EE5585 Data Compression	March 28, 2013

Lecture 17

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Solution to Assignment 2

Problem 4 Let X be zero mean and σ^2 variance Gaussian random variable. That is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

and let the distortion measure be squared error. Find the optimum reproduction points for 1-bit scalar quantization and the expected distortion for 1-bit quantization. Compare this with the rate-distortion function of a Gaussian random variable.

Solution Let x be the source and $\hat{x}$ be quantizer output. Our goal is to find optimal reproduction points -a and a (**figure 1**) such that the distortion is minimize in terms of MSE. i.e. minimize $D = E[(x - \hat{x})^2]$.

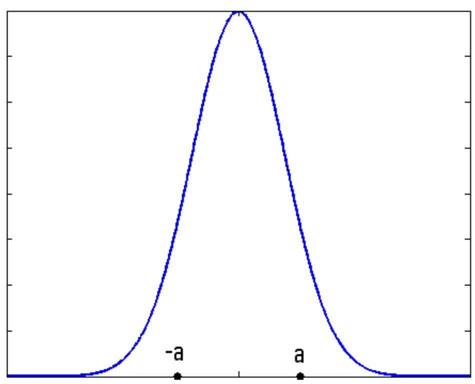

Figure 1: PDF of problem 4 with quantization level **a** and **-a**

$$\begin{aligned}
D = E[(x-\hat{x})^2] &= \int_{-\infty}^{0} f_X(x)(x+a)^2 dx + \int_{0}^{\infty} f_X(x)(x-a)^2 dx \\
&= 2\int_{0}^{\infty} f_X(x)(x-a)^2 dx \\
&= 2\left[\int_{0}^{\infty} a^2 f_X(x)dx + \int_{0}^{\infty} x^2 f_X(x)dx - 2a\int_{0}^{\infty} x f_X(x)dx\right] \\
&= a^2 + \sigma^2 - \frac{4a\sigma^2}{\sqrt{2\pi\sigma^2}}\int_{0}^{\infty} e^{-y}dy \quad (\text{Let } y = -\frac{x^2}{2\sigma^2}) \\
&= a^2 + \sigma^2 - \frac{4a\sigma^2}{\sqrt{2\pi\sigma^2}}
\end{aligned}$$

Now we take partial differentiation of $E[(x-\hat{x})^2]$ with respect to a and set it to zero,

$$\frac{\partial E[(x-\hat{x})^2]}{\partial a} = 2a - \frac{4\sigma^2}{\sqrt{2\pi\sigma^2}} = 0$$
$$\Rightarrow a^* = \sigma\sqrt{\frac{2}{\pi}}$$

So **two optimum reproduction points are** $a^* = \sigma\sqrt{\frac{2}{\pi}}$ and $-a^* = -\sigma\sqrt{\frac{2}{\pi}}$. Substitute $a^* = \sigma\sqrt{\frac{2}{\pi}}$ back to expectd distortion expression we just found, and the **expected distortion**

$$D = E[(x-\hat{x})^2] = \sigma^2\frac{2}{\pi} + \sigma^2 - 4\sigma^2\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}$$
$$= \frac{\pi-2}{\pi}\sigma^2$$

We know that binary rate-distortion function of a Gaussian random variable is

$$R(D) = \frac{1}{2}\log_2\frac{\sigma^2}{D}$$
$$\Rightarrow D_{opt} = \frac{\sigma^2}{2^{2R(D)}}$$

In our case rate $R = 1$, so that $D_{opt} = \frac{\sigma^2}{4} < \frac{\pi-2}{\pi}\sigma^2$, **which means the distortion rate bound is smaller than the distortion in our case**.

Problem 6 Let $\{X\}_{i=o}^{\infty}$ be an i.i.d binary sequence with probability of 1 being 0.3. Calculate $F(01110) = P(0.X_1X_2X_3X_4X_5 < 0.01110)$. How many bits of $F = 0.X_1X_2X_3X_4X_5....$ can be known for sure if it is not known how the sequence 0.01110... continues?

Solution

$$\begin{aligned} F(01110) &= Pr(0.X_1X_2X_3X_4X_5 < 0.01110) \\ &= Pr(X_1 = 0, X_2 < 1) + Pr(X_1 = 0, X_2 = 1, X_3 < 1) + Pr(X_1 = 0, X_2 = 1, X_3 = 1, X_4 < 1) \\ &= 0.7^2 + 0.7^2 \cdot 0.3 + 0.7^2 \cdot 0.3^2 \\ &= 0.6811 \end{aligned}$$

From the source, we have only observed the first five bits, 01110 ,which can be continued with an arbitary sequence. However for an arbitary sequence that starts with the sequence 01110 we have

$$F(01110000\dots) \le F(01110X_6X_7X_8\dots) \le F(01110111\dots).$$

We know that

$$F(01110000\dots) = F(01110) = (0.6811)_{10} = (0.101011100101...)_2$$

However, we also know that $F(01110111\dots) = F(01111)$.

$$\begin{aligned} F(01111) &= Pr(0.X_1X_2X_3X_4X_5 < 0.01111) \\ &= 0.7^2 + 0.7^2 \cdot 0.3 + 0.7^2 \cdot 0.3^2 + 0.7^2 \cdot 0.3^3 \\ &= (0.69433)_{10} = (0.101100011011...)_2 \end{aligned}$$

So comparing binary representation of $F(01110)$ and $F(01111)$ we observe that **we are sure about 3 bits: 101**.

Problem 3 Consider the following compression scheme for binary sequences. We divide the binary sequences into blocks of size 16. For each block if the number of zeros is greater than or equal to 8 we store a 0, otherwise we store a 1. If the sequence is random with probability of zero 0.9, compute ther rate and average distortion (Hamming metric). Compare your result with the corresponding value of rate distortion function for binary sources.

Solution It is easy to see that **rate** $= \mathbf{R(D)} = \mathbf{\frac{1}{16}}$. We can find the optimum distortion D_{opt} when the rate is $\frac{1}{16}$ according to binary rate-distortion function,

$$R(D) = h(p) - h(D_{opt})$$

where h(.) is binary entropy function and p is the source probability of zero.

$$\begin{aligned} D_{opt} &= h^{-1}\left[h(p) - R(D)\right] \\ \Rightarrow D_{opt} &= h^{-1}\left[h(0.1) - \frac{1}{16}\right] \\ \Rightarrow D_{opt} &\approx 0.08 \end{aligned}$$

In our source compression scheme, the distortion is summarized in Table 1. Assume that two quantization levels are 000... and 111... respectively

Number of 0's in block	Encoding	Probability	Distortion
0	1	$\binom{16}{0}(0.1)^{16}$	0
1	1	$\binom{16}{1}(0.1)^{15}\cdot(0.9)^{1}$	1
2	1	$\binom{16}{2}(0.1)^{14}\cdot(0.9)^{2}$	2
3	1	$\binom{16}{3}(0.1)^{13}\cdot(0.9)^{3}$	3
4	1	$\binom{16}{4}(0.1)^{12}\cdot(0.9)^{4}$	4
5	1	$\binom{16}{5}(0.1)^{11}\cdot(0.9)^{5}$	5
6	1	$\binom{16}{6}(0.1)^{10}\cdot(0.9)^{6}$	6
7	1	$\binom{16}{7}(0.1)^{9}\cdot(0.9)^{7}$	7
8	0	$\binom{16}{8}(0.1)^{8}\cdot(0.9)^{8}$	8
9	0	$\binom{16}{9}(0.1)^{7}\cdot(0.9)^{9}$	7
10	0	$\binom{16}{10}(0.1)^{6}\cdot(0.9)^{10}$	6
11	0	$\binom{16}{11}(0.1)^{5}\cdot(0.9)^{11}$	5
12	0	$\binom{16}{12}(0.1)^{4}\cdot(0.9)^{12}$	4
13	0	$\binom{16}{13}(0.1)^{3}\cdot(0.9)^{13}$	3
14	0	$\binom{16}{14}(0.1)^{2}\cdot(0.9)^{14}$	2
15	0	$\binom{16}{15}(0.1)^{1}\cdot(0.9)^{15}$	1
16	0	$\binom{16}{16}(0.9)^{16}$	0

Table 1: Distortion Summary

Refer to the table we can compute the **average distortion**,

$$D_{ave} = E[d(x,\hat{x})] = \sum_{i=0}^{16} Pr(\text{Number of 0's} = i) \cdot d(x_i, \hat{x}_i) = 1.6$$

So the normalized distortion $D_{norm} = \frac{D_{ave}}{16} = 0.1$, **which is larger than D_{opt} we found before.**

Problem 1 Design a 3-bit uniform quantizer(by specifying the decision boundaries and reconstruction levels) for a source with following pdf:

$$f_X(x) = \frac{1}{6} e^{-\frac{|x|}{3}}$$

Solution Let Δ be decision boundary and function $Q(.)$ be uniform quantizer. Since it is a 3-bit uniform quantizer, we need $2^3 = 8$ reconstruction levels as following (Note we only consider positive region here because the distribution is symmetric):

$$Q(x) = \begin{cases} \frac{\Delta}{2} & 0 \leq x \leq \Delta \\ \frac{3\Delta}{2} & \Delta < x \leq 2\Delta \\ \frac{5\Delta}{2} & 2\Delta < x \leq 3\Delta \\ \frac{7\Delta}{2} & 3\Delta < x < \infty \end{cases}$$

Therefore, the expression of Mean Squared Quantization Error becomes

$$\begin{aligned} \sigma_q^2 = E\left[(X - Q(x))^2\right] &= \int_{-\infty}^{\infty} (x - Q(x))^2 f_X(x)dx \\ &= 2\sum_{i=1}^{i=3} \int_{(i-1)\Delta}^{i\Delta} \left(x - \frac{2i-1}{2}\Delta\right)^2 f_X(x)dx + 2\int_{3\Delta}^{\infty} \left(x - \frac{7\Delta}{2}\right)^2 f_X(x)dx \end{aligned}$$

To find the optimal value of Δ. we simply take a derivative of this equation and set it equal to zero,

$$\frac{\partial \sigma_q^2}{\partial \Delta} = -\sum_{i=1}^{i=3}(2i-1)\int_{(i-1)\Delta}^{i\Delta} \left(x - \frac{2i-1}{2}\Delta\right) f_X(x)dx - 7\int_{3\Delta}^{\infty} \left(x - \frac{7\Delta}{2}\right)^2 f_X(x)dx = 0$$

Substitute $f_X(x) = \frac{1}{6}e^{-\frac{|x|}{3}}$ into above equation. After some calculus and algebra, we get

$$\boldsymbol{\Delta \approx 3.101}$$

The optimuml 3-bit uniform quantizer shows in **figure 2**

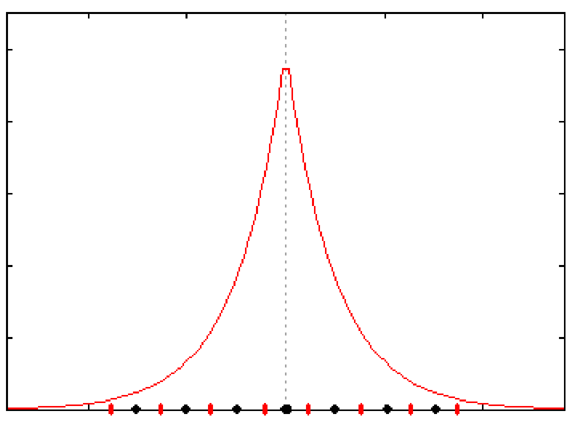

Figure 2: Black dots represent optimuml quantization levels spaced by $\Delta \approx 0.3101$. Red dots represent optimum reconstruction levels, which are set in the middle of two adjacent quantization levels

Problem 2 What is the mean and variance of the random variable of Problem 1 above? Derive the mean of the output of uniform quantizer you designed for the above problem. What is the mean of the optimum quantizer for this distribution?

Solution

$$\begin{aligned}
\textbf{Mean:} \quad & E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{6} e^{-\frac{|x|}{3}} dx = 0 \\
\textbf{Variance:} \quad & E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = 2\int_{0}^{\infty} x^2 \cdot \frac{1}{6} e^{-\frac{x}{3}} dx = 18 \\
\textbf{Optimum Quantizer Mean:} \quad & E[Q(x)] = E[X] = 0
\end{aligned}$$

The **mean of the output of designed uniform quantizer** is

$$\begin{aligned}
E[Q(x)] &= \sum_{i=-4}^{i=4} Pr\left(Q(x) = \frac{2i-1}{2}\Delta\right)\left(\frac{2i-1}{2}\Delta\right) \\
&= 0 \quad \text{(since it is a odd function)}
\end{aligned}$$

where $Pr\left(Q(x) = \frac{2i-1}{2}\Delta\right) = \int_{(i-1)\Delta}^{i\Delta} f_X(x)dx$

Problem 5 Consider a source X uniformly distributed on the set {1,2,...m}. Find the rate distortion function for this source with Hamming distance; that is, $\mathrm{d}(x,\hat{x}) = 0$ when $x = \hat{x}$ and $\mathrm{d}(x,\hat{x}) = 1$ when $x \neq \hat{x}$.

Solution Solution will be given in the next class.

1 Kolmogorov Complexity

Let us briefly talk about the concept of *Kolmogorov complexity*. First, we should ask ourself "what is a computer?". Generally the computer is a *Finite State Machine(FSM)*, consisting of read tape, output tape and work tape. (**figure 3**). This is called a Turing machine (Alan Turing).

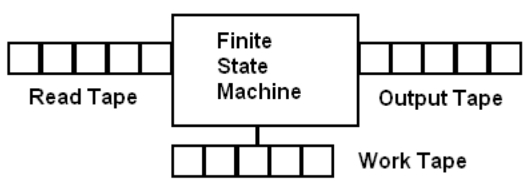

Figure 3: Illustration of a simple Finite State Machine

A Turing machine is able to simulate any other computer (*Church-Turing Thesis*). Any real-world computation can be translated into an equivalent computation involving a Turing machine.

A sequence 010101.... ("01" repeat 10000 times) can be translated to the sentence "Repeat "01" 10000 times". Another example, a sequence 141592... can be translated to the sentence "First 6 digits after the point in π."

Now let us define Kolmogorov complexity. For any sequence x

$$K_U(x) = \min_{p:U(p)=x} l(p)$$

where p is a computer program that computes x and halts, U is a computer.

Instructor note: Incomplete. Please consult your notes.