ROBUSTNESS OF FEEDBACK SYSTEMS

Tryphon Georgiou
Electrical & Computer Engineering
University of Minnesota

Joint work with Malcolm C. Smith (University of Cambridge)
and with Ian Fialho (Boeing), and S. Varigonda (UTRC)

○ WHAT KINDS OF PERTURBATIONS SHOULD A FEEDBACK SYSTEM TOLERATE?

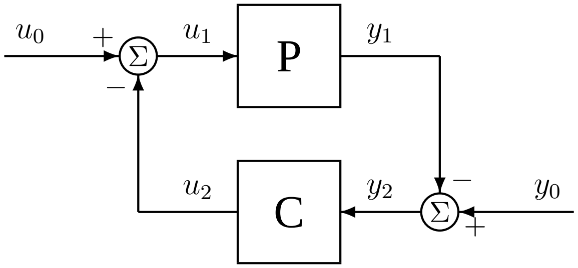
• Uncertainty in “coefficients”, small time-delays, changes in model order, dynamics…

• Lessons from adaptive control:
Parameter “drift”, high-gain instability, high-frequency instability, adverse effects of “fast” adaptation, sufficient excitation, etc.

What is causing problems?

Can the feedback system tolerate such uncertainties?

• Lessons from the linear theory.



• THE NUSSBAUM UNIVERSAL ADAPTIVE CONTROLLER:

$$\text{Plant } \mathbf{P}: \quad \dot{x}(t) = ax(t) + bu_1(t), \quad y_1(t) = x(t)$$

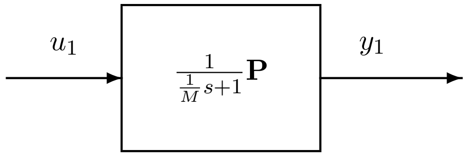
$$\text{Controller } \mathbf{C}: \quad u_2(t) = x(t)\theta^2(t)\cos(\theta(t)), \quad \dot{\theta}(t) = y_2^2(t)$$

○ For $u_0 = y_0 = 0$, and any $x(0), a, b$: $x(t) \to 0$.

○ However, [P,C] is I/O UNSTABLE!
The input-to-error map is $\mathcal{L}_\infty$-unbounded:
For $a = 0$, $b = 1$, $u_0 = \epsilon \neq 0$,

$$\begin{aligned} \dot{x}(t) &= x(t)\theta^2(t)\cos(\theta(t)) + \epsilon \\ \dot{\theta}(t) &= x^2(t). \end{aligned}$$

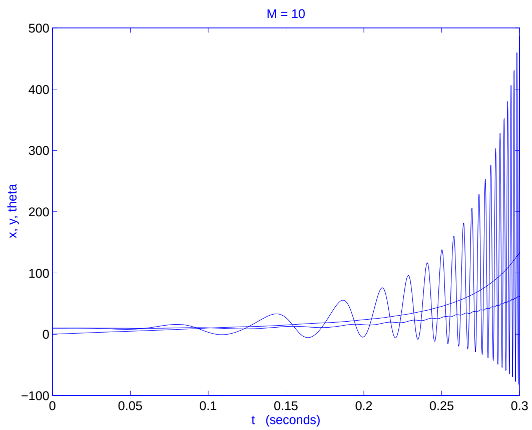
If $\theta(t) <$ bound, then $x(t) \in L_2$ and $\dot{x}(t) - \epsilon \in L_2$, which cannot happen.
Then $\theta(t) \to \infty$ and then it follows that $x(t) \to \infty$ as well.



With a perturbed plant $\mathbf{P}_1$:

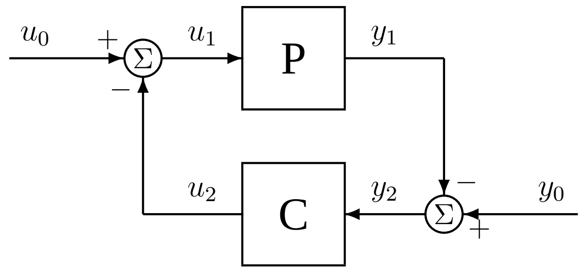
$$\begin{aligned} \dot{x}(t) &= u_1(t) \\ \dot{y}(t) &= M(x(t) - y(t)) \\ y_1(t) &= y(t), \end{aligned}$$

even the autonomous system is unstable (x, y, θ vs. t):



$$dx/d\theta = \theta^2 \cos\theta / y,\ dy/d\theta = M(x-y)/y^2.$$

$$\begin{aligned} \dot{\theta} &\sim \theta^{3/2} \\ x(\theta) &= \theta^{5/4}(a_1 \sin(\theta) + a_2) + \theta^{3/4}(a_6 \sin(\theta) + a_7) + \dots \\ y(\theta) &= \theta^{3/4} a_3 + \theta^{1/4} a_5 + \theta^{-1/4}(a_4 \cos(\theta) + a_8) + \dots \end{aligned}$$



- MODEL REFERENCE ADAPTIVE CONTROLLER:

$$\text{Plant } \mathbf{P}: \quad \dot{x}(t) = ax(t) + bu_1(t)$$
$$y_1(t) = x(t)$$

$$\text{Controller } \mathbf{C}: \quad u_2(t) = -\theta(t)y_2(t)$$
$$\dot{\theta}(t) = \gamma y_2^2(t)$$

where γ chosen so that $\gamma b > 0$.

For $u_0 = y_0 = 0$, globaly stable, $V(x, \theta) := x^2 + b(\theta - a/b)^2/\gamma$ a Lyapunov function,
but [P,C] is again I/O UNSTABLE!

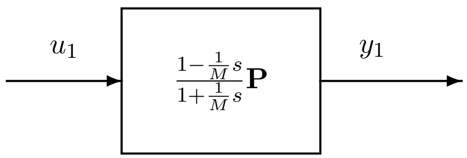
◦ The input-to-error map is $\mathcal{L}_\infty$-unbounded:

For $a = 0$, $b = 1$, $\gamma = 1$, and $u_0 = \epsilon > 0$, and $y_0 \equiv 0$:

$$\begin{aligned} \dot{x} &= \epsilon - \theta x, \\ \dot{\theta} &= x^2. \end{aligned}$$

Then $x(t) \to 0$ while $\theta(t) \to \infty$.

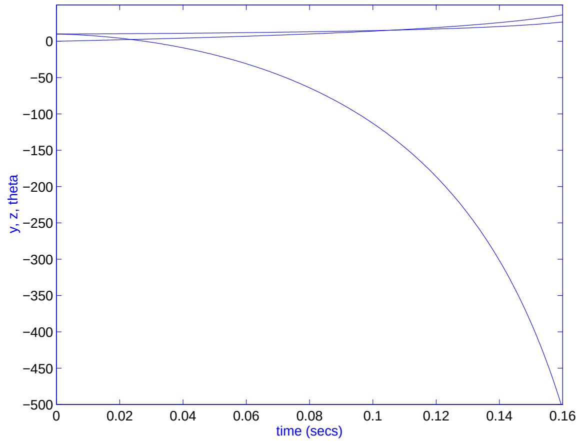
Thus, $u_0 \equiv \epsilon$ on $[0, T]$, $y_0 \equiv 0$ on $[0, T)$ and $y_0(T) = \epsilon$
give $u_1(T) = \epsilon + \theta(T)(\epsilon - x(T))$ — arbitrarily large.



With a perturbed plant $\mathbf{P}_1$:

$$\begin{aligned}\dot{y} &= z + \theta y,\\ \dot{z} &= -M(z + 2\theta y),\\ \dot{\theta} &= y^2.\end{aligned}$$

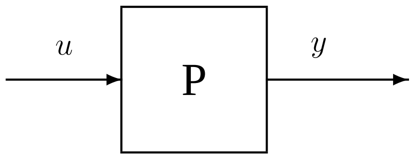
the autonomous system with the MRAC $\mathbf{C}$ gives (y, z, θ versus t with $M = 10$):

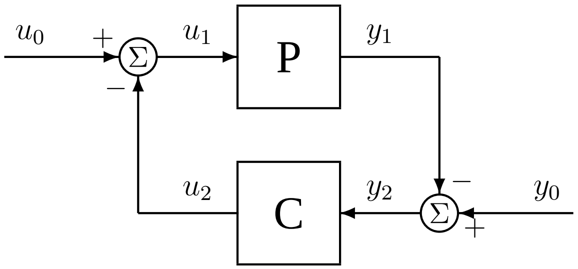


$dy/d\theta = (z + \theta y)/y^2$, $dz/d\theta = -M(z + 2\theta y)/y^2$

$$\begin{aligned}\dot{\theta} &\sim \theta^2\\ y(\theta) &= a_1\theta + a_0 + \theta^{-1}(a_{-1,2}(\log\theta)^2 + a_{-1,1}\log\theta + a_{-1,0})\ldots,\\ z(\theta) &= b_1\theta + b\log\theta + b_0 + \ldots.\end{aligned}$$

- GRAPH OF P: $\mathcal{G}_P = \{$ Collection of input/output pairs$\}$





- NOTION OF DISTANCE:

  The Gap $\delta(\mathbf{P}_1, \mathbf{P}_2) =$ opening between graphs.

- THEOREM:

  $[\mathbf{P}, \mathbf{C}]$ and $[\mathbf{P}_1, \mathbf{C}]$ behave similarly $\Leftrightarrow \delta(\mathbf{P}, \mathbf{P}_1)$ is small enough.

- THEOREM:

  If $\delta(\mathbf{P}, \mathbf{P}_1) <$ robustness margin , then $[\mathbf{P}_1, \mathbf{C}]$ is stable.

- COMPUTATION: ($\mathcal{L}_2$-SIGNALS)

  for gap, margins, and optimal controllers relies on $\mathcal{H}_\infty$-theory.

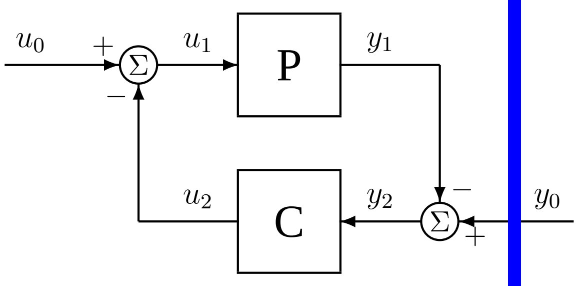
- CONTROLLER DESIGN:

  Glover-McFarlane loopshaping/Weighted-gap optimization.

- FRAMEWORK FOR ROBUSTNESS ANALYSIS OF NONLINEAR SYSTEMS

- UNCERTAINTY:
  - NOT TIED TO A PARTICULAR REPRESENTATION
  - ALLOW FOR UNSTABLE, DISTRIBUTED PARAMETER, ETC. SYSTEMS

- IS THERE A NATURAL METRIC TOPOLOGY?
  - SO THAT, "CLOSENESS OF MODELS" $\sim$ "SIMILAR RESPONSE"?
  - ROBUSTNESS $\sim$ CLOSED-LOOP STABILITY?



○ FEEDBACK STABILITY:
For any (u_0, y_0) there exist unique signals $u_1, u_2 \in \mathcal{U}$ and $y_1, y_2 \in \mathcal{Y}$:

$$\begin{aligned} u_0 &= u_1 + u_2, \\ y_0 &= y_1 + y_2, \quad \text{with} \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} \in \mathcal{G}_{\mathrm{P}}, \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \in \mathcal{G}_{\mathrm{C}}. \end{aligned}$$

○ CLOSED-LOOP MAPPINGS: For $\mathcal{M} := \mathcal{G}_{\mathrm{P}}$, $\mathcal{N} := \mathcal{G}_{\mathrm{C}}$, $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$

$$\begin{aligned} \boldsymbol{\Sigma}_{\mathcal{M},\mathcal{N}} &: \mathcal{M} \times \mathcal{N} \to \mathcal{W} : \left(\begin{pmatrix} u_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \right) \to \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \\ \mathbf{H}_{\mathrm{P},\mathrm{C}} &: \mathcal{W} \to \mathcal{M} \times \mathcal{N} : \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \to \left(\begin{pmatrix} u_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \right) \\ \boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} &: \mathcal{W} \to \mathcal{M} : \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \to \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} \end{aligned}$$

○ PARALLEL PROJECTION: $\boldsymbol{\Pi}(\boldsymbol{\Pi} w_1 + (\mathbf{I} - \boldsymbol{\Pi}) w_2) = \boldsymbol{\Pi} w_1$.

◦ PERFORMANCE/DEGREE OF STABILITY:
Quantified using norms, gains of closed-loop mappings:

$$\begin{aligned} \|\mathbf{F}|_{\mathcal{X}}\| &:= \sup_{\substack{x\in\mathcal{X},\tau>0 \\ \|x\|_\tau\neq 0}} \frac{\|\mathbf{F}x\|_\tau}{\|x\|_\tau} \\ \|\mathbf{F}|_{\mathcal{X}}\|_\Delta &:= \sup_{\substack{x_1,x_2\in\mathcal{X},\tau>0 \\ \|x_1-x_2\|_\tau\neq 0}} \frac{\|\mathbf{F}x_1-\mathbf{F}x_2\|_\tau}{\|x_1-x_2\|_\tau} \\ g[\mathbf{F}](\alpha) &:= \sup_{x\in\mathcal{X}_1,\ \tau>0,\ \|x\|_\tau\leq\alpha} \|\mathbf{F}x\|_\tau . \end{aligned}$$

∘ NONLINEAR GAP:

$$\vec{\delta}(\mathcal{X},\mathcal{Y}) := \begin{cases} \inf\{\|(\mathbf{\Phi}-\mathbf{I})|_{\mathcal{X}}\| : \mathbf{\Phi} \text{ bijective, from} \mathcal{X} \text{ onto } \mathcal{Y},\ \mathbf{\Phi}0 = 0\}, \\ \infty \text{ if no such operator } \mathbf{\Phi} \text{ exists,} \end{cases}$$

$$\delta(\mathcal{X},\mathcal{Y}) := \max\left\{\vec{\delta}(\mathcal{X},\mathcal{Y}), \vec{\delta}(\mathcal{Y},\mathcal{X})\right\}.$$

∘ THEOREM: The gap defines a metric topology: $d(\cdot,\cdot) := \log(1+\delta(\cdot,\cdot))$ is metric

∘ THEOREM: If a system $\mathbf{P}_1$ is such that

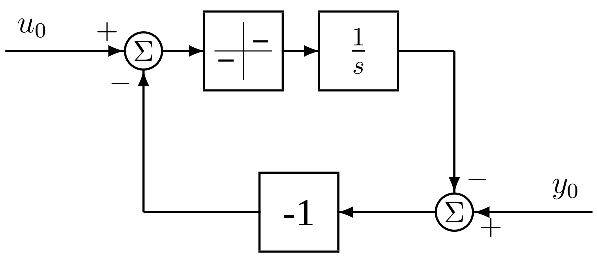
$$\vec{\delta}(\mathcal{G}_{\mathrm{P}}, \mathcal{G}_{\mathrm{P}_1}) < \|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}\|^{-1},$$

then $\mathbf{H}_{\mathrm{P}_1,\mathrm{C}}$ is stable and

$$\|\mathbf{\Pi}_{\mathcal{M}_1//\mathcal{N}}\| \leq \|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}\| \frac{1+\vec{\delta}(\mathcal{G}_{\mathrm{P}}, \mathcal{G}_{\mathrm{P}_1})}{1-\|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}\|\vec{\delta}(\mathcal{G}_{\mathrm{P}}, \mathcal{G}_{\mathrm{P}_1})}.$$

∘ THEOREM: Open-loop uncertainties which correspond to small closed-loop errors are precisely those that are small in the gap:

$$\{\|\mathbf{H}_{\mathrm{P},\mathrm{C}} - \mathbf{H}_{\mathrm{P}_i,\mathrm{C}}\| \to 0\} \Leftrightarrow \{\delta(\mathcal{G}_{\mathrm{P}}, \mathcal{G}_{\mathrm{P}_i}) \to 0\}.$$



- ESTIMATION OF GAP BETWEEN NOMINAL $\mathbf{P}$:

$$\begin{aligned} \dot{x}(t) &= \text{sat}(u_1(t)), \quad x(0) = 0, \\ y_1(t) &= x(t), \end{aligned}$$

AND PERTURBED $\mathbf{P}_1$:

$$\dot{x}(t) = \text{sat}(u(t-h)), \quad x(0) = 0.$$

Using $\mathbf{\Phi} \begin{pmatrix} u(t) \\ x(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ x(t-h) \end{pmatrix}$, follows that

$$\vec{\delta}(\mathcal{G}_{\text{P}}, \mathcal{G}_{\text{P}_1}) \leq \|\mathbf{I} - \mathbf{\Phi}\| = h.$$

(It can be shown that in fact $\vec{\delta}(\mathcal{G}_{\text{P}}, \mathcal{G}_{\text{P}_1}) = h$.)

- ROBUSTNESS MARGIN: $\|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}\|^{-1} = 0.25$, predicting $\mathcal{L}_\infty$-induced norm stability for $h < 0.25$.

- STABILITY ON BOUNDED SETS

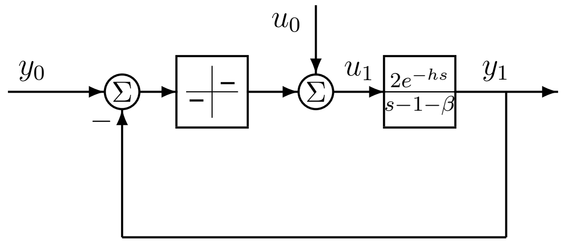
- GAP:

$$\vec{\delta}_{\mathcal{S}_r}(\mathcal{G}_{\mathrm{P}}, \mathcal{G}_{\mathrm{P}_1}) := \begin{cases} \inf \{ \|(\mathbf{\Phi} - \mathbf{I})|_{\mathcal{M}\cap\mathcal{S}_r}\| : \mathbf{\Phi} \text{ maps } \mathcal{G}_{\mathrm{P}} \to \mathcal{G}_{\mathrm{P}_1}, \mathbf{\Phi}0 = 0 \}, \\ \infty \text{ if no such operator } \mathbf{\Phi} \text{ exists.} \end{cases}$$

where $\mathcal{S}_r := \{w \in \mathcal{W} : \sup_\tau \|w\|_\tau < r\}$.

- ROBUSTNESS MARGIN:
  Let $\|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}|_{\mathcal{S}_r}\| = \alpha$, $\vec{\delta}_{\mathcal{S}_{\alpha r}}(\mathcal{G}_{\mathrm{P}}, \mathcal{G}_{\mathrm{P}_1}) = \gamma$.
  If $\gamma < \alpha^{-1}$, then $\mathbf{H}_{\mathrm{P}_1,\mathrm{C}}$ is bounded on $\mathcal{S}_{r(1-\alpha\gamma)}$ and

$$\|\mathbf{\Pi}_{\mathcal{M}_1//\mathcal{N}}|_{\mathcal{S}_{r(1-\alpha\gamma)}}\| \leq \frac{\alpha(1+\gamma)}{1-\alpha\gamma}.$$



- ROBUST STABILIZATION OF AN UNSTABLE LINEAR SYSTEM WITH SATURATION, OVER A BOUNDED SET OF DISTURBANCES:

$$\dot{x}(t) = (1+\beta)x(t) + 2u_1(t-h), \quad y_1(t) = x(t) \text{ with nominal } \beta = h = 0.$$

○ NORM OF PARALLEL PROJECTION: Over a maximal radius $r = 1/3$

$$\left\| \left(\begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \to \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} \right) \Big|_{\mathcal{S}_{1/3}} \right\| = 6.$$

○ GAP ESTIMATE ($\mathcal{L}_\infty$-gap between $\mathbf{P}$ and $\mathbf{P}_1$ with $\beta \neq 0$, $h \neq 0$):

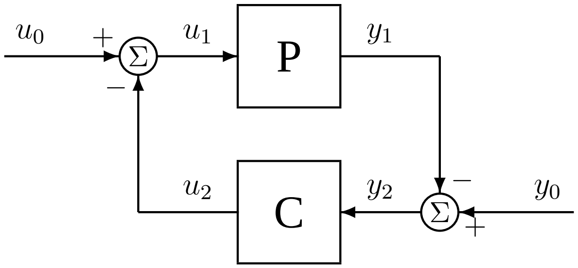
$$\mathbf{\Phi} := \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} (V,\ U)$$

where $(V,\ U) := (1,\ 1)$, $\begin{pmatrix} M_1 \\ N_1 \end{pmatrix} := \begin{pmatrix} \frac{s-1-\beta}{s+1} \\ \frac{2e^{-hs}}{s+1} \end{pmatrix}$, gives

$$\| (\mathbf{I} - \mathbf{\Phi})|_{\mathcal{G}_\mathrm{P}} \| =: \gamma = \max\{2\beta, 8(1-e^{-h})\}.$$

○ ROBUSTNESS MARGIN:

The perturbed system is stable on $\mathcal{S}_{\frac{1}{3}(1-6\gamma)}$ provided that $\gamma = \max\{2\beta, 8(1-e^{-h})\} < 1/6$.



- SYSTEMS WITH POTENTIAL FOR FINITE-TIME ESCAPE.
  MOTIVATING EXAMPLE:

$$\begin{aligned} \dot{x}(t) &= x^2(t) + u_1(t), \quad \text{with } x(0) = 0, \\ y_1(t) &= x(t). \end{aligned}$$

◦ ROBUST STABILITY CAN ONLY BE LOCAL:

$$\begin{aligned} \dot{x}(t) &= x^2(t) + u_0(t-\tau) - u_2(t-\tau), \\ u_2(t) &= \mathbf{C}(y_0(t) - x(t)). \end{aligned}$$

is not globally stabilizable by any causal controller.

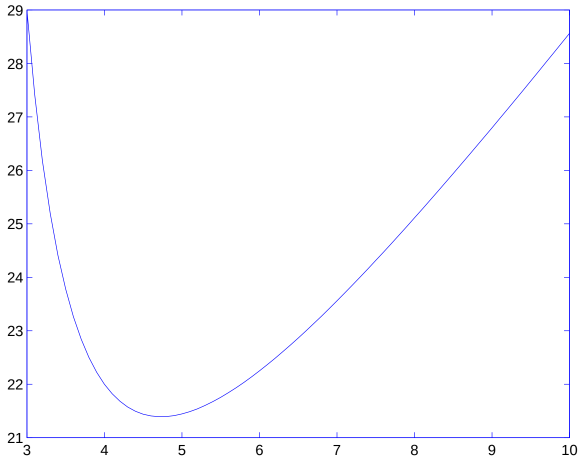
◦ ROBUSTNESS ANALYSIS FOR **C**: $u_2(t) = y_2^2(t) - ky_2(t)$.

$\cdots$

○ PARALLEL PROJECTION: we compute analytically

$$\begin{aligned}\|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}|_{\mathcal{S}_r}\| &= \max\left\{\frac{1+k-r}{k-2r}, 1+\frac{(1+2k-3r)(r+k^2-3r^2)}{(k-2r)^2}\right\} \\ &=: f(r,k)\end{aligned}$$

• As $k \downarrow 2r$ and as $k \uparrow \infty$, $f(r,k) \to \infty$.



$\|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}|_{\mathcal{S}_1}\| = f(1,k)$ vs. k

○ PERTURBATION ESTIMATE: With $\mathbf{\Phi}\binom{u_1(t)}{x(t)} = \binom{u_1(t)}{x(t-\tau)}$, we compute

$$\|(\mathbf{\Phi}-\mathbf{I})\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}|_{\mathcal{S}_r}\| \leq \frac{2\tau(k(1+k)-2r^2)}{k-2r} =: g(r,k).$$

• The perturbed system is guaranteed to be stable on $\mathcal{S}_{r(1-g(r,k_1))}$ provided $g(r,k_1) < 1$.

∘ $g(r,k)$ is minimal for $k = k_1(r) := 2r + \sqrt{2r^2 + 2r}$.

∘ Using $k = k_1(r)$, the perturbed system is stable for any

$$\tau \leq \tau_0 = \frac{r-1}{2r\left\{4r + 1 + 2\sqrt{2r^2 + 2r}\right\}}.$$

∘ E.g., τ_0 maximal at 0.015 for $r = 2.14$, giving $k_1(r) = 7.9$ and

$$\|\mathbf{\Pi}_{\mathcal{M}_1//\mathcal{N}}|_{\mathcal{S}_1}\| \leq 89.1.$$

- ANALYSIS USING GAIN FUNCTIONS.

○ THEOREM: If ∃ causal bijective $\mathbf{\Phi} : \mathcal{M} \rightarrow \mathcal{M}_1$, and $\epsilon(\cdot) \in \mathcal{K}_\infty$:

$$g[\mathbf{I} - \mathbf{\Phi}] \circ g[\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}](\alpha) \leq (1+\epsilon)^{-1}(\alpha) \text{ for all } \alpha \geq 0,$$

then

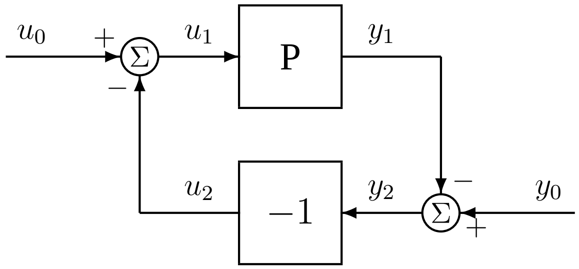
$$g[\mathbf{\Pi}_{\mathcal{M}_1//\mathcal{N}}](\alpha) \leq g[\mathbf{\Phi}] \circ g[\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}] \circ (1+\epsilon^{-1})(\alpha).$$

○ THEOREM...
...if $g[\mathbf{\Pi}_{\mathcal{M}_1//\mathcal{N}} - \mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}](\alpha)$ is "small" then there exists a $\mathbf{\Phi} : \mathcal{M} \rightarrow \mathcal{M}_1$: $g[\mathbf{\Phi} - \mathbf{I}](\alpha)$ is "small".

○ EXAMPLE: A gf-STABLE SYSTEM WITH CUBIC NONLINEARITY.

$$\begin{aligned} \dot{x}(t) &= -x(t)^3 + u_1(t), \quad x(0) = 0 \\ y_1(t) &= x(t) \end{aligned}$$

with negative feedback $u_2(t) = -y_2(t)$.



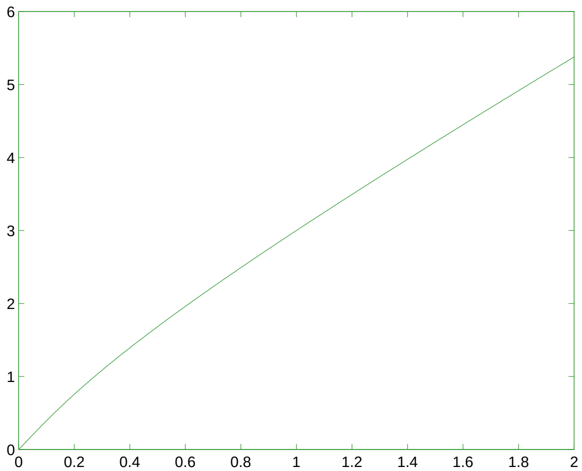
◦ ROBUSTNESS MARGIN/PARALLEL PROJECTION:
Closed loop: $\dot{x}(t) = -x(t)^3 - x(t) + v_0(t)$ with $v_0 = u_0 + y_0$.

◦ Mapping $v_0 \to x$:

$$\begin{aligned}\sup_{\|v_0\|_\infty \le 2\alpha} \|x\|_\infty &\le \inf\{|x| \;:\; x\dot{x} < 0 \text{ for all } |v_0| \le 2\alpha\} \\ &= \ldots = f(2\alpha)\end{aligned}$$

where $f(2\alpha)$ is the unique real root of the equation $x^3 + x = 2\alpha$.

◦ Similarly, $g[\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}](\alpha) = 2\alpha + f(2\alpha)$.



$g[\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}](\alpha)$ vs. α

- TIME-DELAY PERTURBATION:

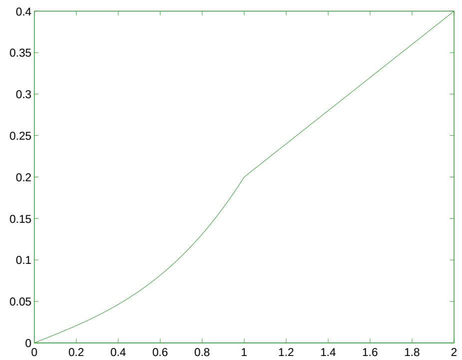
$$\begin{aligned} \dot{x}(t) &= -x(t)^3 + u_1(t), \quad x(0) = 0 \\ y_1(t) &= x(t-h). \end{aligned}$$

Taking $\mathbf{\Phi}\binom{u_1(t)}{x(t)} = \binom{u_1(t)}{x(t-h)}$:

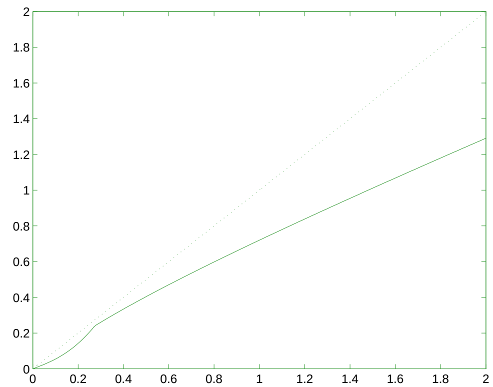
$$g[\mathbf{I} - \mathbf{\Phi}](\alpha)\ldots \;\leq\; h \sup \{\|\dot{x}\|_\tau \;:\; \|u_1\|_\tau \leq \alpha\} \leq 2\alpha h.$$

Also:

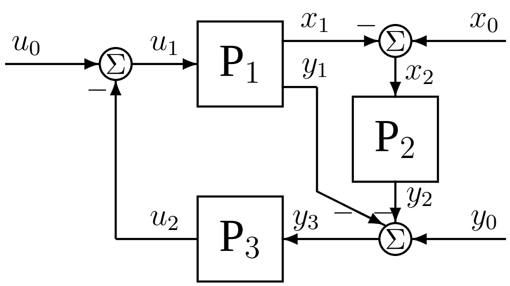
$$g[\mathbf{I} - \mathbf{\Phi}](\alpha)\ldots \;\leq\; h(\alpha^3 + \alpha)$$



$g[\mathbf{I} - \mathbf{\Phi}](\alpha)$ vs. α for $h = 0.1$



$g[\mathbf{I} - \mathbf{\Phi}] \circ g[\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}](\alpha)$ vs. α for $h = 0.12$



- STABILITY/ROBUSTNESS:
  Ambient space: $\mathcal{W} = \mathcal{U} \times \mathcal{X} \times \mathcal{Y} \supset \mathcal{G}_{\mathrm{P}_i} =: \mathcal{M}_i$.

$$\mathcal{G}_{\mathrm{P}_1} = \begin{pmatrix} u_1 \\ \mathbf{P}_1 u_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ x_1 \\ y_1 \end{pmatrix},$$

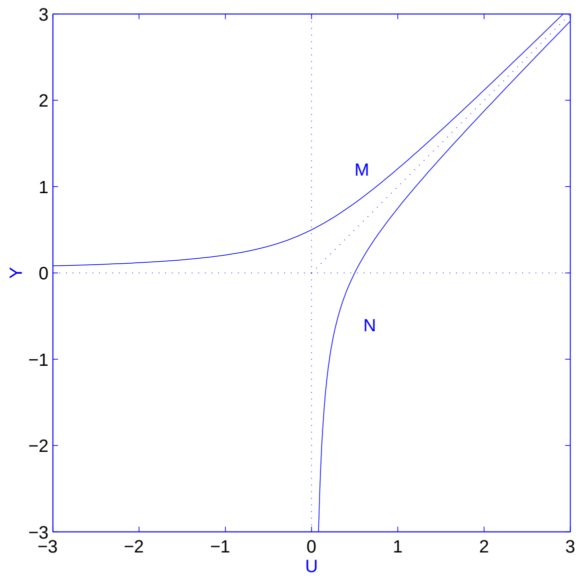
$$\mathcal{G}_{\mathrm{P}_2} = \begin{pmatrix} 0 \\ x_2 \\ \mathbf{P}_2 x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ y_2 \end{pmatrix},$$

$$\mathcal{G}_{\mathrm{P}_3} = \begin{pmatrix} \mathbf{P}_3 y_3 \\ 0 \\ y_3 \end{pmatrix} = \begin{pmatrix} u_3 \\ 0 \\ y_3 \end{pmatrix}.$$

Stability $\sim$ invertibility of

$$\boldsymbol{\Sigma}_{\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3} : \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3 \to \mathcal{W} : (m_1, m_2, m_3) \to m_1 + m_2 + m_3.$$

- THEOREM: If $\boldsymbol{\Sigma}^{-1}_{\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3}$ is stable and $\sum_{i=1}^{3} \delta(\mathcal{M}_i, \mathcal{M}'_i) \|\boldsymbol{\Pi}_i \boldsymbol{\Sigma}^{-1}_{\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3}\| < 1$, then $\boldsymbol{\Sigma}^{-1}_{\mathcal{M}'_1,\mathcal{M}'_2,\mathcal{M}'_3}$ is stable.



○ ACTIVE DIODES:

$\mathcal{U} = \mathcal{Y} = \mathbf{R}$,
$\mathbf{P}u = (u + \sqrt{u^2+1})/2$,
$\mathbf{C}y = (y + \sqrt{y^2+1})/2$

○ $[\mathbf{P}, \mathbf{C}]$ is stable and yet $\mathcal{G}_{\mathbf{P}}$, $\mathcal{G}_{\mathbf{C}}$ do not intersect, neither do they contain the origin

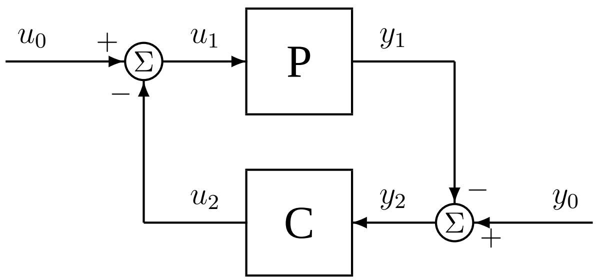
◦ BIASED NORMS:

For $\mathbf{A} : \mathcal{X}_1 \to \mathcal{X}_2$, $\mathcal{X}_i$ $(i = 1, 2)$ signal spaces, $x_0 \in \mathcal{X}_1$,

$$\|\mathbf{A}\|_{(x_0)} := \sup_{\substack{\tau > 0 \\ x_1 \in \mathcal{X}_1 \\ \|x_1 - x_0\|_\tau \neq 0}} \frac{\|\mathbf{A}x_1 - \mathbf{A}x_0\|_\tau}{\|x_1 - x_0\|_\tau}.$$

◦ PROPERTIES:

$$\begin{array}{rl} (i) & \|\mathbf{A}\|_{(x_0)} \geq 0 \text{ and } \|\mathbf{A}\|_{(x_0)} = 0 \Rightarrow \mathbf{A}x = \mathbf{A}x_0 \text{ for all } x \in \mathcal{X}_1 \\ (ii) & \|\lambda\mathbf{A}\|_{(x_0)} = |\lambda| \cdot \|\mathbf{A}\|_{(x_0)} \\ (iii) & \|\mathbf{A} + \mathbf{B}\|_{(x_0)} \leq \|\mathbf{A}\|_{(x_0)} + \|\mathbf{B}\|_{(x_0)} \\ (iv) & \|\mathbf{A}\mathbf{B}\|_{(x_0)} \leq \|\mathbf{A}\|_{(\mathbf{B}x_0)} \cdot \|\mathbf{B}\|_{(x_0)} \\ (v) & \|\mathbf{A}\|_{(x_0)} \geq \|\mathbf{A}\|_{\Delta}. \end{array}$$



• ROBUSTNESS OF STABILITY:

If $\|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}\|_{(z_0)} < \infty$ for some $z_0 \in \mathcal{W}$,
$\mathbf{P}_1$ a perturbed model with $\mathcal{M}_1 := \mathcal{G}_{\mathrm{P}_1}$,
$\mathbf{\Phi} : \mathcal{M} \to \mathcal{M}_1$, with $\|(\mathbf{\Phi} - \mathbf{I})|_{\mathcal{M}}\|_{(g_0)} < \|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}\|_{(z_0)}^{-1}$,

$$g_0 = \mathbf{\Pi}_{\mathcal{M}//\mathcal{N}} z_0, \text{ and } w_0 = \left(\mathbf{I} + (\mathbf{\Phi} - \mathbf{I})\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}\right) z_0,$$

Then

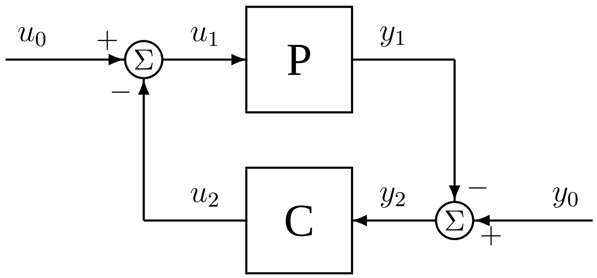
$$\|\mathbf{\Pi}_{\mathcal{M}_1//\mathcal{N}}\|_{(w_0)} \leq \|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}\|_{(z_0)} \frac{1 + \|(\mathbf{\Phi} - \mathbf{I})|_{\mathcal{M}}\|_{(g_0)}}{1 - \|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}\|_{(z_0)} \|(\mathbf{\Phi} - \mathbf{I})|_{\mathcal{M}}\|_{(g_0)}}.$$

$$\dot{x}(t) = -x(t)^3 + \beta x(t-h) + v(t), \text{ and } \beta > 0$$

NOTE: $\dot{x}(t) = -x(t)^3 + \beta x(t)$ has multiple equilibria (0 – unstable, $\pm\sqrt{\beta}$ – stable)

∘ Bring into a feedback framework: $\mathbf{P} = 0$, $\mathbf{C}$ defined by

$$\begin{aligned} \dot{x}(t) &= -x(t)^3 + y_2(t), \qquad x(0) = 0, \\ u_2(t) &= x(t). \end{aligned}$$



∘ Choose $z_0(t) = \begin{pmatrix} 0 \\ r \end{pmatrix}$ with $r > 0$, and compute

$$\|\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}\|_{(z_0)} = 1 + \frac{4}{3} r^{-2/3}$$

∘ Consider $\mathbf{P}_1$ defined by:

$$y_1(t) = \beta \cdot u_1(t-h).$$

For

$$\mathbf{\Phi} \ : \ \begin{pmatrix} u_1(t) \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} \frac{1}{1+\beta} u_1(t) \\ \frac{\beta}{1+\beta} u_1(t-h) \end{pmatrix},$$

it follows that

$$\|\mathbf{I} - \mathbf{\Phi}\|_{(g)} = \frac{\beta}{1+\beta},$$

for any given $g \in \mathcal{M}$.

• APPLY ROBUSTNESS THEOREM:

$$\|\mathbf{\Pi}_{\mathcal{M}_1//\mathcal{N}}\|_{(w_0)} \leq \frac{(1+2\beta)(1+\frac{4}{3}r^{-2/3})}{1-\frac{4}{3}\beta r^{-2/3}},$$

for $w_0 = (\mathbf{I} + (\mathbf{\Phi} - \mathbf{I})\mathbf{\Pi}_{\mathcal{M}//\mathcal{N}}) \begin{pmatrix} 0 \\ r \end{pmatrix}$.

• Bound finite and independent of h (when $\beta < \frac{3}{4} r^{2/3}$)

• GAIN FUNCTION/INDUCED NORM NUMERICAL COMPUTATION.

For general

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)) \\ y(t) &= h(x(t), u(t)). \end{aligned}$$

and any $\alpha \geq 0$ the "value function"

$$V_a(x) = \sup_{\|u\|_\infty \leq a, x(0)=x} \|h(x(\tau), u(\tau))\|_\infty$$

is the smallest lower semicontinuous viscosity solution of

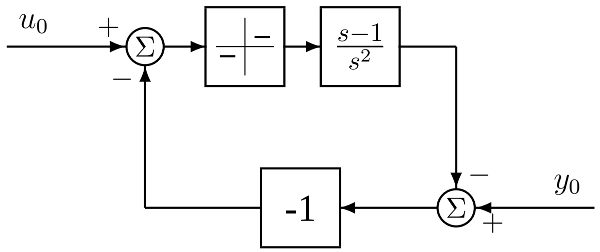
$$\max \left\{ \max_{|u| \leq a} |h(x,u)| - V(x), \max_{|u| \leq a} \frac{\partial V}{\partial x}(x) \cdot f(x,u) \right\} = 0.$$

Then

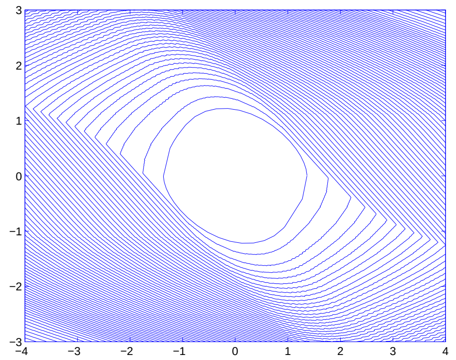
$$g[\mathbf{P}](a) = V_a(0)$$

$$\|\mathbf{P}|_{\mathcal{S}_\alpha}\| = \sup_{a \in (0,\alpha)} \frac{V_a(0)}{a}$$

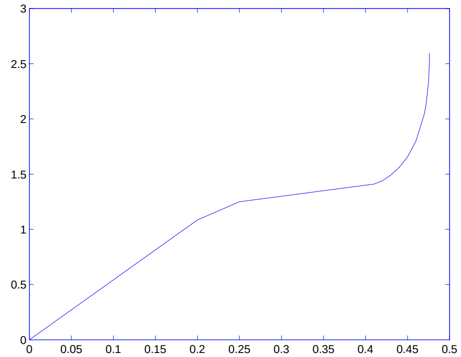
∘ Numerical schemes using dynamic programming.

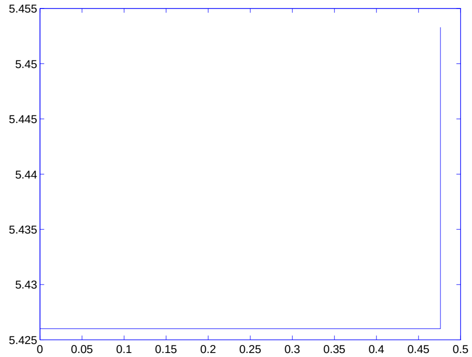


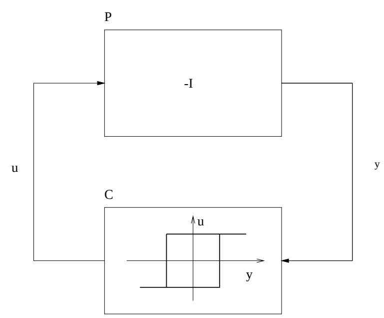
Level curves of $V_a(x)$ for $a = 0.5$



$g[\mathbf{\Pi}_{\mathcal{M},\mathcal{N}}](a)$ and $\|\mathbf{\Pi}_{\mathcal{M},\mathcal{N}}|_{\mathcal{S}_\alpha}\|$ are:







- ROBUSTNESS OF "OSCILLATORY BEHAVIOUR"?

○ SIGNALS:

$$\begin{aligned}\mathrm{Lip}[0,\infty) &= \{y(t),\ t\in[0,\infty)\ :\ y(0)=0,\ \text{and} \\ &\qquad \mathrm{C}_T = \sup\{\frac{|y(s)-y(t)|}{|s-t|}\ :\ s\neq t, s,t\in[0,T)\} < \infty, \}.\end{aligned}$$

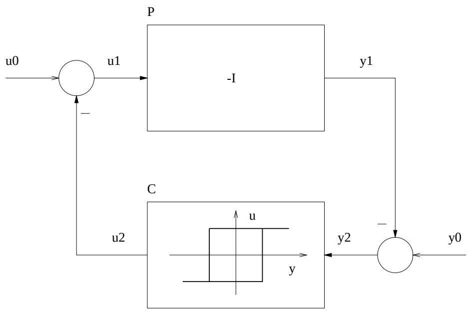
$$\begin{aligned}\mathcal{U} &= \mathcal{L}_\infty[0,\infty), \\ \mathcal{Y} &= \{y(t)\in C[0,\infty)\ :\ y(0)=0\}.\end{aligned}$$

○ SYSTEMS:

$$\mathbf{P}\ :\ u(t)\mapsto y(t)=\int_0^t g(t-\tau)u(\tau)d\tau,$$

with $g(t)$ is piecewise Lipschitz

• The range of $\mathbf{P}$ is a linear submanifold of $\mathrm{Lip}[0, \infty)$.



$$y_1(t) = \int_0^t g(t-\tau)\left(u_0(\tau) - \mathbf{C}\left(y_0 - y_1\right)(\tau)\right) d\tau, \tag{1}$$

• For any $u_0 \in \mathcal{U}$ and $y_0 \in \mathcal{Y}$, $\exists$!solution $y_1 \in \mathcal{Y}$.

The remaining signals in the feedback loop satisfy: $u_1, u_2 \in \mathcal{U}$ and $y_2 \in \mathcal{Y}$.

- For $w_i \in \mathcal{W} := \mathcal{U} \times \mathcal{Y}$ $(i = 1, 2)$ define:

$$d(w_1(t), w_2(t)) := \inf\{\|w_1(t) - w_2(\sigma(t))\|_\infty + \sup_t \frac{|\sigma(t) - t|}{t} \ : \ \sigma \in \mathcal{K}_\infty\},$$

$\mathcal{K}_\infty$ the set of continuous monotonically non-decreasing functions σ of $t \in [0, \infty]$ with $\sigma(0) = 0$ and $\sigma(\infty) = \infty$

Notation: $\sigma w(t) := w(\sigma(t))$

$\mathbf{P}$ a the negative integrator, $\mathbf{P}_1$, and $\mathbf{C}$ be the relay-hysteresis.
their graphs denoted by $\mathcal{M}, \mathcal{M}_1, \mathcal{N}$, respectively.
If there exists a surjective map $\mathbf{\Phi}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}_1$ such that

$$\|(\mathbf{I} - \mathbf{\Phi}_{\mathcal{M}})|_{\mathcal{M}}\| \le \epsilon < \frac{1}{3},$$

then there exists a function $\sigma \in \mathcal{K}_\infty$ such that

$$\sup_t \frac{|\sigma(t) - t|}{t} \le \frac{4\epsilon(1-\epsilon)}{(1-2\epsilon)^2}, \tag{2}$$

and the response of the two feedback systems $[\mathbf{P}, \mathbf{C}]$ and $[\mathbf{P}_1, \mathbf{C}]$ with zero external excitation signals satisfy

$$\|\sigma \mathbf{\Pi}_{\mathcal{M},\mathcal{N}} 0 - \mathbf{\Pi}_{\mathcal{M}_1 \| \mathcal{N}} 0\|_\infty \le \frac{2\epsilon}{1-\epsilon}. \tag{3}$$

☐ Effect of disturbances on nominal trajectory

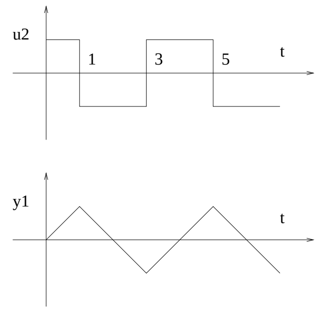


Figure 1: Autonomous response of relay oscillator

☐ Bounds on the forced response and time-scaling function

If a “small” disturbance $w_0 \neq 0$ is applied, the response retains the oscillatory nature, and we construct an appropriate scaling function σ so that $\sigma \mathbf{\Pi}_{\mathcal{M},\mathcal{N}} 0$ is close to $\mathbf{\Pi}_{\mathcal{M},\mathcal{N}} x_0$.

☐ The effect of modelling uncertaity

Global analysis... modified hysteresis

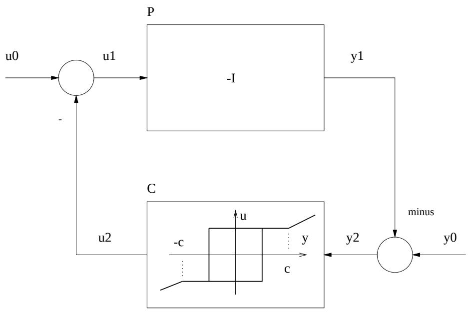


Figure 2: Globally bounded relay oscillator

- The gap topology is natural for studying robustness of stability
  - ◦ Closeness between models $\sim$ Similar closed-loop
  - ◦ Allows comparison between unstable systems, no particular representation,…

- The gap topology may also be the natural topology for studying robustness of oscillators in general.

- Directions:
  - ◦ Computation/estimation of gaps (e.g., L_∞)
  - ◦ Design of robust controllers