf\{\sigma : \tau_\sigma = \tau_s\}$, by definition. Since τ_s is right continuous and strictly increasing, a.s., the claim follows.

Example 3: The case of no temporal uncertainty corresponds to $\tau_s = s$ and $\zeta_t = t$. The Laplace exponent of τ_s is computed directly as $\psi(z) = z$ and the characteristics are $(1, 0)$. Here $\text{dom}(\beta) = \mathbb{C}$.

Example 4: A more interesting temporal noise model, also used as a “business time” model [24], is the inverse Gaussian subordinator. Fix $\gamma > 0$ and $\delta > 0$. Let $C_t = \gamma t + W_t$, where W_t is a standard Brownian motion. The inverse Gaussian subordinator is given by

$$\tau_s = \inf\{t : C_t = \delta s\},$$

with Laplace exponent $\psi(z) = \delta(\sqrt{\gamma^2 + 2z} - \gamma)$. Here $b = 0$ and λ is given by

$$\lambda(dt) = \frac{\delta}{\sqrt{2}\Gamma(1/2)} e^{-\frac{1}{2}\gamma^2 t} t^{-\frac{3}{2}} dt,$$

where Γ is the gamma function. Here, $\text{dom}(\beta)$ corresponds to $\text{Re}z < \gamma^2/2$ and $\beta(A) = \delta(\gamma I - \sqrt{\gamma^2 I - 2A})$, which can be computed from the matrix square root. It can be shown that the inverse process is given by

$$\zeta_t = \sup\{\delta^{-1}C_\sigma : 0 \leq \sigma \leq t\}.$$

See Figure 1.

In the preceding example, the process τ_s has jumps, but the inverse, ζ_t , is continuous. The next proposition generalizes this observation for any strictly increasing subordinator, τ_s .

Proposition 1: The process ζ_t is continuous almost surely.

Proof: Fix $\epsilon > 0$ and $t \geq 0$. Set $s = \zeta_t$. Strict monotonicity of τ_s implies that $[\tau_{\max\{s-\epsilon, 0\}}, \tau_{s+\epsilon}]$ is a nonempty interval, a.s. The inverse property of ζ_t implies (almost surely) that $t \in [\tau_{\max\{s-\epsilon, 0\}}, \tau_{s+\epsilon}]$ and $\zeta_{t'} \in [\max\{s-\epsilon, 0\}, s+\epsilon]$ for all $t' \in [\tau_{\max\{s-\epsilon, 0\}}, \tau_{s+\epsilon}]$. ■

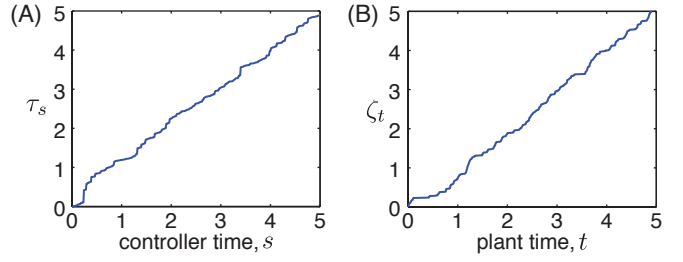


Fig. 1. (A). The inverse Gaussian subordinator, τ_s , with $\gamma = \delta = 2$. The process was simulated by generating independent inverse Gaussians using the method from [25]. (B) The inverse process, ζ_t . Note that the graph of ζ_t can be found from the graph of τ_s by simply switching the axes.

III. MAIN RESULTS

This section presents the main results of the paper. First, given an Itô process, Y_t , a representation of the time-changed process $X_s = Y(\tau_s)$ as a semimartingale with respect to controller time, s , is derived. This representation is then used to derive a general dynamic programming principle for control problems with noisy clocks. As an example, the dynamic programming principle is used to solve a simple portfolio optimization problem under random trade activity rates. Finally, the dynamic programming method is used to solve a noisy-time variant of the linear quadratic regulator problem. All proofs are given in Section IV.

A. Time-Changed Stochastic Processes

This section gives a basic representation theorem for time-changed stochastic processes that will be vital for dynamic programming proofs. The theorem is proved in Subsection IV-A.

Let W_t be a Brownian motion with $\mathbb{E}[W_t W_t^T] = tI$. Let Y be a stochastic process defined by

$$dY_t = F_t dt + G_t dW_t, \quad (9)$$

where F_t and G_t are \mathcal{F}_t^W predictable processes, where $(\mathcal{F}_t^W)_{t \geq 0}$ is the σ -algebra generated by W_t . Furthermore, assume that F_t and G_t are left-continuous with right-sided limits.

For a strictly increasing subordinator, τ_s , let $\mathcal{F}^{\tau, W} = (\mathcal{F}_s^{\tau, W})_{s \geq 0}$ be the smallest filtration such that for all $r \in [0, s]$ and all $t \in [0, \tau_s]$ both τ_r and W_t are measurable.

Theorem 1: Let τ_s be a subordinator characterized by (b, λ) . If the terms of (9) satisfy

- $\int_0^{\tau_s} \|F_t\| dt < \infty$ almost surely and
- $\mathbb{E}[\int_0^{\tau_s} \|G_t\|^2 dt] < \infty$,

then the time-changed process $X_s = Y(\tau_s)$ is an $\mathcal{F}^{\tau, W}$ semimartingale given by

$$X_s = X_0 + b \int_0^s F(\tau_{r-}) dr + \sqrt{b} \int_0^s G(\tau_{r-}) d\tilde{W}_r + \sum_{0 \leq r \leq s} \left(\int_{\tau_{r-}}^{\tau_r} F_t dt + \int_{\tau_{r-}}^{\tau_r} G_t dW_t \right). \quad (10)$$

Here \tilde{W}_s is an $\mathcal{F}^{\tau, W}$ -measurable Brownian motion defined by

$$\sqrt{b}\tilde{W}_s = W(\tau_s) - \sum_{0 \leq r \leq s} (W(\tau_r) - W(\tau_{r-})),$$

satisfying $b\mathbb{E}[\tilde{W}_s \tilde{W}_s^T] = bsI$.

B. Dynamic Programming

This subsection introduces the general control problem studied in this paper. First, the basic notions of controlled time-changed diffusions and admissible systems are defined. Then, the finite-horizon control problem is stated, and the associated dynamic programming verification theorem is stated. The concepts in this subsection are based on the general theory of controlled Markov processes [26]. A nonlinear portfolio problem is solved at the end of the subsection.

Controlled Time-Changed Diffusions. Consider a controlled diffusion

$$dY_t = F(\zeta_t, Y_{t-}, U(\zeta_t))dt + G(\zeta_t, Y_{t-}, U(\zeta_t))dW_t, \quad (11)$$

with state Y and input U . Recall that ζ_t is defined in (8) as the inverse process of a subordinator, τ_s . Let X_s denote the time-changed process, $X_s = Y(\tau_s)$. The processes, X_s is thus a time-changed controlled diffusion.

Admissible Systems For $s \geq 0$, let $\mathcal{F}_s^{\zeta, X}$ be the σ -algebra generated by (s, X_s) , and let $\mathcal{F}^{\zeta, X}$ be the associated filtration.

Let $\mathcal{X} \subset \mathbb{R}^n$ and $\mathcal{U} \subset \mathbb{R}^p$ be a set of states and a set of inputs, respectively. A state and input trajectory (X_s, U_s) is called an *admissible system* if

- $X_s \in \mathcal{X}$ for all $s \geq 0$
- U_s is a càglàd, $\mathcal{F}^{\zeta, X}$ -adapted process such that $U_s \in \mathcal{U}$ for all $s \geq 0$.

Note that the requirement that U_s is càglàd and $\mathcal{F}^{\zeta, X}$ -adapted implies that $U(\zeta_t)$ may depend on the “noisy clock” process, ζ_t , as well as X_r , with $r < \zeta_t$. If $\zeta_t \neq t$, then $U(\zeta_t)$ cannot directly measure t .

Problem 1: The time-changed optimal control problem over time horizon $[0, S]$ is to find a policy U_s that solves

$$\min_U \mathbb{E} \left[\int_0^S c(s, X_s, U_s) ds + \Psi(X_S) \right],$$

where the minimum is taken over all admissible systems (X_s, U_s) .

Given a policy, U , and $(s, x) \in [0, S] \times \mathbb{R}^n$, the cost-to-go function $J(s, x; U)$, is defined by

$$J(s, x; U) = \mathbb{E} \left[\int_s^S c(s, X_r, U_r) dr + \Psi(X_S) \middle| X_s = x \right].$$

Note, then, that the optimal control problem can be equivalently cast as minimizing $J(0, x; U)$ over all admissible systems.

Backward Evolution Operator. As in standard continuous-time optimal control, the *backward evolution operator*,

$$\begin{aligned} \mathcal{A}^u h(s, x) &= \\ \lim_{\sigma \downarrow 0} \frac{1}{\sigma} & (\mathbb{E} [h(s + \sigma, X_{s+\sigma}) | X_s = x, U_r = u] - h(s, x)), \end{aligned} \quad (12)$$

is used to formulate the dynamic programming equations.

To calculate an explicit form for \mathcal{A}^u , an auxiliary stochastic process is introduced. For $(s, x, u) \in [0, S] \times \mathcal{X} \times \mathcal{U}$, define Y_{st}^{xu} by

$$Y_{st}^{xu} = x + \int_0^t F(s, Y_{sr}^{xu}, u) dr + \int_0^t G(s, Y_{sr}^{xu}, u) d\hat{W}_r, \quad (13)$$

where \hat{W}_r is a unit Brownian motion independent of W_t and τ_s .

Now the domain of \mathcal{A}^u is defined. Let \mathcal{D} be the set of $h \in \mathcal{C}_p^{1,2}$ such that there exist K and q satisfying

$$\int_0^\infty |\mathbb{E}_{\hat{W}} [h(s, Y_{st}^{xu})] - h(s, x)| \lambda(dt) < K(1 + \|x\|^q + \|u\|^q) \quad (14)$$

for all $(s, x, u) \in [0, S] \times \mathcal{X} \times \mathcal{U}$.

It will be shown in Subsection IV-B that for $h \in \mathcal{D}$, the backward evolution operator for X_s is given by

$$\begin{aligned} \mathcal{A}^u h(s, x) &= \frac{\partial h(s, x)}{\partial s} + b \frac{\partial h(s, x)}{\partial x} F(s, x, u) \\ &+ \frac{1}{2} b \text{Tr} \left(G(s, x, u)^\top \frac{\partial^2 h(s, x)}{\partial x^2} G(s, x, u) \right) \\ &+ \int_0^\infty (\mathbb{E}_{\hat{W}} [h(s, Y_{st}^{xu})] - h(s, x)) \lambda(dt). \end{aligned} \quad (15)$$

Remark 1: When the dynamics are time-homogeneous, i.e. $F(s, y, u) = F(y, u)$ and $G(s, y, u) = G(y, u)$, and the policy is Markov, $U_s = U(X_{s-})$, the expression for \mathcal{A}^u in (15) is a special case of Phillips’ Theorem [20], [27]. In this case, the formula can be derived using techniques from semigroup theory [27]. The derivation in this paper is instead based on Itô calculus.

Finite Horizon Verification. The following result is a dynamic programming verification theorem for Problem 1. The theorem is proved in Subsection IV-B by reducing it to a special case of finite-horizon dynamic programming for controlled Markov processes [26].

Theorem 2: Assume that there is a function $V \in \mathcal{D}$ that satisfies:

$$\inf_u [c(s, x, u) + \mathcal{A}^u V(s, x)] = 0, \quad (16)$$

$$V(S, x) = \Psi(x), \quad (17)$$

where (16) holds for all $(s, x, u) \in [0, S] \times \mathcal{X} \times \mathcal{U}$ and (17) holds for all $x \in \mathcal{X}$.

Then $V(s, x) \leq J(s, x; U)$ for every admissible system, (X, U) with $X_s = x$.

Furthermore, if a policy U_r^* and associated state process X_r^* , with $X_s^* = x$, satisfy

$$U_r^* \in \arg \min_u [c(r, X_r^*, u) + \mathcal{A}^u V(r, X_r^*)], \quad (18)$$

for almost all $(r, \omega) \in [s, S] \times \Omega$, then $V(s, x) = J(s, x; U^*)$.

Remark 2: The theorem gives a *sufficient* condition for a policy to be optimal. By the definition of admissible systems if (16)-(17) hold and (X_r^*, U_r^*) satisfy (18), then U_r^* is optimal over all causal policies, including those with memory.

Example 5: Consider the problem of maximizing $\mathbb{E} [X_S^\eta]$, with $\eta \in (0, 1)$ subject to the time-changed dynamics

$$\begin{aligned} dY_t &= U(\zeta_t) Y_t (\mu_1 dt + \sigma_1 dW_1(t)) \\ &+ (1 - U(\zeta_t)) Y_t (\mu_2 dt + \sigma_2 dW_2(t)) \\ X_s &= Y(\tau_s), \end{aligned}$$

where $W_1(t)$ and $W_2(t)$ are independent Brownian motions. The problem can be interpreted as allocating wealth between stocks modeled by time-changed geometric Brownian motions: $Z_i(s) = R_i(\tau_s)$, where $dR_i(t) = R_i(t)(\mu_i dt + \sigma_i dW_i(t))$.

Let u^* be the optimal solution and ρ^* be the optimal value of the following quadratic maximization problem:

$$\begin{aligned} \max_u & \left[\frac{1}{2} \eta (\eta - 1) ((u\sigma_1)^2 + ((1-u)\sigma_2)^2) \right. \\ & \left. + \eta (u\mu_1 + (1-u)\mu_2) \right]. \end{aligned}$$

If $\rho^* \in \text{dom}(\beta)$, it can be verified by elementary stochastic calculus that $V(s, x)$ given by

$$V(s, x) = e^{\beta(\rho^*)(S-s)} x^\eta$$

satisfies the dynamic programming equations, (16) and (17), with $\mathcal{X} \times \mathcal{U} = [0, \infty) \times \mathbb{R}$ and max replacing min. The corresponding optimal input is $U_s^* = u^*$.

C. Linear Quadratic Regulators

In this section, Theorem 2 is applied to linear systems with quadratic cost. The result (with no Brownian forcing) was originally presented in [3], using a proof technique specialized for linear systems.

Problem 2: Consider linear dynamics

$$dY_t = (AY_t + BU(\zeta_t))dt + MdW_t, \quad (19)$$

subject to the time change $X_s = Y(\tau_s)$. Here $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{U} = \mathbb{R}^p$.

The time-changed linear quadratic regulator problem over time horizon $[0, S]$ is to find a policy U_s that solves

$$\min_U \mathbb{E} \left[\int_0^S \left(X_s^\top Q X_s + U_s^\top R U_s \right) ds + X_S^\top \Phi X_S \right],$$

over all càglàd, $\mathcal{F}^{\zeta, X}$ -adapted policies. Here Q and Φ are positive semidefinite, while R is positive definite.

The following lemma introduces the mappings used to construct the optimal solution for the time-changed linear quadratic regulator problem. The lemma is proved in Appendix C by showing that each mapping may be computed from $\beta(A)$ for an appropriately defined matrix \bar{A} .

Lemma 2: Let P be an $n \times n$ matrix. If $\{0\} \cup \text{spec}(2A) \subset \text{dom}(\beta)$, then the following linear mappings are well defined:

$$\begin{aligned} F(P) &= b(A^\top P + PA) + \int_0^\infty \left(e^{A^\top t} P e^{At} - P \right) \lambda(dt) \\ G(P) &= bP + \int_0^\infty e^{A^\top t} P \int_0^t e^{Ar} dr \lambda(dt) \\ H(P) &= \int_0^\infty \int_0^t e^{A^\top r} dr P \int_0^t e^{A\rho} d\rho \lambda(dt) \\ g(P) &= \text{Tr} \left(P \left(bMM^\top + \int_0^\infty \int_0^t e^{Ar} MM^\top e^{A^\top r} dr \lambda(dt) \right) \right). \end{aligned}$$

Furthermore, F , G , and H satisfy

$$\begin{aligned} \mathbb{E} \left[e^{A^\top \tau_s} P e^{A\tau_s} \right] &= P + sF(P) + O(s^2) \\ \mathbb{E} \left[e^{A^\top \tau_s} P \int_0^{\tau_s} e^{Ar} dr \right] &= sG(P) + O(s^2) \\ \mathbb{E} \left[\int_0^{\tau_s} e^{A^\top r} dr P \int_0^{\tau_s} e^{A\rho} d\rho \right] &= sH(P) + O(s^2). \end{aligned}$$

Remark 3: The descriptions of F , G , and H in terms of expectations are not required for the proof below. They are given to demonstrate that the formulas in terms of (b, λ) coincide with the formulas from [3].

Example 6: With no temporal noise, the mappings reduce to

$$\begin{aligned} F(P) &= A^\top P + YP, & G(P) &= P, \\ H(P) &= 0, & g(P) &= \text{Tr}(PMM^\top). \end{aligned} \quad (20)$$

Furthermore, since $\beta(z) = z$ is analytic everywhere, these formulas are true for any state matrix, A .

Example 7: Consider an arbitrary strictly increasing subordinator with Laplace exponent ψ . Let $A = \mu$ where μ is a real, non-zero

scalar with $2\mu \in \text{dom}(\beta)$. Let M be a scalar. Combining (2) with the formula $\int_0^t e^{\mu\sigma} d\sigma = \mu^{-1}(e^{\mu t} - 1)$ shows that

$$\begin{aligned} F(P) &= \beta(2\mu)P \\ G(P) &= \mu^{-1}(\beta(2\mu) - \beta(\mu))P \\ H(P) &= \mu^{-2}(\beta(2\mu) - 2\beta(\mu))P \\ g(P) &= \frac{1}{2}\mu^{-1}\beta(2\mu)M^2P. \end{aligned}$$

Theorem 3: Say that $\{0\} \cup \text{spec}(2A) \subset \text{dom}(\beta)$. Define the function $V(s, x) = x^\top P_s x + h_s$ by the backward differential equations

$$\begin{aligned} -\frac{d}{ds} P_s &= Q + F(P_s) - G(P_s)B(R + B^\top H(P_s)B)^{-1}B^\top G(P_s)^\top \\ -\frac{d}{ds} h_s &= g(P_s), \end{aligned}$$

with final conditions $P_S = \Phi$ and $h_S = 0$. The function $V(s, x)$ satisfies dynamic programming equations, (16) and (17), and the optimal policy is given by

$$\begin{aligned} U_s &= K_s X_{s-} \\ K_s &= -(R + B^\top H(P_s)B)^{-1}B^\top G(P_s)^\top. \end{aligned}$$

The policy is computed by minimizing (18). Thus, Theorem 2 implies that linear state feedback is optimal over all causal policies.

A straightforward variation on the proof of Theorem 3 shows that for any linear policy, $U_s = L_s X_{s-}$, the cost-to-go is given by

$$J(s, x; U) = x^\top Z_s x + p_s,$$

where Z_s and k_s satisfy the backward differential equations

$$\begin{aligned} -\frac{d}{ds} Z_s &= Q + F(Z_s) + L_s^\top B^\top G(Z_s)^\top + G(Z_s)BL_s \\ &\quad + L_s^\top (R + B^\top H(Z_s)B)L_s \\ -\frac{d}{ds} p_s &= g(Z_s). \end{aligned}$$

In the following example, these formulas are used in order to compare the performance of the policy from Theorem 3 with the policy $U_s = L_s X_{s-}$, where L_s is the standard LQR gain, not compensating for temporal noise.

Example 8: Consider the system defined by the state matrices

$$A = \begin{bmatrix} 0.75 & 1 \\ 0 & 0.75 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad M = 0,$$

with cost matrices given by

$$R = 0.5, \quad Q = 0, \quad \Phi = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let τ_s be the inverse Gaussian subordinator with $\gamma = \delta = 2$. The condition, $\text{spec}(2A) \subset \text{dom}(\beta)$, is satisfied since $2 \cdot 0.75 = 1.5 < \gamma^2/2 = 2$. Figure 2 compares the optimal policy with the standard LQR policy.

IV. PROOFS OF MAIN RESULTS

A. Proof of Theorem 1

From the definition of X_s ,

$$X_s = X_0 + \int_0^{\tau_s} F_t dt + \int_0^{\tau_s} G_t dW_t. \quad (21)$$

Note that X_s is $\mathcal{F}_s^{\tau, W}$ -measurable. Thus, X_s will be an $\mathcal{F}^{\tau, W}$ semimartingale, provided that 1) $\int_0^{\tau_s} F_t dt$ has finite variation and 2) $\int_0^{\tau_s} G_t dW_t$ is an $\mathcal{F}^{\tau, W}$ martingale.

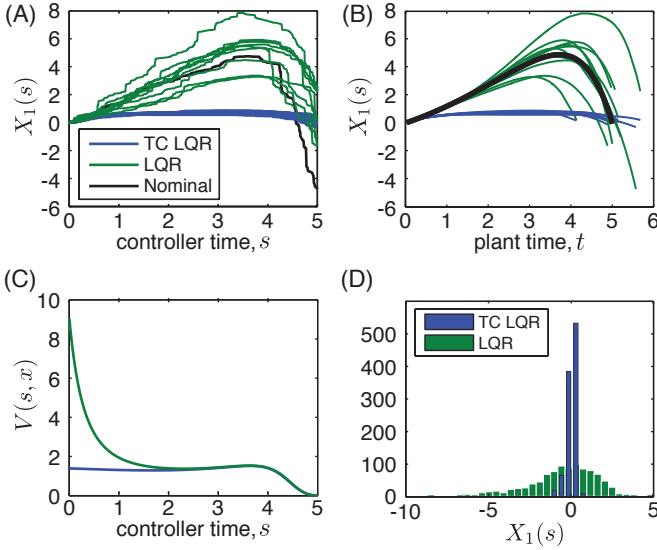


Fig. 2. (A) Plots of $X_1(s)$ under the optimal policy and the LQR policy for 10 realizations of τ_s . The initial condition is $x = [0, 1]^T$. (B) The same plots under time variable t . The black line shows the LQR trajectory with no temporal noise. In the case of no temporal noise, the classical LQR uses high gains near $t = 0$ to produce high-speed trajectories such that Y_1 approaches 0 at final time. In this case, timing errors lead to wide variation in the final position. The optimal policy reduces the speed of the trajectory near $s = 0$ order to minimize the effects of temporal noise. (C) The optimal cost $V(s, x)$ and $J(s, x; U)$ for the LQR policy (i.e. U is optimal without time noise) are plotted for $x = [0, 1]^T$. As expected, $V(s, x) \leq J(s, x; U)$. Furthermore, as the time-horizon increases, the LQR policy depends strongly on timing information, and so temporal noise leads to higher cost as s goes to 0. (D) A histogram of the final positions, $X_1(S)$, for 1000 realizations of τ_s . The optimal controller leads to $X_1(S)$ being tightly distributed around 0, while the LQR controller gives a wide spread of $X_1(S)$ values. The errors in the final position lead to increased cost for the LQR controller.

Finite variation follows since

$$\text{Var} \left(\int_0^{\tau_s} F_t dt \right) \leq \int_0^{\tau_s} \|F_t\|^2 dt < \infty \text{ almost surely.}$$

To prove the martingale property, note that for $0 \leq r \leq s$ we have

$$\begin{aligned} \mathbb{E} \left[\int_0^{\tau_s} G_t dW_t \middle| \mathcal{F}_r^{\tau, W} \right] &= \int_0^{\tau_r} G_t dW_t + \mathbb{E} \left[\int_{\tau_r}^{\tau_s} G_t dW_t \middle| \mathcal{F}_r^{\tau, W} \right] \\ &= \int_0^{\tau_r} G_t dW_t. \end{aligned}$$

Furthermore,

$$\mathbb{E} \left[\left\| \int_0^{\tau_s} G_t dW_t \right\|^2 \right] \leq \mathbb{E} \left[\left\| \int_0^{\tau_s} G_t dW_t \right\|^2 \right] < \infty,$$

where the inequality follows from Jensen's inequality. Thus 2) holds.

Now (10) must be proved. For more compact notation, define the processes H_t and Z_t as

$$H_t = \begin{bmatrix} F_t & G_t \end{bmatrix} \quad Z_t = \begin{bmatrix} t \\ W_t \end{bmatrix}$$

so that X_s may be written as

$$X_s = \int_0^{\tau_s} H_t dZ_t. \quad (22)$$

Note that $Z(\tau_s) = [\tau_s, W(\tau_s)^T]^T$. Since τ_s is a subordinator, $W(\tau_s)$ is a Lévy process on \mathbb{R}^d , with Lévy symbol

$$\eta_{W_\tau}(z) = -\frac{1}{2}bz^T z + \int_{\mathbb{R}^d} (e^{iz^T x} - 1)\mu_{W,\tau}(dx),$$

for some Lévy measure $\mu_{W,\tau}$. (See Theorem 1.3.25 and Theorem 1.3.33, respectively, in [20].) Thus, the continuous part of $W(\tau_s)$ is a Brownian motion with $\mathbb{E} [W(\tau_s)W(\tau_s)^T] = bI$.

Define the \tilde{Z}_s by removing the jumps from $Z(\tau_s)$.

$$\tilde{Z}_s = Z(\tau_s) - \sum_{0 \leq r \leq s} (Z(\tau_r) - Z(\tau_r^-)).$$

It follows that $\tilde{Z}_s = [bs, \sqrt{b}\tilde{W}_s^T]^T$, where \tilde{W}_s is the Brownian motion from the theorem statement. Thus, (10) can be equivalently written as

$$X_s = \int_0^s H(\tau_{r^-}) d\tilde{Z}_s + \sum_{0 \leq r \leq s} \int_{\tau_{r^-}}^{\tau_r} H_s dZ_t \quad (23)$$

So, the proof will follow by deriving (23) from (22). If $b = 0$, then $\tilde{Z}_s = 0$ and $\tau_s = \sum_{0 \leq r \leq s} \tau_r - \tau_{r^-}$. Thus,

$$X_s = \sum_{0 \leq r \leq s} \int_{\tau_{r^-}}^{\tau_r} H_s dZ_t,$$

so, in this case, (10) holds.

Now assume $b > 0$. The cases when τ_s has finite rate ($\lambda((0, \infty)) < \infty$) and infinite rate ($\lambda((0, \infty)) = \infty$) will be treated separately.

Finite Rate. Let $r_0 = 0$ and let r_1, r_2, \dots be the jump times of τ_s . With probability 1, there exist a finite (random) integer L such that L jumps occur over $[0, s]$. Note that (22) may be expanded as

$$\begin{aligned} X_s &= \int_{\tau(r_L)}^{\tau_s} H_t dZ_t \\ &+ \sum_{k=0}^{L-1} \left[\int_{\tau(r_k)}^{\tau(r_{k+1}^-)} H_t dZ_t + \int_{\tau(r_{k+1}^-)}^{\tau(r_{k+1})} H_t dZ_t \right] \end{aligned} \quad (24)$$

Let $s_0^n \leq s_1^n \leq \dots \leq s_{K_n}^n$ be a sequence of partitions such that

$$\begin{aligned} \lim_{n \rightarrow \infty} s_{K_n}^n &= \infty \quad a.s. \\ \lim_{n \rightarrow \infty} \sup \{ |s_{k+1}^n - s_k^n| : k = 0, \dots, K_n - 1 \} &= 0 \quad a.s. \\ \{r_i : r_i \leq s_{K_n}^n\} &\subset \{s_0^n, \dots, s_{K_n}^n\}. \end{aligned}$$

The last condition ensures that the jump times are contained in the partition.

Note that between jumps (i.e. $s \in [r_k, r_{k+1})$), $\tau_s = bs + \tau^d(r_k)$, where τ_s^d is the discontinuous part of τ_s . Since $b > 0$ follows that the sequence $\tau(s_0^n), \tau(s_1^n), \dots$, satisfies the following properties, almost surely:

$$\begin{aligned} \lim_{n \rightarrow \infty} \tau(s_{K_n}^n) &= \infty \\ \lim_{n \rightarrow \infty} \sup \{ |\tau(s_{i+1}^n) - \tau(s_i^n)| : \exists k \text{ s.t. } r_k \leq s_i^n < r_{k+1} \} &= 0 \end{aligned}$$

Using a standard argument from stochastic integration (see Theorem II.21 of [28]), the integral from $\tau(r_k)$ to $\tau(r_{k+1}^-)$ may be evaluated as

$$\begin{aligned} &\int_{\tau(r_k)}^{\tau(r_{k+1}^-)} H_s dZ_t \\ &= \lim_{n \rightarrow \infty} \sum_{r_k \leq s_i^n < r_{k+1}^-} H(\tau(s_i^n)) (Z(\tau(s_{i+1}^-)) - Z(\tau(s_i))) \\ &= \lim_{n \rightarrow \infty} \sum_{r_k \leq s_i^n < r_{k+1}^-} H(\tau(s_i^n)) (\tilde{Z}(s_{i+1}) - \tilde{Z}(s_i)) \\ &= \int_{\tau_k}^{\tau_{k+1}} H(\tau_s^-) d\tilde{Z}_s. \end{aligned} \quad (25)$$

The second equality uses the fact that no jumps occur over (r_k, r_{k+1}) . Equation (23) now follows by combining (24) and (25).

Infinite Rate. Let $\epsilon_n > 0$ be a sequence decreasing to 0, at a rate to be specified later. Define τ_s^n to be the process by removing all jumps of size at most ϵ_n from τ_s :

$$\tau_s^n = bs + \int_{\epsilon_n}^{\infty} tN(s, dt). \quad (26)$$

Let $r_0^n = 0$, and let r_1^n, r_2^n, \dots be the jump times of τ_s^n . Let $L_s^n = \sup\{k : r_k^n \leq s\}$. With probability 1, $L_s^n < \infty$. If ϵ_n are chosen as in Lemma 3 from Appendix A, then X_s may be computed as a limit

$$X_s = \lim_{n \rightarrow \infty} \left[\int_{\tau(r_{L_s^n}^n)}^{\tau_s} H_t dZ_t \right. \\ \left. + \sum_{k=0}^{L_s^n - 1} H(\tau(r_k^n)) (Z(\tau(r_{k+1}^n)) - Z(\tau(r_k^n))) \right. \\ \left. + \sum_{k=0}^{L_s^n - 1} \int_{\tau(r_{k+1}^n)}^{\tau(r_k^n)} H_t dZ_t \right]. \quad (27)$$

Note that $Z(\tau(r_{k+1}^n)) - Z(\tau(r_k^n))$ may be expressed as

$$Z(\tau(r_{k+1}^n)) - Z(\tau(r_k^n)) \\ = \tilde{Z}(r_{k+1}^n) - \tilde{Z}(r_k^n) + \sum_{\substack{r_k^n < r \leq r_{k+1}^n \\ \Delta \tau_r \leq \epsilon_n}} (Z(\tau_r) - Z(\tau_{r-})).$$

Note that the terms in the summation all vanish as $\epsilon_n \rightarrow 0$. Furthermore, $r_{L_s^n}^n \uparrow s$, almost surely. Thus, (27) can be expressed as

$$X_s = \lim_{n \rightarrow \infty} \left[\sum_{k=0}^{L_s^n - 1} H(\tau(r_k^n)) (\tilde{Z}(r_{k+1}^n) - \tilde{Z}(r_k^n)) \right. \\ \left. + \sum_{k=0}^{L_s^n - 1} \int_{\tau(r_{k+1}^n)}^{\tau(r_k^n)} H_t dZ_t \right],$$

and (23) now follows using Theorem II.21 of [28]. \blacksquare

B. Proof of Theorem 2

Theorem 2 is a special case of finite-horizon dynamic programming for controlled Markov processes (Theorem III.8.1 of [26]), provided that the following two conditions hold for all $h \in \mathcal{D}$:

- (i) The backward evolution operator, defined in (12) is given by the formula in (15).
- (ii) If h satisfies

$$\mathbb{E}[|h(S, X_S)| | X_s = x] < \infty, \quad \text{and} \\ \mathbb{E} \left[\int_s^S |\mathcal{A}^{U_r} h(r, X_r)| dr \mid X_s = x \right] < \infty,$$

then the Dynkin formula holds:

$$\mathbb{E}[h(S, X_S) | X_s = x] - h(s, x) \\ = \mathbb{E} \left[\int_s^S \mathcal{A}^{U_r} h(r, X_r) dr \mid X_s = x \right]. \quad (28)$$

First, using Theorem 1, a more explicit formula for X_s is derived, and then using Itô's formula for semimartingales, a formula for $h(s, X_s)$ is given. Using the formula for $h(s, X_s)$, equations (15) and (28) are then proved.

Note that $Y(\tau_{r-}) = X_{r-}$ and for all $t \in [\tau_{r-}, \tau_r]$, $\zeta_t = r$. Therefore

$$F(\zeta(\tau_{r-}), Y(\tau_{r-}), U(\zeta(\tau_{r-}))) = F(r, X_{r-}, U_r), \quad \text{and} \\ F(\zeta_t, Y_t, U(\zeta_t)) = F(r, Y_t, U_r) \quad \text{for all } t \in [\tau_{r-}, \tau_r].$$

The expressions for G are similar. Thus, Theorem 1 implies that X_s is given by

$$X_s = X_0 + b \int_0^s F(r, X_{r-}, U_r) dr + \sqrt{b} \int_0^s G(r, X_{r-}, U_r) d\tilde{W}_r \\ + \sum_{0 \leq r \leq s} \left(\int_{\tau_{r-}}^{\tau_r} F(r, Y_t, U_r) dt + \int_{\tau_{r-}}^{\tau_r} G(r, Y_t, U_r) dW_t \right). \quad (29)$$

Now a formula for $h(s, X_s)$ will be derived. Note that for any càglàd, $\mathcal{F}^{\tau, W}$ -adapted process, Z_s , the stochastic integral with respect to X_s is given by

$$\int_0^s Z_r dX_r = \int_0^s Z_r b F(r, X_{r-}, U_r) dr \\ + \int_0^s Z_r \sqrt{b} G(r, X_{r-}, U_r) d\tilde{W}_r + \sum_{0 \leq r \leq s} Z_r (X_r - X_{r-}).$$

Furthermore, the continuous part of the quadratic variation is given by

$$[X, X]_s^c = \int_0^s \frac{1}{2} b G(r, X_{r-}, U_r) G(r, X_{r-}, U_r)^\top dr.$$

Thus Itô's formula for semimartingales (see [28]) implies that $h(s, X_s)$ is given by

$$h(s, X_s) = h(0, X_0) \\ + \int_0^s \left(\frac{\partial h(r, X_{r-})}{\partial r} + b \frac{\partial h(r, X_{r-})}{\partial x} F(r, X_{r-}, U_r) \right) dr \\ + \int_0^s \frac{1}{2} b \text{Tr} \left(G(r, X_{r-}, U_r)^\top \frac{\partial^2 h(r, X_{r-})}{\partial x^2} G(r, X_{r-}, U_r) \right) dr \\ + \int_0^s \sqrt{b} \frac{\partial h(r, X_{r-})}{\partial x} G(r, X_{r-}, U_r) d\tilde{W}_r \\ + \sum_{0 \leq r \leq s} (h(r, X_r) - h(r, X_{r-})). \quad (30)$$

Now (15) will be derived from (30). Assume that $X_s = x$ and $U_r = u$ for $r \in [s, s+h]$. Then (30) implies that

$$\mathbb{E}[h(s+h, X_{s+h})] = h(s, x) \\ + \mathbb{E} \left[\int_s^{s+h} \left(\frac{\partial h(r, X_{r-})}{\partial r} + b \frac{\partial h(r, X_{r-})}{\partial x} F(r, X_{r-}, u) \right) dr \right] \\ + \mathbb{E} \left[\frac{1}{2} b \text{Tr} \left(G(r, X_{r-}, u)^\top \frac{\partial^2 h(r, X_{r-})}{\partial x^2} G(r, X_{r-}, u) \right) dr \right] \\ + \mathbb{E} \left[\sum_{s < r \leq s+h} (h(r, X_r) - h(r, X_{r-})) \right]. \quad (31)$$

If $r > s$, the Brownian motions W_t for $t \in [\tau_{r-}, \tau_r]$ and \hat{W}_t for $t \in [0, \Delta \tau_r]$ are identically distributed and independent of $\mathcal{F}_s^{\tau, W}$. Therefore, using (13), and given that $X_s = x$ and $U_r = u$, the expectations of the jump terms may be written as

$$\mathbb{E}[h(r, X_r) - h(r, X_{r-})] = \\ \mathbb{E} \left[\mathbb{E}_{\hat{W}} \left[h \left(r, Y_{r, \Delta \tau_r}^{X_{r-}, u} \right) \right] - h(r, X_{r-}) \right].$$

Thus, the term at the bottom of (31) may be evaluated as a Poisson integral:

$$\mathbb{E} \left[\sum_{s < r \leq s+h} (h(r, X_r) - h(r, X_{r-})) \right] \\ = \mathbb{E} \left[\int_s^{s+h} \int_0^\infty \left(\mathbb{E}_{\hat{W}} \left[h \left(r, Y_{rt}^{X_{r-}, u} \right) \right] - h(r, X_{r-}) \right) N(dr, dt) \right] \\ = \mathbb{E}_W \left[\int_s^{s+h} \int_0^\infty \left(\mathbb{E}_{\hat{W}} \left[h \left(r, Y_{rt}^{X_{r-}, u} \right) \right] - h(r, X_{r-}) \right) \lambda(dt) dr \right]$$

where the second is equation follows from Fubini's theorem and (14).

Now, (15) results from combining (12) with (31) and (32).

Turning to (28), since X_s is $\mathcal{F}_s^{\tau, W}$ measurable, it suffices to prove that

$$\mathbb{E} \left[h(S, X_S) - h(s, X_s) - \int_s^S \mathcal{A}^{U_s} h(r, X_r) dr \middle| \mathcal{F}_s^{\tau, W} \right] = 0.$$

Since $X_r(\omega) = X_{r-}(\omega)$ for almost all $(r, \omega) \in [s, S] \times \Omega$, it follows that

$$\begin{aligned} \mathbb{E} \left[\int_s^S \mathcal{A}^{U_s} h(r, X_r) dr \middle| \mathcal{F}_s^{\tau, W} \right] &= \\ \mathbb{E} \left[\int_s^S \mathcal{A}^{U_s} h(r, X_{r-}) dr \middle| \mathcal{F}_s^{\tau, W} \right]. \end{aligned}$$

Combining (15) and (30) and, for brevity, omitting $\mathcal{F}_s^{\tau, W}$ from the expectations, implies that

$$\begin{aligned} \mathbb{E} \left[h(S, X_S) - h(s, X_s) - \int_s^S \mathcal{A}^{U_r} h(r, X_{r-}) dr \right] &= \\ \mathbb{E} \left[\sum_{s < r \leq S} (h(r, X_r) - h(r, X_{r-})) \right] - \\ \mathbb{E} \left[\int_s^S \int_0^\infty (\mathbb{E}_{\hat{W}}[h(r, Y_t^{X_{r-} - U_r})] - h(r, X_{r-})) \lambda(dt) dr \right], \end{aligned} \quad (33)$$

As in the proof of (15), the two terms at the bottom of (33) are equal in expectation. Thus (28) holds and the proof is complete. \blacksquare

C. Proof of Theorem 3

Assume that $V(s, x) = x^\top P_s x + h_s$. Applying the backward evolution operator corresponding to (19) to $V(s, x)$ results in

$$\mathcal{A}^u V(s, x) = x^\top \dot{P}_s x + \dot{h}_s + bx^\top (A^\top P_s + P_s A)x + 2bx^\top P_s B u \quad (34)$$

$$+ b\text{Tr} \left(P_s M M^\top \right) + \int_0^\infty \left(\mathbb{E}_{\hat{W}} \left[Y_t^{xu \top} P_s Y_t^{xu} \right] - x^\top P_s x \right) \lambda(dt).$$

Note that $\mathbb{E}_{\hat{W}} \left[Y_t^{xu \top} Y_t^{xu} \right]$ may be evaluated as

$$\mathbb{E}_{\hat{W}} \left[Y_t^{xu \top} Y_t^{xu} \right] = y_t^{xu} y_t^{xu \top} + \Sigma_t,$$

where y_t^{xu} is the mean of Y_t^{xu} and Σ_t is the covariance. A standard argument in linear stochastic differential equations shows that the mean and covariance are given by

$$\begin{aligned} y_t^{xu} &= e^{At} x + \int_0^t e^{A(t-r)} dr B u, \\ \Sigma_t &= \int_0^t e^{A(t-r)} M M^\top e^{A^\top(t-r)} dr. \end{aligned}$$

Thus, the integral in (34) may be written as

$$\begin{aligned} \int_0^\infty \left(\mathbb{E}_{\hat{W}} \left[Y_t^{xu \top} P_s Y_t^{xu} \right] - x^\top P_s x \right) \lambda(dt) &= \\ \int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} \hat{F}(t, P_s) & \hat{G}(t, P_s) B \\ B^\top \hat{G}(t, P_s)^\top & B^\top \hat{H}(t, P_s) B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \lambda(dt) \\ + \int_0^\infty \int_0^t \text{Tr} \left(P_s e^{A(t-r)} M M^\top e^{A^\top r} \right) dr \lambda(dt), \end{aligned} \quad (35)$$

where the matrices \hat{F} , \hat{G} , and \hat{H} are defined by

$$\begin{aligned} \hat{F}(t, P) &= e^{A^\top t} P e^{At} - P, \\ \hat{G}(t, P) &= e^{A^\top t} P \int_0^t e^{A\rho} d\rho, \\ \hat{H}(t, P) &= \int_0^t e^{A^\top r} dr P \int_0^t e^{A\rho} d\rho. \end{aligned}$$

Combining (34) and (35), and using the linear operators from Lemma 2 gives

$$\mathcal{A}^u V(s, x) = \begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} \dot{P}_s + F(P_s) & G(P_s) B \\ B^\top G(P_s) & B^\top H(P_s) B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \dot{h}_s + g(P_s).$$

Therefore, adding the cost gives

$$\begin{aligned} x^\top Q x + u^\top R u + \mathcal{A}^u V(s, x) &= \dot{h}_s + g(P_s) \\ + \begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} Q + \dot{P}_s + F(P_s) & G(P_s) B \\ B^\top G(P_s) & R + B^\top H(P_s) B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}. \end{aligned}$$

The result now follows from quadratic minimization. \blacksquare

V. DISCUSSION

The work in this paper lays a theoretical foundation for future research on biological motor control [29], [30], finance [31], and multi-agent control [32], [33] in context that the controller is uncertain about the time of the plant. To reason about these problems, theoretical extensions will include time estimation from sensory data, optimal control control with different time horizons, and control with multiple noisy clocks. It will also be useful to consider alternative clock models, and practical methods for controller computation for nonlinear systems.

Time Estimation

This paper focuses on state feedback problems and the optimal solution from dynamic programming only depends on the value of the current state. If the state cannot be perfectly measured, or the clock τ_s does not have independent increments, then control performance might be improved by using the measurement history to estimate time. For example, in option pricing, inferences about the ‘‘business time’’ can be used to estimate the volatility of stock prices [34]. To perceive time, humans appear to integrate sensory cues about the passage of time in a Bayesian manner [35]; humans also appear to incorporate sensory information about the timing of events to improve state estimation [36].

It would also be interesting to test if well-chosen control strategies could improve time estimation. For example the amount that a mass moves in response to a force gives information about how much time has passed. If an improved time estimate could improve control performance, then it is likely that actions which improve timing could be helpful for control.

Variations on the Horizon

This paper studies a controller horizon $[0, S]$, which is an interval of time with respect to the clock measured by the controller. For portfolio optimization, in which the controller measures calendar time, such a horizon is sensible. In human movements, however, different tasks call for different time horizons. When reaching to an object, a natural horizon would be the stopping time describing when the object is touched. For rhythmic movements coordinated with external stimuli such as a metronome, a horizon over real time might be sensible.

Multiple Clocks

In this paper, we assumed that the plant dynamics evolve according to one clock, while the controller can measure a different clock. If the plant consists of numerous subsystems, then each could potentially evolve according to a different clock. This scenario arises in portfolio problems, in which the goal is to allocate wealth between a bank process which accrues interest at known, fixed rate and a stock process that evolves in a variable rate market [22]. Here, the bank

process may be interpreted as evolving with respect to a perfect clock, while the stock process may be viewed as evolving with respect to a noisy clock. In engineering applications, such as mobile sensor networks, multiple autonomous agents with their own clocks solve cooperative control problems. Currently, problems arising from drift between clocks are mitigated by using expensive clocks and time synchronization protocols. The work in this paper will be extended to reduce the need for precision timing and synchronization.

Alternative Clock Models

This paper modeled clocks with Lévy processes largely for mathematical convenience. It would be interesting to develop similar control frameworks based on jitter [12], [13], phase noise [37], stochastic integrals [4], or biological timing models [38].

Efficient Computation

Currently, the dynamic programming method in this paper is limited to special problems that can be solved exactly. It may be possible to solve a broader class of problems by integrating approximate dynamic programming techniques [39]–[41].

VI. CONCLUSION

This paper gives basic results on control with uncertainty in time. The technical backbone of the paper is Theorem 1 which expresses the original plant dynamics in terms of the controller's clock index. Using the new representation, the system becomes a controlled Markov process, and thus existing dynamic programming theory can be applied. Given the dynamic programming equations, time changed versions of linear quadratic control and a nonlinear portfolio problem are solved explicitly.

APPENDIX A

A TECHNICAL LEMMA FOR THEOREM 1

Lemma 3: Let τ_s be an infinite rate subordinator. Let $r_0^n = 0$ and let $r_1^n \leq r_2^n \leq \dots$ be the jump times of τ_s^n , from (26). For any sequence $S_n \rightarrow \infty$ there is a sequence $\epsilon_n \downarrow 0$ such that the following limits hold, almost surely

$$\lim_{n \rightarrow \infty} \sup \{r_i^n : r_i^n \leq S_n\} = \infty \quad (36)$$

$$\lim_{n \rightarrow \infty} \sup \{\tau_{r_i^n} : r_i^n \leq S_n\} = \infty \quad (37)$$

$$\lim_{n \rightarrow \infty} \sup \{r_{i+1}^n - r_i^n : r_i^n \leq S_n\} = 0 \quad (38)$$

$$\lim_{n \rightarrow \infty} \sup \{\tau_{r_{i+1}^n} - \tau_{r_i^n} : r_i^n \leq S_n\} = 0. \quad (39)$$

Proof: First it will be shown that for any sequence $S_n \rightarrow \infty$, a sequence $\epsilon_n \downarrow 0$ can be chosen such that (38) and (39) hold. Then it will be shown that (36) and (37) hold.

Consider (38). Using the Borel-Cantelli lemma, it is sufficient to prove that for some constant $R > 0$, and ϵ_n sufficiently small,

$$\mathbb{P} \left(\sup_{r_i^n \leq S_n} |r_{i+1}^n - r_i^n| \geq \frac{1}{2^n} \right) < \frac{R}{2^n}. \quad (40)$$

For ease of notation, the superscripts on r_i^n and the subscripts on ϵ_n and S_n will be dropped.

With probability 1, τ_s^n has only a finite number of jumps over $[0, S]$, so let $K = \max\{i : r_i \leq S\}$.

Consider (40). Define the function $g(\epsilon)$ by

$$g(\epsilon) = \int_{\epsilon}^{\infty} \lambda(dt).$$

Note that the differences are $r_{i+1} - r_i$ are exponential random variables with rate parameter $g(\epsilon)$. Thus, the event that $r_{i+1} - r_i \geq 1/2^n$ is a Bernoulli random variable with probability $p(\epsilon)$ given by

$$p(\epsilon) = \mathbb{P} \left(r_{i+1} - r_i \geq \frac{1}{2^n} \right) = g(\epsilon) \int_{1/2^n}^{\infty} e^{-g(\epsilon)x} dx = e^{-g(\epsilon)/2^n}.$$

Let J be the geometric random variable defined by

$$J = \min \left\{ i : r_{i+1} - r_i \geq \frac{1}{2^n} \right\}$$

Then the probability of J is given by

$$\mathbb{P}(J = k) = (1 - p(\epsilon))^k p(\epsilon).$$

Using the definitions of K and J , the probability in (40) may be written as

$$\mathbb{P} \left(\sup_{r_i \leq S} |r_{i+1} - r_i| \geq \frac{1}{2^n} \right) = \mathbb{P}(J \leq K).$$

Furthermore, given any constant $M > 0$,

$$\mathbb{P}(J \leq K) \leq \mathbb{P}(J \leq M) + \mathbb{P}(M \leq K). \quad (41)$$

Thus, (40) may be bounded by bounding the terms on the right of (41) separately.

Now $\mathbb{P}(M \leq K)$ will be bounded. Note that K is a Poisson random variable with parameter $Sg(\epsilon)$. Markov's inequality thus shows that

$$\mathbb{P}(M \leq K) \leq \frac{1}{M} \mathbb{E}[K] = \frac{Sg(\epsilon)}{M} \quad (42)$$

The term $\mathbb{P}(J \leq M)$ can be computed exactly as

$$\mathbb{P}(J \leq M) = p(\epsilon) \sum_{k=0}^M (1 - p(\epsilon))^k = 1 - (1 - p(\epsilon))^{M+1}.$$

Thus, (40) will hold if M can be chosen such that

$$Sg(\epsilon)/M < 1/2^n \quad \text{and} \quad 1 - (1 - p(\epsilon))^{M+1} < 1/2^n. \quad (43)$$

Rearranging terms, (43) is equivalent to

$$2^n Sg(\epsilon) < M < \frac{\log(1 - \frac{1}{2^n})}{\log(1 - p(\epsilon))} - 1.$$

Therefore, a suitable constant M exists if

$$(2^n Sg(\epsilon) + 1) \log(1 - p(\epsilon)) > \log \left(1 - \frac{1}{2^n} \right). \quad (44)$$

It will be shown that (44) holds provided that ϵ is sufficiently small. Since τ_s has infinite rate, $\lim_{\epsilon \rightarrow 0} g(\epsilon) = \infty$. Thus, the limit of the left side of (44) may be evaluated by L'Hôpital's rule:

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} (2^n Sg(\epsilon) + 1) \log(1 - p(\epsilon)) &= \lim_{g \rightarrow \infty} \frac{\log(1 - e^{-g/2^n})}{(2^n Sg + 1)^{-1}} \\ &= \lim_{g \rightarrow \infty} \frac{\frac{e^{-g/2^n}/2^n}{1 - e^{-g/2^n}}}{-2^n S(2^n Sg + 1)^{-2}} \\ &= -\frac{1}{4^n S} \lim_{g \rightarrow \infty} \frac{(2^n Sg + 1)^2}{e^{g/2^n} - 1} \\ &= 0. \end{aligned}$$

Thus, when ϵ is sufficiently small, (40) must hold.

Now consider (39). Note that $\tau_{r_{i+1}^n} - \tau_{r_i^n}$ can be expressed as

$$\tau_{r_{i+1}^n} - \tau_{r_i^n} = b(r_{i+1}^n - r_i^n) + \sum_{\substack{r_i^n < u \leq r_{i+1}^n \\ \Delta \tau_u \leq \epsilon_n}} \Delta \tau_u$$

Thus

$$\sup_{r_i^n \leq S_n} |\tau_{r_{i+1}^n} - \tau_{r_i^n}| \leq b \sup_{r_i^n \leq S_n} |r_{i+1}^n - r_i^n| + \sup_{r_i^n \leq S_n} \sum_{\substack{r_i^n < u \leq r_{i+1}^n \\ \Delta\tau_u \leq \epsilon_n}} \Delta\tau_u$$

It has already been shown that the first term on the right converges to 0 almost surely. Thus to prove (39), it suffices to prove that

$$\lim_{n \rightarrow \infty} \sup_{r_i^n \leq S_n} \sum_{\substack{r_i^n < u \leq r_{i+1}^n \\ \Delta\tau_u \leq \epsilon_n}} \Delta\tau_u = 0, \quad (45)$$

almost surely, when $\epsilon_n \downarrow 0$ sufficiently quickly. Again, by the Borel-Cantelli lemma, (45) will follow if ϵ_n is chosen such that

$$\mathbb{P} \left(\sup_{r_i^n \leq S_n} \sum_{\substack{r_i^n < u \leq r_{i+1}^n \\ \Delta\tau_u \leq \epsilon_n}} \Delta\tau_u \geq \frac{1}{2^n} \right) < \frac{R}{2^n}, \quad (46)$$

for some $R > 0$.

As before, suppress the superscripts on r_i^n and the subscripts on ϵ_n and S_n . Recall that $r_{i+1} - r_i$ are exponential random variables with rate parameter $g(\epsilon)$. Furthermore, the jump times of τ_s^n are independent of the small-jumps process

$$\tau_s - \tau_s^n = \sum_{\substack{0 \leq r \leq s \\ \Delta\tau_r \leq \epsilon}} \Delta\tau_r.$$

Define $h(\epsilon)$ by

$$h(\epsilon) = \int_0^\epsilon t \lambda(dt).$$

Let $q(\epsilon)$ be the probability that $\sum_{\substack{r_i < u \leq r_{i+1} \\ \Delta\tau_u \leq \epsilon}} \Delta\tau_u \geq \frac{1}{2^n}$. Define $\hat{q}(\epsilon)$ as the upper bound on $q(\epsilon)$ given by Markov's inequality:

$$q(\epsilon) = \mathbb{P} \left(\sum_{\substack{r_i < u \leq r_{i+1} \\ \Delta\tau_u \leq \epsilon}} \Delta\tau_u \geq \frac{1}{2^n} \right) \leq \quad (47)$$

$$2^n \mathbb{E} \left[\sum_{\substack{r_i < u \leq r_{i+1} \\ \Delta\tau_u \leq \epsilon}} \Delta\tau_u \right] = 2^n \mathbb{E} \left[\int_{r_i}^{r_{i+1}} \int_0^\epsilon t N(dr, dt) \right] = \frac{2^n h(\epsilon)}{g(\epsilon)}.$$

As in the proof of (40), the bound in (46) will be recast as a more tractable inequality.

Let L be the geometric random variable defined by

$$L = \min \left\{ i : \sum_{\substack{r_i < u \leq r_{i+1} \\ \Delta\tau_u \leq \epsilon}} \Delta\tau_u \geq \frac{1}{2^n} \right\}.$$

So L has probability given by $\mathbb{P}(L = k) = (1 - q(\epsilon))^k q(\epsilon)$. As in the proof of (40), for any constant $M > 0$,

$$\mathbb{P} \left(\sum_{\substack{r_i < u \leq r_{i+1} \\ \Delta\tau_u \leq \epsilon}} \Delta\tau_u \geq \frac{1}{2^n} \right) = \mathbb{P}(L \leq K) \leq \mathbb{P}(L \leq M) + \mathbb{P}(M \leq K).$$

The first term on the right can be bounded as

$$\mathbb{P}(L \leq M) = 1 - (1 - q(\epsilon))^{M+1} \leq 1 - (1 - \hat{q}(\epsilon))^{M+1}.$$

Furthermore, as in the proof of (40), if

$$(2^n S g(\epsilon) + 1) \log(1 - \hat{q}(\epsilon)) < \log \left(1 - \frac{1}{2^n} \right), \quad (48)$$

the constant M can be chosen such that

$$\mathbb{P}(L \leq M) + \mathbb{P}(M \leq K) \leq \frac{2}{2^n}.$$

Thus, if (48) holds, then so does (46). Note that $\hat{q}(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$, and thus $\log(1 - \hat{q}(\epsilon)) \rightarrow 0$ as well. Thus, for (48) to hold for sufficiently small ϵ , it suffices to show that $g(\epsilon) \log(1 - \hat{q}(\epsilon)) \rightarrow 0$ as $\epsilon \rightarrow 0$.

Using the power series expansion of $\log(1 - \hat{q}(\epsilon))$ implies that

$$\begin{aligned} |g(\epsilon) \log(1 - \hat{q}(\epsilon))| &= g(\epsilon) \sum_{k=1}^{\infty} \frac{\hat{q}(\epsilon)^k}{k} = g(\epsilon) \sum_{k=1}^{\infty} \frac{2^{nk} h(\epsilon)^k}{g(\epsilon)^k k} \\ &= 2^n h(\epsilon) \sum_{k=0}^{\infty} \frac{\hat{q}(\epsilon)^k}{k+1} \leq 2^n h(\epsilon) \sum_{k=0}^{\infty} \hat{q}(\epsilon)^k = \frac{2^n h(\epsilon)}{1 - \hat{q}(\epsilon)}. \end{aligned}$$

Now $\lim_{\epsilon \rightarrow 0} h(\epsilon) = 0$ implies that $\lim_{\epsilon \rightarrow 0} g(\epsilon) \log(1 - \hat{q}(\epsilon)) = 0$. Therefore (48) holds for sufficiently small ϵ and so (39) is proved.

Now (36) and (37) will be proved. As long as $S_n \rightarrow \infty$, the limit in (36) is immediate from (38) since

$$S_n - \sup\{r_i^n : r_i^n \leq S_n\} \leq \sup\{r_{i+1}^n - r_i^n : r_i^n \leq S_n\}.$$

Now (37) will be proved. If $b > 0$, then (37) follows for any sequence with $S_n \rightarrow \infty$. Thus, assume that $b = 0$. There exists $\hat{\epsilon} > 0$ such that $\lambda((\hat{\epsilon}, \infty)) > 0$. For all n such that $\epsilon_n \leq \hat{\epsilon}$, the following holds.

$$\sup\{\tau_{r_i^n} : r_i^n \leq S_n\} \geq \tau_{S_n}^n \geq \hat{\epsilon} N(S_n, (\hat{\epsilon}, \infty)).$$

Here $N(s, (\hat{\epsilon}, \infty))$ is a Poisson process with rate $\lambda((\hat{\epsilon}, \infty)) > 0$. Thus, the lower bound goes to ∞ almost surely as $S_n \rightarrow \infty$. Therefore (37) holds. ■

APPENDIX B PROOF OF LEMMA 1

First note that β is analytic at z if $\int_0^\infty (e^{zt} - 1) \lambda(dt)$ is. Furthermore, by the Lévy-Itô decomposition,

$$\mathbb{E}[e^{z\tau_s}] = e^{zbs} \mathbb{E} \left[\exp \left(z \int_0^\infty t N(s, dt) \right) \right].$$

Thus, it suffices to prove the lemma for the case that $b = 0$.

It will be shown that β is analytic. Let $y : [0, 1] \rightarrow \mathbb{C}$ be a continuous, piecewise continuously differentiable curve with $y(s) \in \text{dom}(\beta)$, with derivative $y'(s)$ continuous on all but a finite set. Since y is arbitrary, Morera's theorem implies that β is analytic, provided that

$$\oint_C \int_0^\infty (e^{yt} - 1) \lambda(dt) dy = 0. \quad (49)$$

For any $t \geq 0$, $e^{yt} - 1$ is analytic, and so $\oint_C (e^{yt} - 1) dy = 0$. Thus, (49) will hold, provided that the order of integration can be switched. This will be justified by Fubini's theorem.

Since $\frac{dy(s)}{ds} = y'(s)$ almost everywhere, Fubini's theorem can be applied if the following holds:

$$\int_0^1 \int_0^\infty |e^{y(s)t} - 1| |y'(s)| \lambda(dt) ds < \infty. \quad (50)$$

Since $y'(s)$ is continuous, except on a finite set, it follows that there is a constant M such that $|y'(s)| < M$ almost everywhere. Now $e^{y(s)t} - 1$ will be bounded. For $t \in [0, 1]$, the following bound holds

$$|e^{y(r)t} - 1| \leq \sum_{k=1}^{\infty} \frac{t^k |y(r)|^k}{k!} \leq t \left(e^{|y(r)|} - 1 \right). \quad (51)$$

Let $r = \sup\{\operatorname{Re} y(s) : s \in [0, 1]\}$. By continuity, $r < r_{\max}$. The function $e^{y(s)t} - 1$ can also be bounded using the triangle inequality:

$$\left| e^{y(r)t} - 1 \right| \leq |e^{y(s)t}| + 1 \leq e^{rt} + 1. \quad (52)$$

Thus, the integral in (50) can be bounded as

$$\begin{aligned} & \int_0^1 \int_0^\infty \left| e^{y(s)t} - 1 \right| |y'(s)| \lambda(dt) ds \leq \\ & \int_0^1 \left(\int_0^1 t \lambda(dt) \left(e^{|y(r)|} - 1 \right) + \int_1^\infty (e^{rt} + 1) \lambda(dt) \right) M ds. \end{aligned}$$

Note that the integral on the right is finite since $r < r_{\max}$. Thus, (49) holds and β is analytic.

Since β is analytic, the integral in (5) converges for any matrix A with $\operatorname{spec}(A) \subset \operatorname{dom}(\beta)$. See [21].

Now (4) will be proved. The proof is similar to the proof of Theorem 2.3.8 in [20].

First, the function t will be approximated by step functions over $(0, \infty)$. The construction is similar to the approach in the proof of Theorem 1.17 in [42]. Consider a sequence a sequence of integers $j_n \rightarrow \infty$ so that $\gamma_n = 2^{-j_n} \downarrow 0$ at a rate to be specified later. Let $k_n(t)$ be the unique integer such that $k\gamma_n \leq t < (k+1)\gamma_n$. Define the function $\varphi_n(t)$ by

$$\varphi_n(t) = \begin{cases} k_n(t)\gamma_n & t \in (0, n) \\ n & t \geq n. \end{cases}$$

Then $\varphi_n(t)$ is a simple function such that $\varphi_n(t) = 0$ for $t \in (0, \gamma_n)$, $t - \gamma_n < \varphi_n(t) \leq t$ for $t \in [\gamma_n, n]$, and $\varphi_n(t) \leq t$ for $t > 0$. The formula, (4), is a consequence of the following chain of equalities

$$\begin{aligned} & \mathbb{E} \left[\exp \left(z \int_0^\infty t N(s, dt) \right) \right] \\ &= \lim_{n \rightarrow \infty} \mathbb{E} \left[\exp \left(z \int_0^\infty \varphi_n(t) N(s, dt) \right) \right] \quad (53) \end{aligned}$$

$$= \lim_{n \rightarrow \infty} \exp \left(s \int_0^\infty (e^{z\varphi_n(t)} - 1) \lambda(dt) \right) \quad (54)$$

$$= \exp \left(s \int_0^\infty (e^{zt} - 1) \lambda(dt) \right). \quad (55)$$

The first equation is the most challenging, and will be handled last. To prove (54), note that $z\varphi_n(t)$ is a simple function. Thus, there are constants $c_i \in \mathbb{C}$ and disjoint λ -measurable sets, A_i , such that

$$z\varphi_n(t) = \sum_{i=1}^q c_i \mathbf{1}_{A_i}(t).$$

Since $\varphi_n(t) = 0$ over $(0, \gamma_n)$, it follows that 0 is not in the closure of any A_i . Thus, the integral on the right of (53) may be written as

$$\int_0^\infty z\varphi_n(t) N(s, dt) = \sum_{i=1}^q c_i N(s, A_i),$$

where $N(s, A_i)$ are independent Poisson processes with rate $\lambda(A_i)$.

Thus, the expectation on the right of (53) may be calculated as

$$\begin{aligned} & \mathbb{E} \left[\exp \left(z \int_0^\infty \varphi_n(t) N(s, dt) \right) \right] \\ &= \prod_{i=1}^q \mathbb{E} [\exp(c_i N(s, A_i))] \\ &= \prod_{i=1}^q \exp(-s\lambda(A_i)) \sum_{k=0}^\infty \frac{(s\lambda(A_i))^k}{k!} e^{c_i k} \\ &= \prod_{i=1}^q \exp(s\lambda(A_i) (e^{c_i} - 1)) \\ &= \exp \left(s \sum_{i=1}^q (e^{c_i} - 1) \lambda(A_i) \right) \\ &= \exp \left(s \int_0^\infty (e^{z\varphi_n(t)} - 1) \lambda(dt) \right). \end{aligned}$$

Thus, (54) holds.

To prove (55), note that the construction of φ_n implies that $e^{z\varphi_n(t)} - 1$ is 0 for $t < \gamma_n$ and constant for $t \geq n$. Thus, it is absolutely integrable, so by Lebesgue's dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_0^\infty (e^{z\varphi_n(t)} - 1) \lambda(dt) = \int_0^\infty (e^{zt} - 1) \lambda(dt).$$

Therefore, (55) holds.

Now (53) will be proved. First, it will be shown that

$$\lim_{n \rightarrow \infty} \int_0^\infty \varphi_n(t) N(s, dt) = \int_0^\infty t N(s, dt), \quad a.s. \quad (56)$$

Then, dominated convergence will be applied.

Assume that γ_{n-1} is fixed. The difference of the right and left of (56) may be bounded as

$$\begin{aligned} 0 &\leq \int_0^\infty (t - \varphi_n(t)) N(s, dt) \leq \quad (57) \\ & \int_0^{\gamma_{n-1}} t N(s, dt) + \gamma_n \int_{\gamma_{n-1}}^n N(s, dt) + \int_n^\infty (t - n) N(s, dt). \end{aligned}$$

To bound the first term on the right of (57), note that

$$\int_0^{\gamma_{n-1}} t N(s, dt) = \sum_{i=1}^\infty \int_{\gamma_{i+1}}^{\gamma_i} t N(s, dt) < \infty, \quad \text{almost surely.}$$

Thus, the the first term on the right of (57) may be expressed as the tail sum:

$$\int_0^{\gamma_{n-1}} t N(s, dt) = \sum_{i=n-1}^\infty \int_{\gamma_{i+1}}^{\gamma_i} t N(s, dt)$$

which converges to 0 almost surely, provided that $\gamma_n \downarrow 0$ sufficiently quickly. (See [20].)

Now consider the second term on the right of (57). For fixed γ_{n-1} , the next term γ_n may be chosen sufficiently small to give the following probability bound:

$$\begin{aligned} & \mathbb{P} \left(\gamma_n \int_{\gamma_{n-1}}^n N(s, dt) \geq 2^{-n} \right) = \\ & e^{-s\lambda([\gamma_{n-1}, n])} \sum_{k \geq \frac{1}{\gamma_n 2^n}} \frac{(s\lambda([\gamma_{n-1}, n]))^k}{k!} < \frac{1}{2^n}. \end{aligned}$$

The Borel-Cantelli lemma implies that the second term converges to 0 almost surely.

The last term on the right of (57) is 0 if $\tau_s < n$, which holds for sufficiently large n almost surely. Thus (56) holds.

Now it will be shown that Lebesgue's dominated convergence applies to (53). Let $r = \text{Re } z$. Note that the function on the right has magnitude given by

$$\left| \exp \left(z \int_0^\infty \varphi_n(t) N(s, dt) \right) \right| = \exp \left(r \int_0^\infty \varphi_n(t) N(s, dt) \right). \quad (58)$$

Thus, it suffices to show that the term on the right has finite expectation. If $r \leq 0$, then the term is bounded above by 1 and so finiteness is immediate. So, consider the case that $r > 0$.

The expectation will be bounded using monotone convergence. Note that $\varphi_n(t) \leq \varphi_{n+1}(t)$. Indeed, recall that $\gamma_n = 2^{-j_n}$ for an integer j_n . Thus $\gamma_n = 2^{j_{n+1} - j_n} \gamma_{n+1}$. By construction $k_n(t)$ satisfies $k_n(t) \gamma_n = k_n(t) 2^{j_{n+1} - j_n} \gamma_{n+1} \leq t$. It follows that $k_n(t) 2^{j_{n+1} - j_n} \leq k_{n+1}(t)$. It follows that $\varphi_n(t) \leq \varphi_{n+1}(t)$ for all $t \geq 0$.

The monotone convergence theorem now implies that (53) holds for $z = r$. Since (54) and (55) have already been proved, it follows that the magnitude from (58) has expectation bounded by $e^{s\beta(r)}$, which is finite.

Finally, (6) will be proved. Since β is analytic on $\text{dom}(\beta)$, it follows that both sides of (4) must be analytic as well. Say that $\text{spec}(A) \subset \text{dom}(\beta)$ and consider a contour C , contained in $\text{dom}(\beta)$, which encloses $\text{spec}(A)$. Using the holomorphic functional calculus (see [21]), (6) can be derived as follows:

$$\begin{aligned} \mathbb{E} \left[e^{A\tau_s} \right] &= \frac{1}{2\pi i} \oint_C \mathbb{E} \left[e^{y\tau_s} \right] (yI - A)^{-1} dy \\ &= \frac{1}{2\pi i} \oint_C e^{s\beta(y)} (yI - A)^{-1} dy \\ &= e^{s\beta(A)}. \end{aligned}$$

APPENDIX C PROOF OF LEMMA 2

Define the matrix Z by

$$Z = \begin{bmatrix} I \\ 0 \end{bmatrix} P \begin{bmatrix} I & 0 \end{bmatrix}$$

and define the matrix \tilde{A} by

$$\tilde{A} = \begin{bmatrix} A & I \\ 0 & 0 \end{bmatrix}.$$

Note that $e^{\tilde{A}t}$ is given by

$$e^{\tilde{A}t} = \begin{bmatrix} e^{At} & \int_0^t e^{Ar} dr \\ 0 & I \end{bmatrix}.$$

Thus, the matrix-valued mappings may be written as

$$\begin{aligned} \begin{bmatrix} F(P) & G(P) \\ G(P)^\top & H(P) \end{bmatrix} &= \\ b \left(\tilde{A}^\top Z + Z \tilde{A} \right) + \int_0^\infty \left(e^{\tilde{A}^\top t} Z e^{\tilde{A}t} - Z \right) \lambda(dt) \end{aligned}$$

Since $e^{\tilde{A}^\top t} \otimes e^{\tilde{A}t} = e^{\tilde{A}^\top \oplus \tilde{A}t}$, the equation may be vectorized as

$$\begin{aligned} \text{vec} \left(\begin{bmatrix} F(P) & G(P) \\ G(P)^\top & H(P) \end{bmatrix} \right) \\ &= \left(b \tilde{A}^\top \oplus \tilde{A}^\top + \int_0^\infty \left(e^{\tilde{A}^\top \oplus \tilde{A}t} - I \right) \lambda(dt) \right) \text{vec}(Z) \\ &= \beta(\tilde{A}^\top \oplus \tilde{A}^\top) \text{vec}(Z). \end{aligned}$$

Thus according to Lemma 1, F , G , and H are well defined, as long as $\text{spec}(\tilde{A}^\top \oplus \tilde{A}^\top) \subset \text{dom}(\beta)$. By construction, the spectrum is given by

$$\begin{aligned} \text{spec}(\tilde{A}^\top \oplus \tilde{A}^\top) &= \text{spec}(\tilde{A}^\top) + \text{spec}(\tilde{A}^\top) \\ &= \{0\} \cup \text{spec}(A) \cup (\text{spec}(A) + \text{spec}(A)). \end{aligned}$$

Let $r = \max\{\text{Re } \mu : \mu \in \text{spec}(A)\}$. If $r \leq 0$, then the maximum real part of any eigenvalue of $\tilde{A}^\top \oplus \tilde{A}^\top$ is 0. If $r > 0$, then the corresponding maximum real part must be $2r$. Since $\{0\} \cup \text{spec}(2A) \subset \text{dom}(\beta)$, it follows that $\text{spec}(\tilde{A}^\top \oplus \tilde{A}^\top) \subset \text{dom}(\beta)$, and so the mappings are defined.

Furthermore, the relevant expectations may be vectorized and evaluated using (6):

$$\begin{aligned} \text{vec} \left(\mathbb{E} \left[e^{\tilde{A}^\top \tau_s} Z e^{\tilde{A}^\top \tau_s} \right] \right) \\ &= \mathbb{E} \left[e^{\tilde{A}^\top \oplus \tilde{A}^\top \tau_s} \right] \text{vec}(Z) \\ &= \text{vec}(Z) + s\beta(\tilde{A}^\top \oplus \tilde{A}^\top) \text{vec}(Z) + O(s^2). \end{aligned}$$

The proof for g is similar, noting that

$$\text{vec} \left(\int_0^t e^{At} M M^\top e^{A^\top t} dt \right) = \begin{bmatrix} I & 0 \end{bmatrix} \left(e^{\hat{A}t} - I \right) \begin{bmatrix} 0 \\ I \end{bmatrix} \text{vec}(M M^\top),$$

where

$$\hat{A} = \begin{bmatrix} A \oplus A & I \\ 0 & 0 \end{bmatrix}.$$

REFERENCES

- [1] S. M. LaValle and M. B. Egerstedt, "On time: Clocks, chronometers, and open-loop control," in *IEEE Conference on Decision and Control*, 2007.
- [2] S. G. Carver, E. S. Fortune, and N. J. Cowan, "State-estimation and cooperative control with uncertain time," in *American Control Conference*, 2013.
- [3] A. Lamperski and N. J. Cowan, "Time-changed linear quadratic regulators," in *European Control Conference*, 2013.
- [4] A. E. D. Veraart and M. Winkel, "Time change," in *Encyclopedia of Quantitative Finance*. Wiley, 2010, vol. 4, pp. 1812–1816.
- [5] P. K. Clark, "A subordinated stochastic process model with finite variance for speculative prices," *Econometrica*, vol. 41, pp. 135–155, 1973.
- [6] T. Ané and H. Geman, "Order flow, transaction clock, and normality of asset returns," *The Journal of Finance*, vol. 55, no. 5, pp. 2259–2284, 2000.
- [7] P. Carr and L. Wu, "Time-changed Lévy processes and option pricing," *Journal of Financial Economics*, vol. 71, pp. 113–141, 2004.
- [8] D. M. Eagleman, "Human time perception and its illusions," *Current Opinion in Neurobiology*, vol. 18, no. 2, 2008.
- [9] M. Jazayeri and M. N. Shadlen, "Temporal context calibrates interval timing," *Nature Neuroscience*, vol. 13, no. 8, 2010.
- [10] T. E. Hudson, L. T. Maloney, and M. S. Landy, "Optimal compensation for temporal uncertainty in movement planning," *PLoS Computational Biology*, vol. 4, no. 7, 2008.
- [11] H. J. Kushner and L. Tobias, "On the stability of randomly sampled systems," *IEEE Transactions on Automatic Control*, vol. 14, no. 4, pp. 319–324, 1969.
- [12] B. Wittenmark, J. Nilsson, and M. Törngren, "Timing problems in real-time control systems," in *American Control Conference*, 1995.
- [13] J. Skaf and S. Boyd, "Analysis and synthesis of state-feedback controllers with timing jitter," *IEEE Transactions on Automatic Control*, vol. 54, no. 3, pp. 652–657, 2009.
- [14] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138–162, 2007.
- [15] M. Adès, P. E. Caines, and R. P. Malhamé, "Stochastic optimal control under poisson-distributed observations," *IEEE Transactions on Automatic Control*, vol. 45, no. 1, pp. 3–13, 2000.
- [16] N. M. Freris and P. R. Kumar, "Fundamental limits on synchronization of affine clocks in networks," in *IEEE Conference on Decision and Control*, 2007.

- [17] O. Simeone, U. Spagnolini, Y. Bar-Ness, and S. H. Strogatz, "Distributed synchronization in wireless networks," *IEEE Signal Processing Magazine*, vol. 25, no. 5, pp. 81–97, 2008.
- [18] C. Lorand and P. H. Bauer, "Stability analysis of closed-loop discrete-time systems with clock frequency drifts," in *American Control Conference*, 2003.
- [19] R. Singh and V. Gupta, "On LQR control with asynchronous clocks," in *IEEE Conference on Decision and Control and European Control Conference*, 2011.
- [20] D. Applebaum, *Lévy Processes and Stochastic Calculus*. Cambridge University Press, 2004.
- [21] N. J. Higham, *Functions of Matrices: Theory and Computation*. SIAM, 2008.
- [22] J. Cvitanić, V. Polimenis, and F. Zapatero, "Optimal portfolio allocation with higher moments," *Annals of Finance*, vol. 4, pp. 1–28, 2008.
- [23] D. B. Madan, P. P. Carr, and E. C. Chang, "The variance gamma process and option pricing," *European Finance Review*, vol. 2, pp. 79–105, 1998.
- [24] O. E. Barndorff-Nielsen, "Processes of normal inverse Gaussian type," *Finance and Stochastics*, vol. 2, pp. 41–68, 1998.
- [25] J. R. Michael, W. R. Schucany, and R. W. Haas, "Generating random variates using transformations with multiple roots," *The American Statistician*, vol. 30, no. 2, pp. 88–90, 1976.
- [26] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, 2nd ed. Springer, 2006.
- [27] K. Sato, *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, 1999.
- [28] P. E. Protter, *Stochastic Integration and Differential Equations*, 2nd ed. Springer, 2004.
- [29] E. Todorov and M. I. Jordan, "Optimal feedback control as a theory of motor coordination," *Nature Neuroscience*, vol. 5, no. 11, 2002.
- [30] C. Harris and D. Wolpert, "Signal-dependent noise determines motor planning," *Nature*, vol. 394, no. 6695, pp. 780–784, 1998.
- [31] L. C. G. Rogers, *Optimal Investment*. Springer, 2013.
- [32] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Transactions on Automatic Control*, vol. 48, pp. 988–1001, 2003.
- [33] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, pp. 1520–1533, 2004.
- [34] H. Geman, D. B. Madan, and M. Yor, "Stochastic volatility, jumps and hidden time changes," *Finance and Stochastics*, vol. 6, pp. 63–90, 2002.
- [35] M. B. Ahrens and M. Sahani, "Observers exploit stochastic models of sensory change to help judge the passage of time," *Current Biology*, vol. 21, pp. 1–7, 2011.
- [36] M. M. Ankaralı, H. T. Şen, A. De, A. M. Okamura, and N. J. Cowan, "Haptic feedback enhances rhythmic motor control performance by reducing variability, not convergence time," *Journal of Neurophysiology*, 2014, in press.
- [37] A. Hajimiri and T. H. Lee, "A general theory of phase noise in electrical oscillators," *IEEE Journal of Solid-State Circuits*, vol. 33, no. 2, pp. 179 – 194, 1998.
- [38] S. Grondin, "Timing and time perception: A review of recent behavioral and neuroscience findings and theoretical directions," *Attention, Perception, & Psychophysics*, vol. 72, no. 3, pp. 561–582, 2010.
- [39] E. Todorov and W. Li, "A generalized iterative lqg method for locally-optimal feedback control of constrained nonlinear systems," in *American Control Conference*, 2005.
- [40] E. Theodorou, J. Büchli, and S. Schaal, "Reinforcement learning of motor skills in high dimensions: A path integral approach," in *IEEE International Conference on Robotics and Automation*, 2010.
- [41] J. B. Lasserre, D. Henrion, C. Prieur, and E. Trélat, "Nonlinear optimal control via occupation measures and LMI-relaxations," *SIAM Journal of Control and Optimization*, vol. 47, no. 4, pp. 1643 – 1666, 2008.
- [42] W. Rudin, *Real and Complex Analysis*. WCB/McGraw-Hill, 1987.